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Tense Logic and the Theory of Linear Order

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Philosophy

by

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ABSTRACT OF THE DISSERTATION

Tense Logic and the Theory of Linear Order

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In the past few years several tense operators (like P and F, reading: 'it was the case that...' and 'it will be the case that...,' respectively) have been closely investigated. Axiom-systems for such operators were given and in many cases the deductive completeness of these systems was shown.

No attention was paid, however, to the definitional completeness of such systems: All truth-functional operators are expressible in terms of, say, $\neg$ and $\wedge$. In a similar fashion one might ask whether all tense operators are expressible in terms of given ones (e.g. P and F, together with sentential connectives). But what is a tense operator?

Notice the natural correspondence between the formulae

Pq_1 (where q_1 is a propositional variable) and

$(\exists t)(t < t_0 \wedge Q_1(t))$.

In the second formula the individual variables are thought of as ranging over time (with t_0 "representing" the present) and $<$ as being the temporal order. Similarly between

Sq_1q_2 (reading: 'it has been the case that q_2 ever since it was the case that q_1') and $(\exists t)(t < t_0 \wedge Q_1(t) \wedge \forall t'(t < t' < t_0 \rightarrow Q_2(t')))$, and between

Uq_1q_2 (reading: 'it will be the case that q_2 until it is the case that q_1') and $(\exists t)(t_0 < t \wedge Q_1(t) \wedge \forall t'(t_0 < t' < t \rightarrow Q_2(t')))$.

In this way we associate with each formula ϕ of 1st order logic s.th.

1) ϕ contains one binary relation constant '$<$' and further only monadic predicate constants $Q_1, \ldots, Q_n$,

2) ϕ has exactly one free variable t_0,

an n-place tense operator.

In view of the familiarity of P and F one might wish that all tense operators were expressible in terms of those and sentential connectives. However, if time is dense, the 2-place operator S is not expressible in terms of any set of 1-place operators together with sentential connectives. On the other hand, if time is complete (i.e.

if every nonempty set of moments which has a lower (upper) bound has a greatest lower (least upper) bound), every tense operator is expressible in terms of S, U and sentential connectives. This last result gives a normal form for 1st order formulae with properties 1) and 2).

CHAPTER I

INTRODUCTION

The Sentence

(1) 'Greece is a kingdom,' is true now (May 10, 1968); but it has been false before; and who knows if it will be true tomorrow? This property--of being true at some moments and false at others--(1) shares with many, if not most, sentences of English, and other natural languages. The dependency of the truth of a sentence on the moment at which it is asserted, is a special case of a more general phenomenon: the truth of a sentence may depend on the circumstances under which it is asserted. The time of its assertion is only one aspect of these circumstances; but other aspects may be relevant as well. So will the truth of the sentence: 'I am home' depend not only on the time of assertion but also on the place where it is asserted and the person who asserts it.

Tense logic, however, is concerned only with the way in which the truth of a sentence depends on the time of its assertion. Its aim is to study how the truth of a sentence at one moment is related to the truth of certain other sentences at that or other moments; and in particular to analyze the meaning (or function) of those expressions by means of which we convert a sentence--or combine several sentences--into an assertion about the truth of that sentence--or these sentences at the same or other times. An example of such an expression is 'it was the case that,' which will convert any English sentence s into the sentence 'it was the case that s'; here

(2) the latter sentence is true at a moment t if and only if s is true at some moment previous to t.

It should be clear that the truth of the statement (2) depends exclusively on the meaning of the expression 'it was the case that.' (2) is paradigmatic of the semantic relationships that tense logic deals with; and its close correspondence to the meaning of 'it was the case that' exemplifies the correspondences that exist in general between semantical relationships like the one expressed by (2) on the one hand and expressions like 'it was the case' on the other. This indicates that studying relationships like (2) or expressions like 'it was the case that' really amounts to the same thing.

Other English expressions with which tense logic deals are e.g.

(3) 'it will be the case that,' 'it is being the case that,' 'until,' 'before,' 'after,' 'while.'

Besides them English contains still other devices which form out of a given sentence an assertion about its truth at other moments, but which cannot be identified with particular words or word groups. They are the <u>tenses</u>. As a matter of fact they constitute the most common means of forming out of sentences assertions about their truth at other times; which justifies the name of this area of logic.

In ordinary logic sentences are either true or false. More precisely, any interpretation for a language L of, say, first order logic, will assign to each sentence of L a definite truth value, either T(ruth)(or 1) or F(alsehood) (or 0).

This fact conflicts with what we have observed so far about sentences like (1). Indeed, according to the view embodied in ordinary model theory, (1) is not really a sentence at all, since its truth value is not definite but fluctuates with time; on this view (1) is rather like a formula with one free variable, ranging over times, which is satisfied by a moment whenever, as we say, (1) "is true at" that moment.

Indeed we can symbolize sentence like (1) within first order logic, if we treat them as such formulae. But the theory implicit in such a treatment is unsatisfactory in various ways. In the first place it is rather odd that, as this theory suggests, most English sentences--among them the grammatically most central examples--should not really be sentences at all. I.e. the theory proposes a concept of 'sentence' which seems to have very little to do with the intuitive meaning of the word. In the second place the theory leads us to regard what appear to be n-place relations in English as (n + 1)-place relations. Thus the expression 'is a kingdom'--apparently a 1-place predicate--has to be analyzed as standing for a 2-place relation between countries and times. In this way we can indeed symbolize (1), e.g. by

(4) $K(G,t)$,

where K stands for 'is a kingdom at,' G stands for 'Greece,' and t is a variable that takes moments as values.

This symbolization raises a third difficulty. (4) is meaningful only if times exist. And so this approach commits us to the assumption of the existence of time, in a sense which is prior to language. This commitment is a rather strong one, which some will be reluctant to make. It would be desirable to have a logical theory of that fragment of language in which moments are not mentioned explicitly--and of which (1) and similar sentences are part--which is not based on the assumption that times exist--with the option of developing, on the basis of that theory, a concept of a moment, which could then be used for a semantical account of the part of language where moments are mentioned explicitly.

The three points raised here make the approach suggested above quite unacceptable. Let us therefore abandon ordinary logic as the framework for tense logic. Rather we will accept the lesson that English seems to teach us and recognize that the truth of a sentence may vary with time.

That the truth of a sentence may vary with time, though,--as we said we would assume,--not with other aspects of the context, we can express alternatively by saying that the things expressed by sentences, propositions, are functions from times to truth values rather than simple truth values. If propositions are functions from times to truth values rather than truth values, the number of propositional functions--functions from propositions to propositions will be much larger than that of the so-called truth functions, the functions from truth values to truth values. For example there are only four 1-place truth functions. But if time is infinite then the number of 1-place propositional functions is infinite also. Some of these propositional functions will correspond in a natural way to truth functions. Thus corresponds to the truth function N,--defined by $N(0) = 1$, $N(1) = 0$ - the propositional function N^*, defined by $N^*(p) = \lambda tN(p(t))$. However, most propositional functions do not correspond in this manner to truth functions. As an example may serve the propositional function PAST, defined by:

PAST $(p)(t) = 1$ if and only if there is a t' before t such that $p(t) = 1$. N^* may be called truthfunctional in so far as for any proposition p and moment t the value of $N^*(p)$ at t depends only on the value of p at t (but not on the values of p at other moments). Let us therefore put in general: An n-place propositional function O is <u>truthfunctional</u> if for any propositions $p_1,\ldots,p_n$ and moment t the value of $O(p_1,\ldots,p_n)$ depends only on the values of $p_1,\ldots p_n$ at t. Clearly the truth functions and the truth-functional propositional functions correspond to each other in a one to one way.

Some of the truth functions correspond to expressions of English, like e.g. 'it is not the case that,' 'and,' 'or.' Indeed they may be regarded as the meanings of these expressions. Thus we may regard the function N as the meaning of the expression 'it is not the case that.' If, however, propositions are functions from times to truth values instead of simple truth values, then we should not regard the truth functions themselves as the meanings of such expressions but rather the corresponding truth-functional propositional functions; e.g. the meaning of 'it is not the case that' should be the function N^* rather than the function N. We saw already that many propositional functions are not truth-functional. One may wonder if they too can be regarded as the meanings of certain English expressions. For some this is indeed the case. An example is the function PAST mentioned above, which can be regarded as the meaning of the expression 'it was the case that' or, alternatively, of the past tense. Indeed, if s is any sentence of English, then both the sentence 'it was the case that s' and the past tense of s are true at a moment t if and only if s is true at some moment previous to t. In a similar way we can identify the meanings of the other expressions listed in (3) with non-truth-functional propositional functions; and in general this will be the case for any expression of the class that we indicated, however vaguely, by the examples given in (3).

As we said before, the analysis of expressions like 'it was the case that' is one of the central goals of tense logic. Thus a formal system will be a suitable framework for tense logical investigations only if such expressions can be represented within it. Now both the grammatical function of these expressions and the remarks made above about their meanings show their similarity to the sentential connectives. It is therefore natural to develop a formal system for tense logic in which these expressions are represented as sentential operators. If such a system contains some such operators as primitives then others will in general be expressible in terms of them--in the same way as many--in fact all--sentential connectives can be expressed within a system of propositional calculus the primitive connectives of which are implication and negation. On the other hand composition of truth-functional propositional functions will always produce functions which are again truth-functional. Thus any operator which can be expressed within a system in which the primitive operators are truth functional will again be truth-functional; and a system in which non truth-functional operators are expressible must have such non-truth-functional operators among its primitives.

In fact, several people have developed in the recent past formal systems of tense logic by adding to a given system of ordinary logic certain non-truth-functional operators, so-called tense operators. Many of these systems are propositional calculi, of which only formation rules, axioms and inference rules are specified, but for which no model theory is given. A tense predicate logic was developed by Cocchiarella; he gave besides formation rules, axioms and inference rules also a model theory and showed the deductive completeness of his system. The first system of tense logic which was adequate for its purposes, as we have tried to outline them here, was given by Prior. His system was originally a propositional calculus without formal semantics. Cocchiarella later extended the system to a predicate logic, provided a model theory and proved deductive completeness. In this paper we will pay exclusive attention to the Prior-Cocchiarella system and to systems which are like it except that their primitive tense operators may be different. Presently we will consider only the propositional part of the Prior-Cocchiarella system, essentially Prior's original system together with the natural model theory, which was first developed by Montague. We will call this part 'TL_0.'

TL_0 is characterized as follows:

a) <u>Vocabulary</u>:

sentential constants: $q_0, q_1, \ldots$

sentential operators: 1-place: $\neg$, P, F

2-place: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

b) <u>Formulae</u>:

(i) q_j is a formula

(ii) If ζ , η are formulae, then $\neg\eta$, $(\zeta \wedge \eta)$, $(\zeta \vee \eta)$, $(\zeta \rightarrow \eta)$, $(\zeta \leftrightarrow \eta)$, $P\zeta$ and $F\zeta$ are formulae.

c) Let $\mathcal{T}$ be a binary structure--i.e. a pair $\langle T, < \rangle$, where T is a nonempty set and $<$ is a binary relation on T [1]. (We think of T as the set of moments of time and of $<$ as the earlier-later relation between moments.)

A possible interpretation for TL_0 relative to $\mathcal{T}$ is a pair $\langle Q, \mathcal{R} \rangle$, where Q is a sequence [2] of subsets of T and $\mathcal{R}$

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[1]By a <u>sequence</u> we will always understand a sequence of length ω.

[2]Henceforth $\mathcal{T}$ and $\langle T, < \rangle$ will be binary structures. Moreover, it is always assumed that $\mathcal{T} = \langle T, < \rangle$.

is the function with domain $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, P, F\}$ and range consisting of 1- and 2-place functions from $\wp(T)^3$ into $\wp(T)$ such that

(i) $\mathcal{R}_{\neg}(J) = T - J$, for $J \subseteq T$.

(ii) $\mathcal{R}_{\wedge}(J,K) = J \cap K$ for $J, K \subseteq T$.

(iii) $\mathcal{R}_{\vee}(J,K) = J \cup K$, for $J, K \subseteq T$.

(iv) $\mathcal{R}_{\rightarrow}(J,K) = (T - J) \cup K$, for $J, K \subseteq T$.

(v) $\mathcal{R}_{\leftrightarrow}(J,K) = ((T - J) \cup K) \cap ((T - K) \cup J)$, for $J, K \subseteq T$.

(vi) $\mathcal{R}_P(J) = \{t \in T: \bigvee_{t' \in J} t' < t\}$, for $J \subseteq T$.

(vii) $\mathcal{R}_F(J) = \{t \in T: \bigvee_{t' \in J} t < t'\}$, for $J \subseteq T$.

d) Truth:

For any possible interpretation $\mathfrak{A} = \langle Q, \mathcal{R} \rangle$, formula η of TL_0 and $t \in T$ " η is true at t in $\mathfrak{A}$ " is defined by the following two clauses:

(1) q_j is true at t in $\mathfrak{A}$ iff $t \in Q_j$;

(2) if Z is an n-place sentential operator of TL_0 and $\eta_0, \ldots, \eta_{n-1}$ are formulae of TL_0, $Z\eta_0, \ldots, \eta_{n-1}$ is true at t in $\mathfrak{A}$ iff $t \in i(\{t' \in T: \eta_0$ is true at t' in $\mathfrak{A}\}, \ldots, \{t' \in T: \eta_{n-1}$ is true at t' in $\mathfrak{A}\})$.

(N.B.: in the above clause n is of course either 1 or 2; we gave the definition in this--presently unnecessarily--general form for later reference.)

e) Validity:

A formula η of TL_0 is $\mathcal{T}$-valid iff for all possible interpretations $\mathfrak{A}$ for TL_0 relative to $\mathcal{T}$ and all $t \in T$

[3] For any set A $\wp(A)$ is to be the power set of A.

η is true at t in $\mathfrak{A}$. Two formulae η and ζ are $\mathcal{J}$ -equivalent iff $\eta \leftrightarrow \zeta$ is $\mathcal{J}$-valid.

The valid formulae of ordinary propositional logic are the formal counterparts of those English sentences which are true in view of the meanings of the sentential connectives alone. Similarly the valid formulae of TL_0 ought to be defined in such a way that they are the counterparts of those English sentences which are, as we will say, tense valid, i.e. true just in view of the meanings of the sentential connectives and the past and future tenses. Which English sentences are tense valid depends, however, on the character of time. For example, if time is linear, then a sentence 'If it will be the case that it was the case that s, then either it is the case that s or it was the case that s or it will be the case that s' is tense valid; but such a sentence is not tense valid if time is not linear.

Thus, which formulae of TL_0 should be called valid depends on the properties of time. Tense logic should perhaps rather investigate, for certain properties of time, which sentences are tense valid just in view of those properties. For this purpose we will identify properties of the earlier-later relation with classes of binary structures, and thus come to the following definition of validity relative to a class of binary structures.

Definition: Let $\mathcal{K}$ be a class of binary structures. A formula η is $\mathcal{K}$-valid iff for every $\mathcal{T} \in \mathcal{K}$ η is $\mathcal{T}$-valid. We have to decide which classes $\mathcal{K}$ of binary structures to consider. This decision is inevitably a bit arbitrary. We could take all such classes into account. It seems more natural, however, to assume once and for all that time has certain properties, of which we are firmly convinced, and which are, as it were, implicit in the meanings of the words 'earlier' and 'later.' In fact we will assume that the ordering of time is linear; i.e., we consider $\mathcal{K}$-validity only for classes $\mathcal{K}$ all members of which are linear orderings.

Definition: A binary structure $\langle T, < \rangle$ is a linear ordering iff

(i) $<$ is asymmetric

(ii) $<$ is transitive

and (iii) $<$ is connected on T.

The assumptions that the earlier-later relation is asymmetric and transitive seem to be beyond controversy. Some people may question the plausibility of the assumption that the ordering is connected. We personally feel however that this assumption is also firmly rooted in our intuition of time and of the meanings of tenses.

For future reverence we introduce the following particular linear orderings and classes of linear orderings:

Lin: the class of all linear orderings

Den: the class of all <u>dense</u> linear orderings; i.e., the class of all linear orderings $\langle I, < \rangle$ such that if $i, j \in I$ and $i < j$ then there is a $k \in I$ such that $i < k$ and $k < j$.

Des: the class of all <u>discrete</u> linear orderings; i.e., the class of all linear orderings $\langle I, < \rangle$ such that for all $i \in I$ it is the case that if there is a $j \in I$ such that $i < j$ then there is a $j' \in I$ such that $i < j'$ and for no $k \in I$ $i < k$ and $k < j'$, and if there is a $j \in I$ such that $j < i$ then there is a $j' \in I$ such that $j' < i$ and for no $k \in I$ $j' < k$ and $k < i$.

Com: the class of <u>complete</u> linear orderings; i.e., the class of all linear orderings $\langle I, < \rangle$ such that every nonempty subset of I which has a $<$-lower bound has a $<$-greatest lower bound and every nonempty subset of I which has a $<$-upper bound has a $<$-least upper bound.

$\mathcal{I}n$: the linear ordering $\langle In, <_{\mathcal{I}n} \rangle$ where In is the set of integers and $<_{\mathcal{I}n}$ is the natural ordering of the integers.

$\mathcal{R}a$: the linear ordering $\langle Ra, <_{\mathcal{R}a} \rangle$, where Ra is the set of natural numbers, and $<_{\mathcal{R}a}$ is the natural ordering of the rational numbers.

$\mathcal{R}e$: the linear ordering $\langle Re, <_{\mathcal{R}e} \rangle$, where

$\mathcal{R}e$ is the set of real numbers and $<_{\mathcal{R}e}$ is the natural ordering of the real numbers.

It follows from a result by A. Ehrenfeucht[4] that for all of the classes $\mathcal{K}$ defined above the set of $\mathcal{K}$ - valid formulae of TL_0 is decidable.

Probably the first person to present complete axiom systems for the sets of $\mathcal{K}$- valid formulae for several classes $\mathcal{K}$ was Prior; he had, however, no proofs of their completeness. Cocchiarella appears to have been first to prove the completeness of a particular axiom system for the Lin-valid formulae of TL_0.[5] His method of proof also yields complete axiomatizations for the sets of Lin-, Den-, and $\mathcal{R}$ a-valid sentences.

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[4]A Ehrenfeucht, "Decidability of the Theory of the Linear Order Relation," A.M.S. Notices, Vol. 6, No. 3, Issue 38 (June 1959) pp. 556. See also: H. Läuchli and J. Leonard, "On the Elementary Theory of Linear Order," Fundamenta Mathematica, Vol. 53 (1966), pp. 109-115.

[5]Nino B. Cocchiarella, "Tense and Modal Logic: A Study in the Topology of Temporal Reference" (unpublished dissertation, University of California, Los Angeles, 1966). See also: Nino B. Cocchiarella, "A Completeness Theorem for Tense Logic," Abstract, The Journal of Symbolic Logic, Vol. 31 (1966), p. 689.

CHAPTER II

TENSES AND THEIR EXPRESSIBILITY

The observations made in Chapter I show, in our opinion, that the system TL_0 provides an adequate explanation of the function of the simple past and future tenses. However, as we remarked, there are many other tenses and tense-like expressions in English (cf. (3) on p. 2), and we want an explanation of <u>their</u> function as well.

There is a way in which a formal representation of these other English tense operators can be obtained quite easily: We simply add to the vocabulary of TL_0 new sentential operators of the right number of places and change every possible interpretation $\langle Q, \mathcal{R} \rangle$, relative to $\mathcal{T}$, to a possible interpretation $\langle Q, \mathcal{R}' \rangle$ where $\mathcal{R}'$ is an extension of $\mathcal{R}$ whose domain includes these new operators and which assigns to each such operator the meaning (relative to $\mathcal{T}$) of the corresponding English tense operator.

Let us consider as an example the English expression

(1) . . . 'it was the case that . . . before it was the case that . . . .'

This expression clearly functions as a binary tense operator, and its meaning is the 2-place function $B_{\mathcal{T}}$ from $\mathcal{S}$(T) into $\mathcal{S}(T)$ given by

$$B_{\mathcal{T}}(J,K) = \{t \in T : \bigvee_{t' \in J} \bigvee_{t'' \in K} (t'' < t \text{ and } t' < t'')\}.$$

We can represent (1) by extending TL_0 to a language TL_0' as follows:

(i) we extend the vocabulary of TL_0 with a 2-place sentential operator Z (and adapt, of course, the definition of a formula accordingly); and

(ii) we modify the interpretations $\langle Q, \mathcal{R} \rangle$ of TL_0, relative to $\mathcal{T}$, in such a way that domain $\mathcal{R} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, P, F, Z\}$ and $\mathcal{R}(Z) = B_{\mathcal{T}}$.

The truth-definition of p. 10 will then automatically provide the correct notion of truth for TL_0'. This definition of truth is in agreement with the intended meaning of Z; indeed, one easily verifies that according to this definition it is the case for arbitrary formulae η_0, η_1 of TL_0', interpretations $\mathfrak{A}$ for TL_0', relative to $\mathcal{T}$, and $t \in T$ that

(2). . . $Z\eta_0\eta_1$ is true at t in $\mathfrak{A}$ if and only if there is a t" before t and t' before t" such that η_1 is true at t" in $\mathfrak{A}$ and η_0 is true at t' in $\mathfrak{A}$.

There is, however, another way in which we can represent (1) in TL_0, without introducing a new sentential operator. For there is for any two formulae η_0 and η_1 of TL_0 another formula of TL_0--viz. $P(\eta_1 \wedge P\eta_0)$--the truth of which depends on the truth of η_0 and η_1 in exactly the same way as the truth of $Z\eta_0\eta_1$ depended on the truth of η_0 and η_1. Indeed, one easily verifies that for any possible interpretation $\mathfrak{A}$ for TL_0 relative to $\mathcal{T}$

(3) . $P(\eta_0 \wedge P\eta_1)$ is true at t in $\mathfrak{A}$ iff there is a t" before t and a t' before t" such that η_1 is true at t" in $\mathfrak{A}$ and η_0 is true at t' in $\mathfrak{A}$. (3) justifies us in saying that the complex expression $P(\eta_1 \wedge P\eta_0)$ represents

(4) 'it was the case that η_0 before it was the case that η_1' in TL_0; or, in other words, that there is a 2-place function f which maps formulae of TL_0 into formulae of TL_0 such that for any two formulae η_0 and η_1 of TL_0, $f(\eta_0, \eta_1)(= P(\eta_1 \wedge P\eta_0)!)$ represents (4). This function f is of course completely characterized by the formula $P(q_1 \wedge Pq_0)$, since for any two formulae η_0 and η_1 of TL_0 $f(\eta_0, \eta_1) = [P(q_1 \wedge P\ q_0)]\ \frac{\eta_0}{q_0}\ \frac{\eta_1}{q_1}$ [1].

If a syntactic function--say, of n places,--can be characterized in this manner by any formula of TL_0 at all, then it can also be characterized by a formula of TL_0 which contains the propositional constants $q_0, \ldots, q_{n-1}$ and no others. Let us therefore restrict our attention to

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[1]For any formulae $\eta, \eta_0, \ldots, \eta_n$ and propositional constants $q_0, \ldots, q_n$ we understand by $[\eta]\ \frac{\eta_0}{q_0} \ldots \frac{\eta_n}{q_n}$ the result of substituting in η η_0 for q_0, η_1 for q_1, ..., η_n for q_n.

formulae of this special kind.

Definition 1. A formula η of TL_0 is called an n-place schema if it contains the propositional constants $q_0, \ldots, q_{n-1}$ and no others.

Thus the formula $P(q_1 \wedge Pq_0)$ is indeed a schema. Moreover, the remarks above suggest that we can regard this schema as representing the English expression (1).

An alternative, though essentially equivalent, justification for this is given by the fact that for any possible interpretation $\mathfrak{A}$ for TL_0, relative to $\mathcal{T}$,

(5). . . . $B_{\mathcal{T}}(\{t \in T: q_0$ is true at t in $\mathfrak{A}\}, \{t \in T:$ q_1 is true at t in $\mathfrak{A}\}) = \{t \in T: P(q_1 \wedge Pq_0)$ is true at t in $\mathfrak{A}\})$.

Generalizing from this example we can say that an n-place schema η of TL_0 represents an (English) tense operator S (relative to $\mathcal{T}$) if the meaning M of S (relative to $\mathcal{T}$) is such that

$\{t \in T : \eta$ is true at t in $\mathfrak{A}\} = M(\{t \in T : q_0$ is true at t in $\mathfrak{A}\}, \ldots, \{t \in T : q_{n-1}$ is true at t in $\mathfrak{A}\})$.

We can now rephrase the question whether all English tense operators are expressible within TL_0 as the question if each tense operator is represented by a schema of TL_0. The answer to this question depends of course on the properties of time. Nonetheless it seems to be almost unambiguously negative. For the present progressive tense --whose meaning Pr , relative to $\mathcal{T}$, is defined by

: $\mathrm{Pr}(J) = \{ t \in T : \bigvee_{t', t'' \in T} (t' < t$ and $t < t''$ and $t' \in J$ and $t'' \in J$ and $\bigwedge_{t''' \in T}$ (if $t' < t''' < t''$ then $t''' \in J$)) $\}$, for all $J \subseteq T$,--is represented, relative to $\mathcal{T}$, by no 1-place schema of TL_0, if only $\mathcal{T}$ is a linear ordering and T is infinite. We will prove this assertion in Chapter IV.

In view of this negative result it is natural to look for other tense operators such that within a tense logic with those operators as primitive operators all English tense operators can be represented. We have not yet explained what it means for an operator to be representable in such an arbitrary logic. But the generalization from (5) to this more comprehensive notion is rather straightforward. Let us remark first that the question whether a tense operator is representable (relative to $\mathcal{T}$) in TL_0 depends only on its meaning (relative to $\mathcal{T}$) and the meanings (relative to $\mathcal{T}$) of the primitive operators of TL_0. And this should be the case not just for TL_0, but for tense logics with other primitive operators as well. Therefore we will concentrate on the <u>meanings</u>, relative to $\mathcal{T}$, of the tense operators,--i.e., on functions (in general n-place) from $\mathcal{S}(T)$ into $\mathcal{S}(T)$.

We will define when such a function is ($\mathcal{T}$-) <u>expressible</u> in terms of other such functions. Then an English tense operator can be said to be <u>representable</u> (relative to $\mathcal{T}$) within a tense logical system TL if its meaning (relative to $\mathcal{T}$) is ($\mathcal{T}$-) expressible in terms of the meanings (relative to $\mathcal{T}$) of the primitive operators of TL.

Before we proceed to define $\mathcal{T}$-expressibly, however, let us first determine which functions from $\mathcal{S}$ (T) into $\mathcal{S}$(T) should be regarded as meanings (relative to $\mathcal{T}$) of English tense operators. It is of course absurd to suppose that all such functions are the meanings (relative to $\mathcal{T}$) of English tense operators (especially if T is infinite, which is a natural assumption). One way to attempt to answer this question would be to make up an inventory of all the tense operators of English. But such an investigation would be cumbersome and not very illuminating. We will rather use a more formal device to single out a natural class of functions from $\mathcal{S}$ (T) into $\mathcal{S}$ (T) and then later discuss how this class is related to the class of meanings, relative to $\mathcal{T}$, of English tense operators. We remark that for each of the tense operators mentioned so far its meaning f, relative to $\mathcal{T}$, can be defined by

$f(Q_0,\ldots\ldots,Q_{n-1}) = \{ t \in T : \phi(t,Q_0,\ldots.,Q_{n-1}) \}$, where ϕ is an expression that involves besides t and the Q_i's only the symbol $<$ and other time variables, and is built up from these elements by means of sentential connection and quantification over time. A brief investigation of English confirms that this is the case for a wide variety of English tense operators. This suggests how we can define a class of tenses which is both natural and comprehensive. In order to specify this class we first define a particular first order language L.

Definition 2. L is the first order language characterized by the following vocabulary:

Variables: $t_0, t_1, \ldots$

Logical constants: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ (connectives);

$\forall$, $\exists$ (quantifiers)

Non-logical constants: a) $<$ (2 - place predicate constant)

b) $P_0, P_1, \ldots$ (all 1 - place predicate constants

(Formulae of L and models for L are defined in the usual way.)

We will pay particular attention to those formulae of L which contain the variable t_0 free and have no other free variables, and which, for some number n, contain the predicate constants $P_0, \ldots, P_{n-1}$, and no other 1 - place predicate constants. We will call such formulae n - place tense defining formulae.

Definition 3. a) For any n - place tense defining formula ϕ of L the $\mathcal{T}$- tense defined by ϕ (in symbols: $Te_{\mathcal{T}}(\phi)$) is the n - place function f from $\mathcal{S}(T)$ to $\mathcal{S}(T)$ such that for $Q_0, \ldots, Q_{n-1} \subseteq T$, $f(Q_0, \ldots, Q_{n-1}) = \{ t \in T$: t satisfies ϕ in $\langle T, <, \quad Q_0, \ldots, Q_{n-1}, \ldots \rangle$.

b) An n - place function from $\mathcal{S}(T)$ into $\mathcal{S}(T)$ is a <u>first order definable tense</u> if it is the tense defined by some tense defining formula of L.

We will from now on restrict our attention to first order definable tenses, and we will often refer to these simply as 'tenses.'

Now that we have specified the set of tenses with which we want to deal, let us reconsider the problem whether there exists a finite number of tenses in terms of which all other tenses are expressible.

Before defining, for any $\mathcal{T}$ - tense f and set of $\mathcal{T}$-tenses S, 'f is expressible in terms of S' we introduce the notions of a <u>sentential language</u> for a set of tense defining formulae of L and of a <u>sentential language</u> for a set of $\mathcal{T}$-tenses.

<u>Definition 4</u>. Let S be a set of tense defining formulae.

a) A <u>sentential language</u> for S is a one-one function G with range S and with a domain consisting of symbols different from $q_0, q_1, \ldots$.

b) Let G be a sentential language for S.

(i) Formulae of G are defined by:

1) q_j is a formula

2) If $Z \in \text{Dom } G$, $G(Z)$ is an n-place tense defining formula and $\eta_0, \ldots, \eta_{n-1}$ are formulae of G, then $Z\,\eta_0 \ldots \eta_{n-1}$ is a formula of G.

(ii) A formula of G is an <u>n-place schema</u> if it contains all the constants $q_0, \ldots, q_{n-1}$ and no others.

(iii) An interpretation determined by G, relative to $\mathcal{T}$, is a pair $\langle Q, \mathcal{R} \rangle$, where

a) Q is a sequence of subsets of T.

b) $\mathcal{R}$ is a function.

c) Dom $\mathcal{R}$ = Dom G.

d) For $Z \in$ Dom $\mathcal{R}$, $\mathcal{R}_Z = Te_{\mathcal{T}}(G(Z))$

For η a formula of G, $\mathfrak{A}$ an interpretation for G relative to $\mathcal{T}$ and $t \in T$, 'η <u>is true at</u> t <u>in</u> $\mathfrak{A}$' is defined as under d) of the characterization of TL_0 (cf. Chapter I, p. 10). Similarly part e) of that characterization gives the notions of $\mathcal{T}$-<u>validity</u> and $\mathcal{T}$-<u>equivalence</u> for formulae of G.

<u>Definition 5</u>. Let S be a set of $\mathcal{T}$-tenses

a) G is a <u>sentential language</u> for S if and only if there is a set S' of tense defining formulae such that G is a sentential language for S' and there is a one-one function D from S onto S' such that for all f in S f is the $\mathcal{T}$-tense $\mathcal{T}$-expressed by D(f).

b) Let G be a sentential language for S.

1) $\mathcal{R}_{\mathcal{T}}$ (G) is the function $\mathcal{R}$ such that Dom $\mathcal{R}$ = Dom G and for $Z \in$ Dom $\mathcal{R}$ $\mathcal{R}(Z)$ is the $\mathcal{T}$-tense defined by G(Z) (If throughout a given discussion the structure $\mathcal{T}$ is fixed, we will often write $\mathcal{R}$ instead of $\mathcal{R}_{\mathcal{T}}$).

2) Let η be an n-place schema. The $\mathcal{T}$-tense η <u>expressed by</u> η <u>relative to</u> G is the n-place function from $\mathscr{S}$ (T) into $\mathscr{S}$ (T) such that for any subsets $Q_0, \ldots Q_{n-1}$ of T, $f(Q_0, \ldots Q_{n-1}) = \{ t \in T$: is true at t in $\langle \langle Q_0, \ldots, Q_{n-1}, \ldots \rangle, \mathcal{R}(G) \rangle \}$

c) Let f be an n-place $\mathcal{T}$-tense. Then f is <u>$\mathcal{T}$-expressible in terms</u> of S if there is a sentential language G for S and an n-place schema of G such that f is the $\mathcal{T}$-tense expressed by η relative to G.

In Chapter I we were primarily interested in an absolute notion of validity for formulae of TL_0, rather than the relativized notion of $\mathcal{T}$- validity. Similarly, in the present context we would like to find a natural absolute notion of expressibility rather than the relative notion of $\mathcal{T}$- expressibility defined above. As in Chapter I we can, in order to obtain such an absolute notion, either identify time with a particular linear ordering $\mathcal{T}$ and thus define <u>expressibility</u> as $\mathcal{T}$ - expressibility; or else consider various classes of linear orderings. Our objection against the first alternative is the same as before. We will again pursue the second alternative.

<u>Definition 6</u>. Let $\mathcal{K}$ be a class of linear orderings.

a) Let ϕ be an n-place tense defining formula of L.

The n - <u>place</u> $\mathcal{K}$ - <u>tense-function defined by</u> ϕ (in symbols:

$\mathrm{Te}_{\mathcal{K}}(\phi))$ is the function f with domain $\mathcal{K}$, where for every $\mathcal{I} = \langle I, < \rangle \in \mathcal{K}$ $f_{\mathcal{I}}$ is an n-place function from $\mathcal{S}(T)$ to $\mathcal{S}(T)$ such that for all $Q_0, \ldots, Q_{n-1} \subseteq I$, $f_{\mathcal{I}}(Q_0, \ldots, Q_{n-1}) = \{ i \in I : i$ satisfies ϕ in $\langle I, <, Q_0, \ldots, Q_{n-1}, \ldots \rangle \}$.

b) An n-place (first order definable) $\mathcal{K}$ - tense-function is a function f such that for some n-place tense defining formula ϕ f is the n-place $\mathcal{K}$ - tense-function defined by ϕ.

When giving names to particular $\mathcal{T}$ - tenses and $\mathcal{K}$ - tense-functions, we will follow this policy: We attach names primarily to tense defining formulae. For example we call the formula $\neg P_0(t_0)$ 'NOT' and the formula $(\exists t_1)(t_1 < t_0 \wedge P_0(t_1))$ 'PAST.' We then denote, for any class $\mathcal{K}$ of linear orderings, the $\mathcal{K}$ - tense-function defined by such a tense defining formula, by the name of that formula with subscript $\mathcal{K}$. Thus the $\mathcal{K}$ - tense-function defined by NOT will be denoted by '$\mathrm{NOT}_{\mathcal{K}}$' and the $\mathcal{K}$ - tense defined by PAST, by '$\mathrm{PAST}_{\mathcal{K}}$.' In contexts where no ambiguity regarding $\mathcal{K}$ can arise we will often omit the subscript $\mathcal{K}$.

Let us now give names to a few tense defining formulae which will play a prominent role in what follows:

NOT is the formula $\neg P_0(t_0)$

AND is the formula $(P_0(t_0) \wedge P_1(t_0))$

PAST is the formula $(\exists\, t_1)(t_1 < t_0 \wedge P_0(t_1))$

FUTURE is the formula $(\exists\, t_1)(t_0 < t_1 \wedge P_0(t_1))$

SINCE is the formula $(\exists\, t_1)(t_1 < t_0 \wedge P_0(t_1)$

$$\wedge\ (\forall\, t_2)(t_1 < t_2 \wedge t_2 < t_0 \rightarrow P_1(t_2)))$$

UNTIL is the formula $(\exists\, t_1)(t_0 < t_1 \wedge P_0(t_1) \wedge$

$$(\forall\, t_2)(t_0 < t_2 \wedge t_2 < t_1 \rightarrow P_1(t_2)))$$

<u>Definition 7</u>. Let $\mathcal{K}$ be a class of linear orderings, S a class of $\mathcal{K}$ - tense-functions.

a) G is a <u>sentential language</u> for S if there exists a set S' of tense defining formulae and a one-one function D from S onto S' such that for any $f \in S$, f is the $\mathcal{K}$ - tense-function defined by D(f), and G is a sentential language for S'.

b) Let G be a sentential language for S, η an n-place schema of G. The <u>n-place</u> $\mathcal{K}$ - <u>tense-function expressed by</u> η is the n-place $\mathcal{K}$ - tense-function f such that for any $\mathcal{I} = \langle I, < \rangle \in \mathcal{K}$ and $Q_0, \ldots, Q_{n-1} \subseteq I$,

$$f_{\mathcal{I}}\ (Q_0, \ldots, Q_{n-1}) = \{\, i \in I\colon\ \eta \text{ is true at } i \text{ in } \langle\langle Q_0, \ldots, Q_{n-1}, \ldots\rangle\ , \mathcal{R}_{\mathcal{I}}\ (G)\rangle \}$$

c) Let f be an n-place $\mathcal{K}$ - tense-function. f is $\mathcal{K}$ - <u>expressible in terms of</u> S if and only if there is a sentential language G for S and an n-place schema η of G such that f is the n-place $\mathcal{K}$ - tense-function $\mathcal{K}$ -

expressed by η .

Clearly, if $\mathcal{K}$ is a class of linear orderings, $\mathcal{K}' \subseteq \mathcal{K}$, S is a set of $\mathcal{K}$- tense-functions and f is a $\mathcal{K}$-tense-function which is $\mathcal{K}$-expressible in terms of S, then the $\mathcal{K}'$- tense-function $f \restriction \mathcal{K}'$ is $\mathcal{K}'$ -expressible in terms of the set $\{g \restriction \mathcal{K}' : g \in S\}$ of $\mathcal{K}'$ - tense-function. In particular, if all Lin - tense-functions are expressible in terms of some given Lin - tense-functions $f_0, \ldots, f_{r-1}$, then for any class $\mathcal{K}$ of linear orderings it is the case that all $\mathcal{K}$ - tense-functions are $\mathcal{K}$ - expressible in terms of the $\mathcal{K}$ - tense-functions $f_0 \restriction \mathcal{K}, \ldots, f_{r-1} \restriction \mathcal{K}'$. Unfortunately we are unable to exhibit a finite collection of Lin - tense-functions in terms of which all other Lin - tense-functions are Lin - expressible. The question whether such a collection can be exhibited at all, is open. However,

Theorem 1. All Com - tense-functions are Com - expressible in terms of the Com - tense-functions NOT_{Com}, AND_{Com}, $SINCE_{Com}$ and $UNTIL_{Com}$.

The proof of theorem 1 will be given in Chapter III. As this proof will show, theorem 1 can be strengthened to

Theorem 2. There is a primitive recursive function Exp from tense defining formulae to schemata of TL_1, such that for any tense defining formula ϕ Exp (ϕ) Com - expresses the Com - tense-function defined by ϕ .

Theorems 1 and 2 show the importance of sentential languages for the set NOT, AND, SINCE and UNTIL. Indeed, it will be convenient in the sequel to be able to refer to one particular such language. Therefore we define

Definition 8. TL_1 is the sentential language $\langle \neg$,NOT$\rangle$, $\langle \wedge$,AND$\rangle$, $\langle$ S , SINCE $\rangle$, $\langle$ U, UNTIL $\rangle$ for the set $\{$ NOT, AND, SINCE, UNTIL $\}$.

Our basic goal was to design a tense logical system in which all English tense operators can be represented. In the last few pages we have been dealing only with the rather abstract notion of a first order definable tense. The relevance of that discussion--and in particular of theorems 1 and 2--for our original problem depends on the relationship between this notion and the tense operators of English. Part of this relationship is clarified by theorems 1 and 2 themselves. Indeed, they tell us that--provided time is complete--all first order definable tenses are expressible by what we are justified in calling English tense operators. It suffices to notice that the tenses SINCE and UNTIL are the meanings of the English expressions.

(5) 'It has been the case that ... since it was the case that ...' and

(6) 'It will be the case that ... until it will be the case that ...,' respectively. Thus by theorem 2 we can for each first order definable tense effectively find a complex English expression, built up with the help of the words 'not' and 'and' and the expressions (5) and (6), of which this tense is the meaning.

This still leaves open the question whether all English tense operators are representable in a language like TL_1. This question is of course not without ambiguity for it depends on what English expressions we are prepared to regard as tense operators. One easily verifies that indeed a very large number of expressions which are naturally classified as tense operators because of their function have first order definable tenses as their meanings. Yet there are expressions which deserve to be regarded as tense operators but which are nonetheless not representable within TL_1. The words 'mostly' and 'usually' are examples of such expressions. The impossibility of representing these particular expressions stems from the fact that their meanings involve a measure on time in an essential manner. In fact this seems to be the case whenever an English expression that should be regarded as a tense operator cannot be represented within TL_1:

For each of the English expressions which we have encountered so far and which, though naturally regarded as tense operators, cannot be represented within TL_0, the impossibility of representing it stems from the measure-theoretic aspects of its meaning.

Thus far we have only considered sentential tense logics. We will now turn briefly to predicate tense logic. We first consider Cocchiarella's system, of which we have already presented the sentential part, viz. TL_0. This system--to which we will refer as TL_2--is characterized as follows:

Definition 9.

Vocabulary: Variables $v_0, v_1, \ldots$

Logical constants $\neg$, P, F (1-place sentential operators)

$\wedge, \rightarrow, \leftrightarrow$ (2-place sentential operators)

$\exists, \forall$ (quantifiers)

Non-logical constants $q_0, q_1, \ldots$ (sentential constants)

$Q^1_0, Q^1_1, \ldots, Q^2_0, Q^2_1, \ldots$ (1-place, 2-place,...predicate letters; the superscript gives the number of places.)

Formulae:

1) q_1 is a formula.

2) If $x_0, \ldots, x_{n-1}$ are variables then $Q^n_i\, x_0, \ldots, x_{n-1}$ is a formula and $x_0 < x_1$ is a formula.

3) If ϕ, ψ are formulae then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$, $P\phi$, $F\phi$, are formulae

4) If ϕ is a formula and x is a variable then $(\exists x)\phi$, and $(\forall x)\phi$ are formulae.

Semantics: A <u>possible interpretation for</u> TL_2 <u>relative to</u> $\mathcal{T}$, is a triple $\langle A, F, \mathcal{R} \rangle$ such that

1) A is a function with domain T and a range consisting of sets, at least one of which is not empty.

2) F is a function the domain of which is the set of predicate letters and sentential letters of TL_2.

3) $F(q_1)$ is a proposition.

4) $F(Q^n_1)$ is a function with domain T such that for $t \in T$ $F(Q^n_1)(t)$ is an n-place relation on $\bigcup \mathcal{Q}A$.[3]

5) $\mathcal{R}$ is a function with domain $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, P, F\}$ and range consisting of 1- and 2-place functions from $\mathcal{S}(T)$ to $\mathcal{S}(T)$ such that

(i) $\mathcal{R}_\neg(J) = T - J$ for $J \subseteq T$

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[3]For any function of $\mathcal{Q}f$ is the range of f.

(ii) $\mathcal{R}_{\wedge}(J,K) = J \cap K$ for $J,K \subseteq T$

(iii) $\mathcal{R}_{\vee}(J,K) = J \cup K$ for $J,K \subseteq T$

(iv) $\mathcal{R}_{\rightarrow}(J,K) = (T - J) \cup K$ for $J,K \subseteq T$

(v) $\mathcal{R}_{\leftrightarrow}(J,K) = ((T - J) \cup K) \cap ((T - K) \cup J)$ for $J,K \subseteq T$

(vi) $\mathcal{R}_{P}(J) = \{ t \in T : \bigvee_{t' \in J} (t' <_T t) \}$ for $J \subseteq T$

(vii) $\mathcal{R}_{F}(J) = \{ t \in T : \bigvee_{t' \in J} (t <_T t') \}$ for $J \subseteq T$

If $\mathfrak{A}$ is a possible interpretation for TL_2 then by $U_{\mathfrak{A}}$ we understand the set U such that for some A, F, $\mathcal{R}$, $\mathfrak{A} = \langle A, F, \mathcal{R} \rangle$ and $U = \bigcup \bigcap A$

<u>Satisfaction</u>: Let $\mathfrak{A} = \langle A, F, \mathcal{R} \rangle$ be a possible interpretation for TL_2, relative to $\mathcal{T}$.

For $t \in T$, a sequence $\underline{a}$ of elements of U, and a formula ϕ of TL_2 '$\underline{a}$ sat(isfies) ϕ at t in $\mathfrak{A}$' is defined as follows:

(i) $\underline{a}$ sat q_1 at t in $\mathfrak{A}$ iff $t \in F(q_1)$.

(ii) $\underline{a}$ sat $Q_1^n v_{k_0}, \ldots v_{k_{n-1}}$ at t in $\mathfrak{A}$ iff

$$\langle \underline{a}_{k_0}, \ldots, \underline{a}_{k_{n-1}} \rangle \in F(Q_1^n)(t).$$

(iii) $\underline{a}$ sat $\neg \phi$ at t in $\mathfrak{A}$ iff $t \in \mathcal{R}_{\neg}(\{ t' \in T:$ $\underline{a}$ sat ϕ at t' in $\mathfrak{A} \})$,

(iv) $\underline{a}$ sat $(\phi \wedge \psi)$ at t in $\mathfrak{A}$ iff $t \in \mathcal{R}_{\wedge}(\{t' \in T:$ $\underline{a}$ sat ϕ at t' in $\mathfrak{A}\}$, $\{ t' \in T:$ $\underline{a}$ sat ψ at t' in $\mathfrak{A} \})$

(v) $\underline{a}$ sat $(\phi \vee \psi)$ at t in $\mathfrak{A}$ iff $t \in \mathcal{R}_{\vee}($ $\{t' \in T: \underline{a}$ sat ϕ at t' in $\mathfrak{A}\}$, $\{t' \in T: \underline{a}$ sat ψ at t' in $\mathfrak{A}\})$

(vi) $\underline{a}$ sat $(\phi \rightarrow \psi)$ at t in $\mathfrak{A}$ iff $t \in \mathcal{R}_{\rightarrow}($ $\{t' \in T: \underline{a}$ sat ϕ at t' in $\mathfrak{A}\}$, $\{t' \in t: \underline{a}$ sat ψ at t' in $\mathfrak{A}\})$

(vii) $\underline{a}$ sat $(\phi \leftrightarrow \psi)$ at t in $\mathfrak{A}$ iff $t \in \mathcal{R}_{\leftrightarrow}($ $\{t' \in T: \underline{a}$ sat ϕ at t' in $\mathfrak{A}\}$, $\{t' \in T: \underline{a}$ sat ψ at t' in $\mathfrak{A}\})$

(viii) $\underline{a}$ sat Pϕ at t in $\mathfrak{A}$ iff $t \in \mathcal{R}_{\underline{P}}(\{t' \in T:$ $\underline{a}$ sat ϕ at t' in $\mathfrak{A}\})$.

(ix) $\underline{a}$ sat Fϕ at t in $\mathfrak{A}$ iff $t \in \mathcal{R}_{\underline{F}}(\{t' \in T:$ $\underline{a}$ sat ϕ at t' in $\mathfrak{A}\})$

(x) $\underline{a}$ sat $(\forall v_1)\phi$ at t in $\mathfrak{A}$ iff for all $b \in U_{\mathfrak{A}}$, $\underline{a}\frac{b}{1}$ sat ϕ at t in $\mathfrak{A}$.

(xi) $\underline{a}$ sat $(\exists v_1)\phi$ at t in $\mathfrak{A}$ iff for some b $\in U_{\mathfrak{A}}$, $\underline{a}\frac{b}{I}$ sat ϕ at t in .

(By $\underline{a}\frac{b}{I}$ we understand the sequence which is like $\underline{a}$ except that the i-th member is b.)

As before we say, for any class $\mathcal{K}$ of binary structures, that a formula η of TL_2 is $\mathcal{K}$ - $\underline{\text{valid}}$ if or every $\mathcal{J} = \langle I, < \rangle$ and possible interpretation $\mathfrak{A}$ for TL_2 relative to $\mathcal{J}$, sequence $\underline{a}$ of members of $U_{\mathfrak{A}}$ and $i \in I$, $\underline{a}$ sat η at i in $\mathfrak{A}$.

Cocchiarella succeeded in axiomatizing the set of Lin-valid formulae of TL_2. However, not for all classes $\mathcal{K}$ which we considered before is the set of $\mathcal{K}$-valid formulae of TL_2 axiomatizable. Indeed, D. Scott has recently shown that the set of $\mathcal{I}n$ - valid formulae and the set of $\mathcal{R}e$ - valid formulae of TL_2 are not even arithmetical.

In view of theorem 1 it is more natural to develop a predicate tense logic the primitive sentential operators of which denote, besides some sentential connectives, the tenses SINCE and UNTIL.

We obtain one such system--let us call it 'TL_3'--by extending TL_1 in the same way as we extended TL_0 to obtain TL_2. If we assume that the ordering of time is complete, then TL_3 is, like TL_1, a system in which all first order definable tenses can be represented.

If we extend TL_3 to a system TL_3' which contains the new n-place sentential operator Z, and let the n-place Com-tense-function f be the 'meaning' of Z (i.e. we change the notion of a possible interpretation $\langle A, F, \mathcal{R} \rangle$, relative to $\mathcal{T}$, so that the domain of $\mathcal{R}$ includes Z and $\mathcal{R}_Z = f_{\mathcal{T}}$), then every formula ϕ of TL_3' will be Com-equivalent to a formula ϕ' of TL_3.

We obtain ϕ' as follows. Let η be an n-place schema of TL_1 which Com - expresses f. (By theorem 1 there is such a schema!) We replace every part of ϕ of the form $Z\psi_0 \ldots \psi_{n-1}$ by the formula $[\eta]\frac{\psi_0}{q_0} \ldots \frac{\psi_{n-1}}{q_{n-1}}$. One easily verifies for the formula ϕ' which results from those replacements, that if $\mathfrak{A}$ is a possible interpretation for TL_3', $t \in T$ and a is a sequence of elements of $U_{\mathfrak{A}}$, then a satisfies ϕ' at t in $\mathfrak{A}$ iff a satisfies ϕ at t in $\mathfrak{A}$.

This functional completeness property makes TL_3 a very natural system for tense logic. In particular it is to be preferred over TL_2 as its expressing force is so much larger and its syntax and semantics are hardly more complicated. As a matter of fact many simple English tense operators which occur frequently and whose meaning is crucial to many important philosophical arguments, can be represented in TL_3 but not in TL_2. Examples of such expressions are: the expressions (5) and (6), the present progressive tense and the expressions:

'It has been the case for some time that...'

'It will be the case for some time that...'

'It has been the case that ... since the last time that it was the case that...'

'It will be the case that ... before the first time that it will be the case that ...'

The representability of these tense operators within TL_3 follows--if we assume that time is complete--from the easily verifiable fact that their meanings are first order definable. That these operators cannot be represented within TL_2 is harder to show. Moreover, the non-representability of such an operator within TL_2, relative to $\mathcal{T}$, can be shown only if $\mathcal{T}$ satisfies certain conditions (for example, if $\mathcal{T}$ is a linear ordering and T is finite then all the expressions above <u>can</u> be represented within TL_2, relative to $\mathcal{T}$). Some theorems related to this matter are given in Chapter IV.

In the same chapter we will also show that a system simpler than TL_3 (e.g. a system where all the primitive tense operators other than the truth functional connectives are 1-place), but with the same expressive power, does not exist.

From the last two paragraphs we may conclude that TL_3 is inferior to no other systems which are like it except for the meanings of their primitive sentential operators. In view of the distinguished position that TL_3 occupies among all systems of this type, it would be of particular interest to give, for various classes $\mathcal{K}$ of linear orderings, complete axiomatizations for its $\mathcal{K}$-valid formulae. Of course axiomatization of the set of

$\mathcal{K}$ - valid formulae of TL_3 is impossible for all those classes $\mathcal{K}$ for which the set of $\mathcal{K}$ - valid formulae of TL_2 can not be axiomatized. On the other hand it is easily seen that the sets of Lin - valid, of Den - valid, of $\mathcal{R}$a - valid and of Dis - valid formulae of TL_3 (as well as the sets S of $\mathcal{K}$ - valid formulae of several other subclasses $\mathcal{K}$ of Lin) can be axiomatized. However, no interesting and intuitively satisfying axiomatization of any of these sets has been given yet.

The importance of theorems 1 and 2 for the problem of finding a simple and adequate system for tense logic, depends on the plausibility of the assumption that time is complete. Of the most natural hypotheses about the character of time, two--viz. that it is like the integers and that it is like the real numbers--imply that it is complete. Indeed, we have the immediate corollaries of theorem 1:

Corollary 1: All $\mathcal{R}$e - tenses are $\mathcal{R}$e - expressible in terms of the $\mathcal{R}$e - tenses $NOT_{\mathcal{R}e}$, $AND_{\mathcal{R}e}$, $SINCE_{\mathcal{R}e}$ and $UNTIL_{\mathcal{R}e}$

Corollary : All $\mathcal{I}$n- tenses are Re - expressible in terms of the $\mathcal{I}$n - tenses $NOT_{\mathcal{I}n}$, $AND_{\mathcal{I}n}$, $SINCE_{\mathcal{I}n}$ and $UNTIL_{\mathcal{I}n}$.

But if time is like the rational numbers, then its ordering is not complete. And in this case theorem 1 definitely fails. Indeed, it will be shown in Chapter IV that some

$\mathcal{R}$a - tense is not $\mathcal{R}$a expressible in terms of NOT$_{\mathcal{R}a}$, AND$_{\mathcal{R}a}$, SINCE$_{\mathcal{R}a}$ and UNTIL$_{\mathcal{R}a}$. However, since theorem 1 holds for the real numbers the $\mathcal{R}$a - tenses which are not $\mathcal{R}$a - expressible in terms of the above-mentioned tenses must be intimately related to the differences between the real numbers and the rationals, i.e. with the existence or non-existence of least upper and greatest lower bounds. We do not believe that such tenses can be expressed in English without explicit reference to moments. Thus there is good reason to assume that even if time is like the rational numbers those first order tenses which can be expressed by means of English tense operators are expressible in terms of NOT, AND, SINCE and UNTIL.

CHAPTER III

THE MAIN THEOREM

This chapter is concerned exclusively with the proof of Theorem II.1.

Let us notice first that Theorem II.1 can be regarded as a normal form theorem for the tense defining formulae of L. Indeed let $\mathcal{C}$ be the set of formulae defined by:

(i) $P_i(t_0) \in \mathcal{C}$

(ii) If $\phi, \psi \in \mathcal{C}$ then $\neg\phi, (\phi \wedge \psi) \in \mathcal{C}$

(iii) If $\phi, \psi \in \mathcal{C}$ and i, j are the first and second numbers respectively such that t_i and t_j do not occur in ϕ or ψ then

$$(\exists t_i)(t_i < t_0 \wedge [\phi]\frac{t_i}{t_0}) \wedge (\forall t_j)(t_i < t_j \wedge t_j < t_0 \rightarrow [\psi]\frac{t_j}{t_0})) \in \mathcal{C}$$ [1]

and

$$(\exists t_i)(t_0 < t_i \wedge [\phi]\frac{t_i}{t_0} \wedge (\forall t_j)(t_0 < t_j \wedge t_j < t_i \rightarrow [\psi]\frac{t_j}{t_0})) \in \mathcal{C}.$$

(iv) No formula ϕ is in $\mathcal{C}$ except by (i), (ii), and (iii).

Further, let, for any class $\mathcal{K}$ of binary structures and formula ϕ of L, ϕ be $\mathcal{K}$-valid if for every

[1]For any formula ϕ of L and variables x, y of L we understand by $[\phi]\frac{x}{y}$ the result of replacing y by x throughout ϕ.

$\langle I, < \rangle \in \mathcal{K}$, $i \in I$ and sequence Q of subsets of I, i satisfies ϕ in $\langle I, <, Q_0, Q_1, \ldots \rangle$; and let two formulae ϕ and ψ of L be $\mathcal{K}$ - equivalent if $\phi \leftrightarrow \psi$ is $\mathcal{K}$ - valid. For ϕ a formula of L and $\psi \in \mathcal{C}$ we say that ψ $\mathcal{K}$- expresses ϕ iff ϕ and ψ are $\mathcal{K}$ - equivalent .

Then Theorem II.1 is equivalent to the statement (theorem 2 below):

(A) For every tense defining formula ϕ of L there is a tense defining formula ϕ' in $\mathcal{C}$ which is Com-equivalent to ϕ .

In order to show the equivalence of (A) and Theorem II.1 let us introduce the following notions:

Definition 1. $\underline{E}$ is an enumeration of the formulae of L. Let ϕ be a formula of L. Let ψ be a formula of L with exactly one free variable x and let Q be a 1-place predicate letter of L. By $[\phi]\frac{\psi}{Q}$ we understand the formula ϕ' which we obtain by replacing each subformula of ϕ of the form $Q(y)$ by the formula $[\psi']\frac{y}{x}$, where ψ' is the first alphabetic variant of ψ in $\underline{E}$ none of whose bound variables occur in ϕ . We will refer to this sort of substitution as P-substitution.

Definition 2. $\overline{S}$ and $\overline{U}$ are 2-place functions which map the pairs of formulae of L which have exactly one free variable x, (which is the same for both) into formulae of L which have x as their only free variable. They are defined by the following clause:

If ϕ, ψ are formulae of L, x is a variable which occurs free in both ϕ and ψ and neither ϕ nor ψ has free variables other than x, then

$$\bar{S}(\phi, \psi) = (\exists t_i)(t_i < x \wedge [\phi]\frac{t_i}{x} \wedge (\forall t_j)(t_i < t_j \wedge t_j < x \rightarrow [\psi]\frac{t_j}{x}))$$

$$\bar{U}(\phi, \psi) = (\exists t_i)(x < t_i \wedge [\phi]\frac{t_i}{x} \wedge (\forall t_j)(x < t_j \wedge t_j < t_i \rightarrow [\psi]\frac{t_j}{x})),$$

where i and j are the first and second number, respectively, such that t_i and t_j do not occur in ϕ or in ψ. One easily sees that if $\phi, \psi \in \mathcal{C}$, then $\bar{S}(\phi, \psi) \in \mathcal{C}$ and $\bar{U}(\phi, \psi) \in \mathcal{C}$.

<u>Definition 3</u>. D is the one-one function such that

(i) the domain of D is the set of formulae of TL_1;

(ii) the range of D is $\mathcal{C}$;

(iii) $D(q_i) = P_i(t_0)$;

(iv) If η is a formula of TL_1, then $D(\neg \eta) = \neg D(\eta)$

(v) If η and ζ are formulae of TL_1, then $D((\eta \wedge \zeta)) = (D(\eta) \wedge D(\zeta))$, $D(S\,\eta\,\zeta) = \bar{S}(D(\eta), D(\zeta))$, and $D(U\,\eta\,\zeta) = \bar{U}(D(\eta), D(\zeta))$.

One easily verifies that

(R 1) η is an n-place schema of TL_1 if and only if

$D(\eta)$ is an n-place tense defining formula of L.

(R 2) If η is a formula of TL_1, Q is a sequence of subsets of T and $t \in T$ then η is true at t in $\langle Q, \mathcal{R}_{\mathcal{T}}(TL_1)\rangle$ iff t satisfies $D(\eta)$ in $\langle T, <, Q_0, Q_1, \ldots \rangle$. To see that Theorem II.1 implies (A), we argue as follows: Let ϕ be an n-place tense defining formula of L. By Theorem II.1 there is an n-place schema η of TL_1, which Com-expresses the Com-tense-function defined by ϕ, i.e. if $\langle I,<\rangle \in$ Com, $Q_0, \ldots, Q_{n-1} \subseteq I$, $i \in I$, then η is true at i in $\langle\langle Q_0, \ldots Q_{n-1}, \ldots\rangle, \mathcal{R}_{\langle I,<\rangle}(TL_1)\rangle$ iff i satisfies ϕ in $\langle I, <, Q_0, \ldots, Q_{n-1}\rangle$. So by (R 2) i sat ϕ in $\langle I, <, Q_0, \ldots, Q_{n-1}\rangle$ iff i sat $D(\eta)$ in $\langle I, <, Q_0, \ldots, Q_{n-1}, \ldots\rangle$ for any $\langle I,<\rangle \in$ Com, $Q_0, \ldots, Q_{n-1}, \subseteq I$ and $i \in I$. And thus ϕ and the formula $D(\eta)$ in $\mathcal{C}$ are Com-equivalent. To see that (A) implies theorem II.1 let f be an n-place first order definable Com-tense-function. By (A) there is an n-place tense defining formula ϕ in $\mathcal{C}$ which defines f. Let η be the n-place schema of TL_1 such that $D(\eta) = \phi$; then by (R2) $D(\eta)$ Com-expresses f.

Before we give the proof of theorem 2 we will first prove another theorem, which states the existence of a normal form for arbitrary formulae of L (rather than just for the tense defining formulae).

We introduce the following additional notions and conventions.

Convention 1. We say that t_i precedes t_j if and only if $i < j$;

Convention 2. We will often write '$x < y < z$' for '$x<y \wedge y<z$'; also '$x_1 < \dots . < x_r$' for '$x_1 < x_2 \wedge x_2 < x_3 \wedge \dots \wedge x_{r-1} < x_r$'

In analogy to the notion of $\mathcal{K}$-validity and equivalence for formulae of a sentential language for a set of tense defining formulae of L, we put

Definition 4. Let $\mathcal{K}$ be a class of binary structures. A formula ϕ of L is $\mathcal{K}$-consistent if $\neg\phi$ is not $\mathcal{K}$-valid; otherwise ϕ is $\mathcal{K}$-inconsistent.

Until further notice 'valid,' 'equivalent,' 'consistent,' 'inconsistent,' will stand for 'Lin-valid,' 'Lin-equivalent,' 'Lin-consistent' and 'Lin-inconsistent,' respectively.

Definition 5. Assume that ϕ is a formula (of L) in which no variable is bound by two different quantifiers or occurs both free and bound. Then

a) x precedes y in ϕ iff

either (i) x, y free in ϕ and x precedes y or (ii) x free in ϕ and y bound in ϕ

or (iii) x, y bound in ϕ and the quantifier that binds y is in the scope of the quantifier that binds x.

b) x is <u>initial in</u> ϕ if x is bound in ϕ and is preceded by no other bound variables in ϕ. (F is the case we will also refer to the quantifier that binds x as initial).

<u>Definition 6</u>. (i) A formula π of L is a <u>temporal prefix</u> if and only if π is a conjunction of formulae of the forms: $x < y$ or $x = y$.

(ii) Let $\mathcal{K}$ be a class of binary structures. A temporal prefix π is $\mathcal{K}$ -<u>complete</u> if and only if for every temporal prefix π' all the variables of which are among those of π, we have either $\mathcal{K} \models \pi \to \pi'$ or $\mathcal{K} \models \pi \to \neg\pi'$. (Clearly, if π, π' are $\mathcal{K}$ -complete, $\mathcal{K}$ -consistent prefixes with the same variables, then $\mathcal{K} \models \pi \leftrightarrow \pi'$ or $\mathcal{K} \models \pi \to \neg\pi'$). We say that π is <u>complete</u> if $\mathcal{K}$ is Lin-complete.

(iii) Let π be a complete, consistent temporal prefix. Then

a) x is an <u>essential variable in</u> π if and only if x occurs in π and for all y different from x that occur in π and for which Lin $\models$ $\pi \rightarrow x = y$, x precedes y.

b) x <u>comes after</u> y <u>in</u> π if x, y are essential in π and Lin $\models \pi \rightarrow y < x$ and for all z different from x and y that occur in π, Lin $\models \pi \rightarrow \neg(y < z \wedge z < x)$

c) x is <u>the 1^{st} variable</u> in π if and only if x is essential and for no y x comes after y in π. x is the <u>$n+1^{st}$ variable</u> in π iff there is a y such that y is the n^{th} variable in π and x comes after y in π.

<u>$l(\pi)$</u> is the greatest number n such that for some x x is the n^{th} variable in π.

x is the <u>last variable</u> in π iff x is the $l(\pi)^{th}$ variable in π.

d) A temporal prefix π is <u>standard</u> iff

1) π is complete and consistent.

2) No atomic formula occurs more than once in π.

3) Every conjunct $x < y$ occurs before every conjunct $u = v$.

4) If Lin $\models \pi \rightarrow (x < y \wedge y < z)$, then the formula $x < z$ is not a conjunct of π.

5) If $x < y$ and $u < v$ occur in π

and $\mathrm{Lin} \models \pi \rightarrow y < u$, then $x < y$ occurs before $u < v$ in π.

6) If x occurs in π and x precedes y and $\mathrm{Lin} \models \pi \rightarrow x = y$, then π contains no conjunct $y = u$, $y < u$, $u < y$ (for any u!).

7) If $x = y$ occurs in π then x is essential in π. If $x = y$ and $u = v$ occur in π and u comes after x in π then $x = y$ occurs before $u = v$ in π. The formula $x = x$ occurs in π iff x is the only variable of π.

Moreover we will say that a temporal prefix π is (a prefix) in $x_1,\ldots,x_r$ if $x_1,\ldots,x_r$ are the free variables of π.

Remarks:

1) If π, π' are complete prefixes in the same variables, then $\mathrm{Lin} \models \pi \leftrightarrow \pi'$ or $\mathrm{Lin} \models \pi \rightarrow \neg \pi'$.

2) Every consistent prefix whose variables are among $x_1,\ldots,x_r$ is Lin-equivalent to a disjunction of complete, consistent prefixes in $x_1,\ldots,x_r$.

3) If π is a complete, consistent prefix, then there is a unique standard prefix π' in the same variables as π, such that $\mathrm{Lin} \models \pi \leftrightarrow \pi'$.

4) If π, π' are standard prefixes and $\mathrm{Lin} \models \pi \leftrightarrow \pi'$, then $\pi = \pi'$

5) If π is a standard prefix and contains only one variable, x, then π is the formula $x = x$.

Definition 7.

a) For each variable x, $At(x)$ is the set of all conjunctions of formulae of the forms $Q(x)$ and $\neg Q(x)$, together with the formula: $P_0(x) \vee \neg P_0(x)$.

Thus $\lambda x At(x)$ is a function from variables to sets of formulae.

We will now define three similar functions Bef(ore), Aft(er) and Bet(ween) (2-place) by recursion:

b) 1) $Bef_0(x) = Aft_0(x) = Bet_0(x,y) = \emptyset$

2) $Bef_{n+1}(x) = Bef_n(x) \cup \{(\exists z)(z < x \wedge \phi_1 \wedge \ldots \wedge \phi_m),$
$(\forall z)(z < x \rightarrow \phi_1 \vee \ldots \vee \phi_m)$: z is a variable, $m \in \omega$,
$z \neq x \wedge \bigwedge_{i \leq m} (\phi_i \in At(z) \cup Bef(z) \cup Bet_n(z,x)\}$;

$Aft_{n+1}(x) = Aft_n(x) \cup \{(\exists z)(x < z \wedge \phi_1 \wedge \cdots \wedge \phi_m),$
$(\forall z)(x < z \rightarrow \phi_1 \vee \ldots \vee \phi_m)$: z is a variable, $m \in \omega$, $z \neq x$,
$\bigwedge_{i \leq n} (\phi_i \in At(z) \cup Aft_n(z) \cup Bet_n(x,z))\}$;

$Bet_{n+1}(x,y) = Bet_n(x,y) \cup \{(\exists z)(x < z \wedge z < y \wedge \phi_1 \wedge$
$\ldots \wedge \phi_n)$, $(\forall z)(x < z \wedge z < y \rightarrow \phi_1 \vee \ldots \vee \phi_m)$:
z is a variable, $m \in \omega$, $z \neq x, y$,
$\bigwedge_{i \leq n} (\phi_1 \in At(z) \cup Bet_n(x,z) \cup Bet_n(z,y)\}$

$$At = \bigcup_{i \in \omega} At(t_i);$$

$$Bef(x) = \bigcup_n Bef_n(x); \quad Aft(x) = \bigcup_n Aft_n(x); \quad Bet(x,y) = \bigcup_n Bet_n(x,y);$$

$$Bef_n = \bigcup_i Bef_n(t_i); \quad Aft_n = \bigcup_i Aft_n(t_i);$$

$$Bet_n = \bigcup_{i,j} Bet_n(t_i,t_j)$$

$$Bef = \bigcup_n Bef_n; \quad Aft = \bigcup_n Aft_n; \quad Bet = \bigcup_n Bet_n .$$

Definition 8. Let r be a natural number $\neq 0$.

a) A standard formula ϕ in $x_1,\ldots,x_r$ is a disjunction $\bigvee_i \phi_i$ where each ϕ_i is a conjunction of a standard prefix π_i in $x_1,\ldots,x_r$ and a conjunction $\bigwedge_j \phi_{i,j}$ such that if $y_1,\ldots,y_s$ are the essential variables of π_i, listed in the order in which they appear in π_i, then for all j, $\phi_{i,j} \in \bigcup_{k=1}^{s} At(y_k) \cup \bigcup_{k=1}^{s-1} Bet(y_k,y_{k+1}) \cup$

$$Bef(y_1) \cup Aft(y_s)$$

b) ϕ is a standard formula if there are r, $x_1,\ldots,x_r$ such that ϕ is a standard formula in $x_1,\ldots,x_r$.

Theorem 1. Let r be a natural number $\neq 0$.
Every formula of L with free variables $x_1,\ldots,x_r$ is Lin-equivalent to a standard formula in $x_1,\ldots,x_r$.

Proof.

Every formula of L is logically equivalent to a formula of L in prenex normal form with the same free variables. Therefore it suffices to prove the theorem for prenex formulae. We proceed by induction on the number of quantifiers. Let ϕ be a formula of L with free variables $x_1,\ldots,x_r$. We may of course assume that no variable occurs both bound and free in ϕ, or is bound by two different quantifiers in ϕ.

1) ϕ contains no quantifiers. In this case the existence of an equivalent standard formula follows from well-known facts about the propositional calculus and some of our remarks about prefixes:

We bring ϕ in disjunctive normal form ψ. Consider a disjunct δ of ψ. δ will be a conjunction of atomic formulae and negations thereof. We can write δ as $\delta_0 \wedge \delta_1 \wedge \ldots \wedge \delta_r$, where δ_0 consists of those conjuncts of δ that involve the symbols '$=$' or '$<$', and for $i = 1,..,r$, δ_i is the conjunction of the conjuncts of δ of the forms $Q(x_i)$ or $\neg Q(x_i)$. In δ_0 we can replace each conjunct $\neg\, y = z$ by $y<z \wedge z<y$ and each conjunct $\neg\, y<z$ by $y = z \vee z<y$, obtaining thus an equivalent formula δ'_0.

Let us first assume that δ_0 is consistent. Then δ'_0 is also consistent and is therefore, as one easily sees, equivalent to a disjunction of consistent prefixes, and thus, by remarks 2) and 3) on p. 46 also equivalent to a disjunction of standard prefixes, $\bigvee_j \gamma_j$. Thus δ is equivalent to $\bigvee_j (\gamma_j \wedge \bigwedge_k \delta_k)$. The latter formula is a standard formula if at least one of the δ_k $(k = 1,\ldots,r)$ is not empty. But if all δ_k are empty then δ is equivalent to $\bigvee_j (\gamma_j \wedge (P_0(x_1) \vee \neg P_0(x_1)))$. The case where δ_0 is empty then reduces to the case where δ_0 is the formula $x_1 = x_1$. If δ_0 is inconsistent then δ is equivalent to $P_0(x_1) \wedge \neg P_0(x_1)$, which is equivalent to a standard formula. Thus each disjunct of ϕ is equivalent to a standard formula. It follows that ϕ is equivalent to a standard formula.

2) Let ϕ be $(\exists x)\ \psi$, and assume the theorem for ψ . We may assume that ψ contains x; for otherwise ϕ is equivalent to ψ and the theorem follows for ϕ at once. By assumption ψ is equivalent to a standard formula $\bigvee_i (\pi_i \wedge \bigwedge_j \psi_{i,j})$. So ϕ is equivalent to

1)... $$\bigvee_i (\exists x)(\pi_i \wedge \bigwedge_j \psi_{i,j}).$$

It suffices to show that each disjunct of (1) is equivalent to a standard formula in $x_1,\ldots,x_r$. Consider the disjunct

2)... $$(\exists x)\ (\pi \wedge \bigwedge_j \psi_j)$$ of (1).

a) Assume π contains a formula $y = x$. Then this is the only conjunct of π in which x occurs. We omit this conjunct from π and so obtain a standard prefix π' in $x_1,\ldots,x_r$. (In case that π contains no other conjuncts than $y = x$ --i.e. when $r = 1$--, we let π' be the formula $y = y$, which is again a standard prefix in $x_1,\ldots,x_r$.) In this case none of the ψ_j will contain x so that (2) is equivalent to $\pi' \wedge \bigwedge_j \psi_j$, which is clearly a standard prefix in $x_1,\ldots,x_2$.

b) Assume π contains a formula $x = y$. Then x is essential in π. Let z be the first variable (in the list $t_1,t_2,\ldots$) such that $x = z$ occurs in π. We replace x everywhere in π by z and omit the resulting conjunct $z = z$ (unless $r = 1$!). In this way we obtain a standard prefix in $x_1,\ldots,x_r$. Moreover we replace x everywhere by z in the ψ_j, thus obtaining formulae ψ_j'. (2) will then be equivalent to $\pi' \wedge \bigwedge_j \psi_j'$, the latter being again clearly a standard formula in $x_1,\ldots,x_r$

c) Assume π contains no formulae of the forms $y = x$ and $x = y$. So the conjuncts of π in which x occurs are either

(i) $x < y$ (for some unique y) alone or

(ii) $y < x$ (for some unique y) alone or

(iii) $y < x$ and $x < z$ (for some unique y,z)

Let us assume (i). We omit $x < y$ from π and obtain a standard prefix in $x_1,\ldots,x_r$. (Again, if $r = 1$, then we put π' equal to $y = y$.) If no ψ_j contains x then (2) is equivalent to $\pi' \wedge \bigwedge_j \psi_j$, which is easily seen to be standard. Otherwise (2) is equivalent to

$\pi' \wedge \bigwedge_g \psi_g' \wedge (\exists x)(x < y \wedge \bigwedge_h \psi_h'')$, where the ψ_g' are those ψ_j that do not contain x, and the ψ_h'' are the others.

The last formula is again standard in $x_1,\ldots,x_r$. Cases (ii) and (iii) are treated similarly.

3) Let ϕ be $(\forall x)\psi$ and assume the theorem for ψ. So ψ is equivalent to

(3).... $\bigvee_i (\pi_i \wedge \bigwedge_j \psi_{i,j})$. Again we may assume that ψ contains x. With respect to the π_i we distinguish the cases

a) π_i contains a formula $x = y$

b) π_i contains a formula $y = x$

c1) π_i contains no formulae of the forms $x = y$ or $y = x$ and at most one conjunct containing x

c2) π_i contains no formulae of the forms $x = y$ but does contain $y < x$, $x < z$ for some y,z.

In cases (b) and (c1) we obtain a standard prefix π_i' in $x_1,\ldots,x_r$ simply by omitting from π_i the conjuncts that contain x (with the usual provision for the case where $r = 1$). In case (c2) we obtain a standard prefix by replacing $y < x \wedge x < z$ by $y < z$. Further we let in these three cases β_i be the conjunction of the conjuncts of π_i that contain x, and we put $\psi'_{i,j} = \psi_{i,j}$. Suppose that (i) applies to π_i. Let again y be the first variable such that $x = y$ appears in π_i. We obtain π'_i by replacing x everywhere in π_i by y and omitting $y = y$ from the result (unless $r = 1$). Further we put β_i equal to $x = y$ and let $\psi'_{i,j}$ be the formulae which are obtained by substitution of y for x in $\psi_{i,j}$. Then clearly (3) is equivalent to

(4).... $\bigvee_i (\pi'_i \wedge \beta_i \wedge \bigwedge_j \psi'_{i,j})$

In (4) some of the π'_i may be the same even though their indices are different. However, we can rewrite (4) equivalently as

$$\bigvee_k (\rho_k \wedge \bigvee_h (\beta_{k,h} \wedge \bigwedge_j \psi_{k,h,j}))$$

where the ρ_k are all different standard prefixes in $x_1, \ldots, x_r$ and each $\beta_{k,h}$ is equal to some β_i in (4) and each $\psi_{k,h,j}$ to some $\psi'_{i,j}$ in (4). Since the ρ_k are mutually exclusive (i.e. if $j \neq j'$, then Lin $\models \neg(\rho_j \wedge \rho_{j'}))$ and do not contain x, Φ is equivalent to

(5).... $\bigvee_k (\rho_k \wedge (\forall x) \bigvee_h (\beta_h \wedge \bigwedge_j \psi_{k,h,j}))$

It will again be sufficient to show that each disjunct of

(5) is equivalent to a standard formula. Consider the disjunct

(6).... $\rho \wedge (\forall x) \bigvee_h (\beta_h \wedge \bigwedge_j \psi_{h,j})$

Let us assume that $y_1, \ldots, y_s$ are the essential variables of ρ, listed in the order in which they appear in ρ. Again, some of the β_h may be the same, but we can rewrite (6) equivalently as

(7).... $\rho \wedge (\forall x) \bigvee_g (\beta_g \wedge \bigwedge_m \bigvee_p \psi_{g,m,p})$

where all the β_g are different, each β_g is equal to some β_h in (6) and each $\psi_{g,m,p}$ is equal to some $\psi_{h,j}$ in (6). We have Lin $\models \neg(\rho \wedge \beta_g \wedge \beta_{g'})$ If we have, moreover,

(8).... Lin $\models \rho \rightarrow \bigvee_g \beta_g$, then (7) is equivalent to

$$\rho \wedge (\forall x)(\beta_g \rightarrow \bigwedge_m \bigvee_p \psi_{g,m,p}),$$

which in turn is equivalent to

(9).... $\bigwedge_g \bigwedge_m (\rho \wedge (\forall x)(\beta_g \rightarrow \bigvee_p \psi_{g,m,p})$

Suppose that (8) does not hold. Let $z_1, \ldots, z_u$ be those y_q $(q \leqslant s)$ such that x precedes y_q and $z_{u+1}, \ldots, z_s$ the others. β_g is one of the formulae $x < y_1, y_1 < x \wedge x < y_2$, $\ldots, y_{s-1} < x \wedge x < y_s, y_s < x$, $x = z_1, \ldots, x = z_u, z_{u+1} = x, \ldots, z_s = x$.

Since (8) fails, some of the formulae in the list do not occur among the β_g. Let $\gamma_1, \dots, \gamma_p$ be all those. Then (7) is equivalent to

$$(10) \dots \bigwedge_g \bigwedge_m \bigwedge_f (\rho \wedge (\forall x)(\beta_g \to \bigvee_p \psi_{g,m,p}) \wedge (\forall x)(\gamma_f \to (P_0(x) \wedge \neg P_0(x)))$$

We will show that (9) is equivalent to a standard formula. The argument will equally apply to (10). Consider the conjunct

$$(11) \dots \quad \rho \wedge (\forall x)(\beta \to \bigvee_p \psi_p) \qquad \text{of (9).}$$

If β is an equality then x cannot occur in any ψ_p. For either x was not essential in the prefixes π_i of (3) which we 'split' into ρ and β and so the corresponding formulae $\psi_{i,j}$ of (3) do not contain x; but each ψ_p is equal to one of those; or else x is not essential in the π_i that were split into ρ and β. Then the corresponding $\psi_{i,j}$ were immediately replaced by formulae $\psi'_{i,j}$ in which x does not occur; and in this case each ψ_p of (11) occurs among these $\psi'_{i,j}$. So in either case (11) is equivalent to $\bigvee_\rho (\rho \wedge \psi_\rho)$. One easily verifies that each disjunct $\rho \wedge \psi_\rho$ is a standard formula in $x_1 \dots, x_r$ and thus (11) is equivalent to a standard formula in $x_1, \dots, x_r$. Suppose that β consists of inequalities. If none of the ψ_ρ contains x then (11) is equivalent to $\bigvee_\rho (\rho \wedge \psi_p) \vee (\rho \wedge (\forall x)(\beta \to P_0(x) \wedge \neg P_0(x)))$ So suppose that some ψ_ρ does contain x. If all ψ_ρ contain x then (11) is a standard formula. Otherwise (11) is equivalent to

$$\bigvee_q (\rho \wedge \psi_q') \vee (\rho \wedge (\forall x)(\beta \rightarrow \bigvee_t \psi_t'')),$$ where

the ψ_q' are the ψ_p in which x does not occur and the ψ_t'' are the other ψ_p. Again this is a standard formula. One easily sees that since each conjunct of (9) is equivalent to a standard formula which involves no other prefixes than ρ, (9) is itself equivalent to a standard formula. This completes the proof of theorem 1.

We gave theorem 1 primarily because it is needed in our proof of theorem 2. However, theorem 1 has, it seems to us, some interest in itself,--even in the presence of theorem 2, since it gives a normal form for a larger class of formulae and it asserts that such a formula and its normal form are Lin-, rather than Com-equivalent. Theorem 1 deals itself only with formulae which have at least one free variable. However, it has as an immediate consequence corollary1 below, which states a corresponding normal form for sentences of L.

Definition 9. A sentence ψ of L is a standard sentence if and only if there is a variable x and a formula ϕ, standard in x, such that ψ is $(\exists x)\phi$ or ψ is $(\forall x)\phi$.

Corollary 1. Every sentence of L is Lin-equivalent to a boolean combination of standard sentences.

Convention 3. a) If η, ζ are formulae of TL_1 we will write '$S(\eta, \zeta)$' instead of '$S\eta\zeta$' and '$U(\eta, \zeta)$' instead of '$U\eta\zeta$'

b) outermost parentheses will be omitted-- e.g. we write '$\eta \wedge \neg(\zeta \wedge \eta)$' instead of '$(\eta \wedge \neg(\zeta \wedge \eta))$'

We also adopt standard conventions concerning parentheses for formulae of L.

Definition 10. The following are 1-place functions from formulae of TL_1 to formulae of TL_1. They are characterized by:

$P(\eta) = S(\eta, \neg(\eta \wedge \neg\eta))$,
$F(\eta) = U(\eta, \neg(\eta \wedge \neg\eta))$,
$H(\eta) = \neg P(\neg\eta)$,
$G(\eta) = \neg F(\neg\eta)$,
$H'(\eta) = S(\neg(\eta \wedge \neg\eta), \eta)$,
$G'(\eta) = U(\neg(\eta \wedge \neg\eta), \eta)$,
$P'(\eta) = \neg H'(\neg\eta)$ and
$F'(\eta) = \neg G'(\neg\eta)$, where η is an arbitrary formula of TL_1.

(It is worth noticing that the schemata corresponding to the functions P and F--i.e. formulae $S(q_0, \neg(q_0 \wedge \neg q_0))$ and $U(q_0, \neg(q_0 \wedge \neg q_0))$--Lin-express in TL_1 the past and future tenses, respectively.

Indeed, if $\mathcal{T}$ is a linear ordering, $\mathfrak{A}$ a possible interpretation determined by TL_1, relative to $\mathcal{T}$, and $t \in T$

then for any formula η of TL_1 $P(\eta)$ is true at t in $\mathfrak{A}$ iff there is a $t' \in T$, such that $t' < t$ and η is true at t' in $\mathfrak{A}$; and $F(\eta)$ is true at t in $\mathfrak{A}$ if and only if there is a $t' \in T$ such that $t < t'$ and η is true at t' in $\mathfrak{A}$.
In a similar way $H(\eta)$ Lin- expresses the fact that η has always been true in the past, $G(\eta)$ that η will always be true in the future, $H'(\eta)$ that η was true uninterruptedly from some moment in the past up to the present, $P'(\eta)$ that η has been true at past moments infinitely close to the present and $F'(\eta)$ that η will be true at future moments infinitely close to the present).
For each of the functions defined in definition 10 we introduce the analogous 1-place functions from formulae of L to formulae of L which have exactly one free variable.

<u>Definition 11</u>. For any formula ϕ of L with exactly one free variable

$P(\phi) = S(\phi, \neg(\phi \wedge \neg\phi))$,
$F(\phi) = U(\phi, \neg(\phi \wedge \neg\phi))$,
$H(\phi) = \neg P(\phi)$,
$G(\phi) = \neg F(\phi)$,
$H'(\phi) = S(\neg(\phi \wedge \neg\phi), \phi)$,
$G(\phi) = U(\neg(\phi \wedge \neg\phi), \phi)$,
$P'(\phi) = \neg H'(\neg\phi)$,
$F'(\phi) = \neg G'(\neg\phi)$,

Definition 11. Let $\mathcal{K}$ be a class of linear structures, ϕ a formula of L in which only x occurs free. Let $\psi \in \mathcal{C}$. ψ $\mathcal{K}$-expresses ϕ (in terms of AND, NOT, SINCE and UNTIL) if ϕ is $\mathcal{K}$-equivalent to a formula ψ' which comes by proper substitution of x for t_0 in some alphabetic variant of ψ. We say that the formula η of TL_1 $\mathcal{K}$-expresses ϕ if $D(\eta)$ $\mathcal{K}$-expresses ϕ.

We will again often write 'expressed,' 'expressible' etc. instead of 'Lin-expressed,' 'Lin-expressible' etc.. We state the following obvious facts concerning expressibility for formulae of L.

Let ϕ_1, ϕ_2 be formulae of L, $\chi_1, \chi_2 \in \mathcal{C}$ and η_1, η_2 formulae of TL_1.

(R_3) If ϕ_1 is $\mathcal{K}$-expressed by χ_1 (η_1) and ϕ_2 is $\mathcal{K}$-expressed by χ_2 (η_2) then $\neg\phi_1$, $\phi_1 \wedge \phi_2$ $S(\phi_1, \phi_2)$ and $U(\phi_1, \phi_2)$ are $\mathcal{K}$-expressed by $\neg\chi_1$ (η_1), $\chi_1 \wedge \chi_2$ $(\eta_1 \wedge \eta_2)$, $S(\chi_1, \chi_2)$ $(S(\eta_1, \eta_2))$ and $U(\chi_1, \chi_2)$ $(U(\eta_1, \eta_2))$ respectively.

In particular $P(\phi_1)$, $F(\phi_1)$,....are expressed by $P(\chi_1)$ $(P(\eta_1))$, $F(\chi_1)$ $(F(\eta_1))$,....

(R_4) If ϕ_1, ϕ_2 are $\mathcal{K}$-expressed by χ_1 (η_1), χ_2

(η_2) respectively, then $[\phi_1]\frac{\phi_2}{p_1}$ is $\mathcal{K}$-expressed by $[\chi_1]\frac{\chi_2}{p_1}$ ($[\eta_1]\frac{\eta_2}{q_1}$).

We will now proceed to prove

Theorem 2. Every tense defining formula is Com-equivalent to a member of $\mathcal{C}$.

We have to show that every tense defining formula is Com-expressible. In view of theorem 1 it suffices to prove that every standard formula in which only t_0 occurs free is Com-expressible. It will be convenient, however, to prove the slightly stronger statement that for any variable x, the standard formulae which have only x free are Com-expressible.

Every standard formula with one free variable, x say, is equivalent to a boolean combination of members of $At(x) \cup Bef(x) \cup Aft(x)$. Therefore, since all members of At are Com-expressible, it will suffice to show that the classes Bef and Aft consist of Com-expressible formulae. It is natural to try to prove this by induction on the definition of Bef and Aft. Thus, assuming that all formulae of Bef_n and Aft_n are Com-expressible, we want to show the same fact for a formula ϕ of, say, Bef_{n+1}. But ϕ will in general be defined in terms not only of members of $Bef_n \cup Aft_n$ but also of members of Bet_n. So it will be necessary to prove some facts about the class Bet as well. On the other hand we may restrict our attention to the class $\mathcal{E}$ of those members of Bef $\cup$ Aft which do not have proper subformulae that belong to Bef $\cup$ Aft. For assuming that all members of $\mathcal{D}$ are expressible we can show the Com-expressibility of all members of Bef $\cup$ Aft as follows: If $\phi \in \mathrm{Bef}_1 \cup \mathrm{Aft}_1$, then ϕ does not have proper subformulae which belong to Bef $\cup$ Aft and therefore belongs to $\mathcal{D}$. Suppose now that $\phi \in \mathrm{Bef}_{n+1} \cup \mathrm{Aft}_{n+1}$ and that all members of $\mathrm{Bef}_n \cup \mathrm{Aft}_n$ are Com-expressible. Let $\phi_1,\ldots,\phi_k$ be all those proper subformulae of ϕ which belong to Bef $\cup$ Aft and which are not part of a larger proper subformula of ϕ which belongs to Bef $\cup$ Aft. Let $y_1,\ldots,y_k$ be the unique free variables of $\phi_1,\ldots,\phi_k$ respectively. Choose distinct predicate letters $Q_1,\ldots,Q_k$ which do not occur in ϕ. Let ϕ' be obtained by replacing for $i = 1,\ldots,k$ ϕ_i by $Q_i(y_i)$ in ϕ. Then ϕ' is a member of D and therefore expressible. Let η express ϕ'. Further $\phi_1,\ldots,\phi_k$ belong to $\mathrm{Bef}_n \cup \mathrm{Aft}_n$ and thus are expressible by assumption. Let $\eta_1,\ldots,\eta_k$ express $\phi_1,\ldots,\phi_k$ respectively. Then ϕ is expressed by $[\eta]\ \frac{\eta_1}{q_1} \cdots \frac{\eta_k}{q_k}$, where q_i is the propositional constant that corresponds to Q_i (i.e. $\bigwedge_{i \leq k} \bigvee_j (q_i = p_j$ and $Q_i = P_j)$!).

We can restrict our attention even further, namely to these members of $\mathcal{D}$ which begin with an existential quantifier. For if $\phi \in \mathcal{D}$ and ϕ begins with a universal quantifier, then $\neg \phi$ is equivalent to a formula $\psi \in \mathcal{E}$ which begins with an existential quantifier. To verify this claim we observe that we may submit $\neg \phi$ to the following transformations which preserve equivalence: We can move the initial negation sign of $\neg \phi$ inside all the way to the formulae of the forms Q(x) or $\neg$ Q(x), and cancel double negations in front of such formulae, thus obtaining a formula ψ'. In ψ' some existential quantifier may stand in front of a boolean combination of members of At$\cup$Bet which contains disjunction signs, rather than in front of a pure conjunction of such formulae. However, we can bring such boolean combinations in disjunctive normal form and then distribute the existential quantifier over the disjunction. Clearly we have to repeat this only a finite number of times in order to obtain a formula ψ equivalent to ψ' in which all existential quantifiers are followed by conjunctions. One easily verifies that ψ belongs to $\mathcal{E}$.

Thus our task reduces to showing that arbitrary members of Bet have some property I such that if $\phi_1(x,y),\ldots\phi_m(x,y)$ have I, $\psi_1,\ldots,\psi_n \in At(x)$ and $\chi_1,\ldots,\chi_p \in At(y)$, then

12) ... $(\exists\, x)(x < y \wedge \bigwedge_j^n \psi_j \wedge \bigwedge_i^m \phi_i)$ and

$(\exists\, y)(x < y \wedge \bigwedge_k^p \chi_k \wedge \bigwedge_i^m \phi_i)$ are Com-expressible. In order to state the property I we introduce the following sets of formulae:

<u>Definition</u>

of B_n(asic) p(ast) (x,y) and B_n(asic) f(uture) (x,y):

$$Bp_1(x,y) = \{P_i(x) \wedge (\forall z)(x<z<y \rightarrow P_j(z)) \wedge P_k(y) : i,j,k \in \omega \text{ and } x,y,z \text{ are distinct variables}\}$$

$$Bf_1(x,y) = \{P_i(x) \wedge (\forall z)(x<z<y \rightarrow P_j(z)) \wedge P_k(y) : i,j,k \in \omega \text{ and } x,y,r \text{ are distinct variables}\}$$

$$Bp_{n+1}(x,y) = \{P_i(x) \wedge (\exists z)(x<z<y \wedge (\forall v)(x<v<z \rightarrow P_j(v)) \wedge \phi) : i,j \in \omega \text{ and } \phi \in Bp_n(z,y) \text{ and } x,y,z,v \text{ are distinct variables}\}$$

$$Bf_{n+1}(x,y) = \{(\exists z)(x<z<y \wedge \phi \wedge (\forall v)(z<v<y \rightarrow P_j(v))) \wedge P_i(y) : i,j, \in \omega \text{ and } \phi \in Bf_n(x,z) \text{ and } x,y,z,v \text{ are distinct variables}\}$$

$$Bp(x,y) = \bigcup_n Bp_n(x,y); \quad Bf(x,y) = \bigcup_n Bf_n(x,y)$$

$$Bp_n = \bigcup_{i,j} Bp_n(t_i,t_j); \quad Bf_n = Bf_n(t_i,t_j)$$

$$Bp = \bigcup Bp_n \quad ; \quad Bf = \bigcup_n Bf_n$$

<u>Convention</u> . Let ϕ be a formula of L and let x,y be variables.

By $(\exists x < y)\,\phi$ we understand the formula

$(\exists x)\ (x < y \wedge \phi)$ and by $(\exists x > y)\,\phi$ the formula $(\exists x)$ $(y < x \wedge \phi)$.

<u>Definition</u>

Let $Bp_n\ (y) = \bigcup_i \{(\exists t_i < y)\,\phi : \phi \in Bp_n(t_i, y)\}$

$$Bp_n\ (y) = \bigcup_i \{(\exists t_i > y)\,\phi : \phi \in Bf_n(y, t_i)\}$$

With each formula $\phi \in Bp_n \cup Bf_n$ we associate the $2n+1$ - place sequence $S(\phi)$ of predicate letters which lists all the occurrences of such letters in ϕ in order of appearance in ϕ from left to right. Similarly we let $Se(\phi)$ be the sequence of all those occurrences (listed in order of appearance) which are not in the scope of a universal quantifier. Thus for example, if ϕ is the formula

$$P_1(x) \wedge (\exists z)(x<z<y \wedge (\forall u)(x<u<z \rightarrow P_1(u)) \wedge$$
$$P_2(z) \wedge (\forall v)(z < v < y \rightarrow P_3(v)))) \wedge P_2(y),$$
$$S(\phi) = \langle P_1, P_1, P_2, P_3, P_2 \rangle \quad \text{and } Se(\phi) = \langle P_1, P_2, P_2 \rangle .$$

If ϕ is of the form $Q\ (y)$ then we put $S(\phi) =$

$Se(\phi) = \langle Q \rangle$

Definition

Let $\phi \in Bp(x,y) \cup Bf(x,y)$ and $S(\phi) = \langle Q_1, \ldots, Q_{2n-1} \rangle$
Let $\mathfrak{M} = \langle M, <, \underline{P} \rangle$ be a linear structure (i.e. $\langle M, < \rangle$ is a linear ordering and P is a sequence of subsets of M). Then we say that the sequence $\langle t_1, \ldots, t_n \rangle$ of points of M <u>satisfies</u>$'$ ϕ in $\mathfrak{M}$ iff

(i) $\underline{t}_1, \ldots, \underline{t}_2 \ldots < \underline{t}_n$ and

(ii) for $i = 1, \ldots, n$ $\underline{t}_i \in \underline{Q}_{2i-1}$ and

(iii) for $i = 1, \ldots, n-1$, if $\underline{t} \in M$ and $\underline{t}_i < \underline{t} < \underline{t}_{i+1}$ then $\underline{t} \in \underline{Q}_{2i}$.

(Here by $\underline{Q}$ we understand that subset $\underline{P}_i$ of M such that $P_i = Q$). Also, if $\psi = (\exists\, x < y)\phi$ or $\psi = (\exists\, y > x)\phi$ then $\langle \underline{t}_1, \ldots, \underline{t}_n \rangle$ <u>satisfies</u>$'$ ψ in $\mathfrak{M}$ iff $\langle \underline{t}_1, \ldots, \underline{t}_n \rangle$ satisfies$'$ ϕ in $\mathfrak{M}$.

Remarks

1) For each 2n+1 - place sequence S of predicate letters and distinct variables x,y there are $\phi \in Bp_n(x,y)$

and $\psi \in Bf_n(x,y)$ such that $S(\phi) = S(\psi) = S$.

2) If $\phi, \psi \in Bp(x,y) \cup Bf(x,y)$ and $S(\phi) = S(\psi)$ then $\mathrm{Lin} \models \phi \leftrightarrow \psi$.

3) From 1), 2) it follows that for every $\phi \in Bp(x,y)$ $(Bf(x,y))$ there is a $\psi \in Bf(x,y)$ $(Bp(x,y))$ such that $\mathrm{Lin} \models \phi \leftrightarrow \psi$.

4) If $\phi \in Bp(x,y) \cup Bf(x,y)$, $\mathfrak{M} = \langle M, <, P_0; P_1, \ldots \rangle$ an arbitrary linear structure and $\underline{t}_1, \underline{t}_2$ are points of M such that $\underline{t}_1 < \underline{t}_2$, then $\underline{t}_1, \underline{t}_2$ satisfy (in the standard sense of 1st order logic) ϕ in $\mathfrak{M}$ iff there is a sequence $\langle \underline{v}_1, \ldots, \underline{v}_n \rangle$ of points in M which satisfies' ϕ in $\mathfrak{M}$, such that $\underline{v}_1 = \underline{t}_1$ and $\underline{v}_n = \underline{t}_2$. Similarly, if $\phi \in Bp(y)$ $(Bf(y))$ and $\mathfrak{M}$ as above, then $\underline{t}_1$ satisfies ϕ in $\mathfrak{M}$ if and only if there is $\langle \underline{v}_1, \ldots, \underline{v}_n \rangle$ which satisfies ϕ in $\mathfrak{M}$, where $\underline{v}_n = \underline{t}_1$ $(\underline{v}_1 = \underline{t}_1)$.

Definition

a) $\mathrm{Lp}_0(y) = \{ (\forall x)(x < y \rightarrow (\exists z)(x<z<y \wedge P_i(z))) :$
$i \in \omega$ and x,y,z are distinct variables$\}$.

$\mathrm{Lf}_0(y) = \{ (\forall x)(y<x \rightarrow (\exists z)(y<z<x \wedge P_i(z))) :$
$i \in \omega$ and x,y,z are distinct variables$\}$.

$\mathrm{Lp}_{n+1}(y) = \{ (\forall x)(x < y \rightarrow (\exists u)(x<u<y \wedge (\exists v)(u<v<y \wedge \phi)))$
$: \phi \in Bp_n(u,v)$ and $x,y,u,0$ are distinct variables$\}$

$Lf_{n+1}(y) = \{(\forall x)(y < x \rightarrow (\exists u)(y<u<x \wedge$
$(\exists v)(u < v < x \wedge \phi))) : \phi \in Bp_n(u,v)$ and
x,y,u,v are distinct variables $\}$.

$Lp_n = \bigcup_i Lp_n(t_i)$; $Lf_n = \bigcup_i Lf_n(t_i)$;

$Lp(y) = \bigcup_n Lp_n(y)$; $Lf(y) = \bigcup_n Lf_n(y)$; $Lp = \bigcup_n p_n$;
$Lf = \bigcup_n Lf_n$.

b) Let $E(y)$ be the closure of $At(y) \cup Bp(y) \cup$

$Bf(y) \cup Lp(y) \cup Lf(y)$

under conjunction, disjunction, negation and proper substitution of formulae for predicate letters. Let $E = \bigcup_i E(t_i)$

c) The sets of formulae E(lementary) p(ast)(x,y) and E(lementary) f(uture) (x,y) are defined by

$Ep(x,y) = \{\phi :$ there are a $\phi' \in Bp(x,y)$ $n \in \omega$ and $\phi_0, \ldots, \phi_n \in E$ such that $\phi = [\phi'] \frac{\phi_0}{P_0} \ldots \frac{\phi_n}{P_n}\}$

$Ef(x,y) = \{\phi :$ there are $\phi' \in Bf(x,y)$, $n \in \omega$ and $\phi_0, \ldots, \phi_n \in E$ such that $\phi = [\phi] \frac{\phi_0}{P_0} \cdots \frac{\phi_n}{P_n}$

Remarks

(R5) If $\phi \in Ep(x,y)$ $(Ef(x,y))$. Then

$(\exists x < y)\, \phi \in E(y)$ $((\exists y > x)\, \phi \in E(x))$

R6) If $\phi \in Ep\,(x,y)(Ef\,(x,y))$ and $\psi \in E$, then
$[\phi]\frac{\psi}{Q} \in Ep(x,y)$ $(Ef\,(x,y))$

<u>Definition</u> Let ϕ be a formula of L with no other free variables than x and y.

Then ϕ <u>has the property</u> $I(x,y)$ if and only if there is a disjunction $\bigvee_j \phi_j$ of members of $Ep(x,y)$ such that:

(13).... $Com \models x < y \rightarrow (\phi \leftrightarrow \bigvee_j \phi_j)$

<u>Remarks</u>

R 7) If ϕ has the property $I\,(x,y)$ and $\psi \in E$ then $[\phi]\,\frac{\psi}{Q}$ has the property $I(x,y)$

R 8) If $\phi \in E(x)$ and y is any variable other than x, then ϕ has the property $I(x,y)$, since $Com \models x \angle y \rightarrow$
$(\phi \leftrightarrow (\phi \wedge (\forall z)(x \angle z \angle y \rightarrow (P_0(z) \vee \neg P_0(z)) \wedge$
$(P_0(y) \vee \neg P_0(y)))$

For the same reason ϕ has the property $I\,(y,x)$.

<u>Lemma 1</u>. All members of $Bp(y) \cup Bf(y)$ are Lin-expressible.

<u>Proof</u>: We prove that all members of $Bp(y)$ are expressible.

That all members of $Bp(y)$ are expressible is proved in the same way. We define by induction on n the sets of formulae

$Bp_n^*(x,y,i)$ (where x and y are variables and $i \in \omega$):

$$Bp_n^*(x,y,i) = \{(\exists x)(x < y \wedge P_i(x) \wedge (\forall z)(x<z<y \rightarrow P_j(z)) \wedge P_k(y)):$$
$$z \text{ is a variable distinct from x and y and } j,k \in \omega\}$$

$$Bp_{n+1}^*(x,y,i) = \{[\phi]\frac{\psi}{P_j} : j \in \omega, P_j \text{ occurs only once in } \phi \text{ and there is a variable z such that } \phi \in Bp_1^*(x,z,i) \text{ and } \phi \in B_{p_n}(z,y,i)\}$$

$$Bp_n^*(y) = \bigcup_{i,j} Bp^*(t_j,y,i)$$

One easily verifies, by induction on n, that each member of $Bp_n(y)$ is equivalent to a member of $Bp_n^*(y)$. For example,

$$(\exists x)(x < y \wedge P_1(x) \wedge (\exists z)(x<z<y \wedge (\forall u)(x<u<z \rightarrow P_2(u)) \wedge P_3(z) \wedge (\forall v)(z<v<y \rightarrow P_4(v)))) \wedge P_5(y)$$

is equivalent to

$$(\exists z)(z<y \wedge (\exists x)(x<z \wedge P_1(x) \wedge (\forall u)(x<u<z \rightarrow P_2(u))) \wedge P_3(z) \wedge (\forall v)(z<v<y \rightarrow P_4(v))) \wedge P_5(y)$$

It is easy to see that all members of $Bp^*(y) = \bigcup_n Bp_n^*(y)$ are expressible.

For if ϕ is in $Bp_1^*(y)$, then ϕ is $(\exists x)(x<y \wedge P_i(x) \wedge (\forall z)(x<z<y \rightarrow P_j(z))) \wedge P_k(y)$, for some x,z,i,j,k, and thus expressed by $S(q_i,q_j) \wedge q_k$. Since the members of $Bp_{n+1}^*(y)$ are obtained by means of ρ-substitution of a member of $Bp_1^*(y)$ into a member of $Bp_n^*(y)$, the expressibility of the members of $Bp_{n+1}^*(y)$ follows from the expressibility of the members of $Bp_n^*(y)$ by R4.

This completes the proof of lemma 1.

<u>Lemma 2</u>. All members of Lp $\cup$ Lf are Lin-expressible.

<u>Proof</u>: We will prove only that the members of Lp are Lin-expressible. If $\phi \in Lp_0$ then ϕ is of the form $(\forall x)(x<y \rightarrow (\exists z)\ (x<z<y \wedge P_j(z)))$. The latter formula is according to definition 10 equal to P' $(P_j(y)$ and is thus expressible, in view of R 1.

Suppose $\phi \in Lp_1$. Let ϕ be the formula

$$(\forall x)(x<y \rightarrow (\exists u)\ (x<u<y \wedge (\exists v)(u<v<y \wedge P_i(u) \wedge (\forall z)(u<z<v \rightarrow P_j(z)) \wedge P_k\ (v))))$$

Let $\psi = P_i(y) \wedge (\exists v)(y<v \wedge (\forall z)\ (y<z<v \rightarrow P_j(z)) \wedge P_k(v))$

We claim that ϕ is equivalent to:

$$(14)\ldots\ P'\ (\psi) \wedge (\forall x)(x<y \rightarrow (\exists v)(x<v<y \wedge P_k(v)))$$

It is obvious that ϕ implies (14). To see that (14) implies Φ, consider any linear structure $\mathfrak{M} = \langle \underline{M}, <, \underline{P} \rangle$ and a point $\underline{t}_0$ in $\underline{M}$, satisfying (14). Let $\underline{t}_1$ be any point of $\underline{M}$ such that $\underline{t}_1 < \underline{t}_0$. We have to show that

(15)... there are $\underline{t}_2, \underline{t}_3 \in \underline{M}$ such that $\underline{t}_1 < \underline{t}_2 < \underline{t}_3 < \underline{t}_0$ and $\underline{t}_2 \in \underline{P}_i$, $\underline{t}_3 \in \underline{P}_k$ and for all $\underline{t}$ between $\underline{t}_2$ and $\underline{t}_3$, $\underline{t} \in \underline{P}_j$.

Since $\underline{t}_0$ satisfies $P'(\psi)$ in $\mathfrak{M}$ there is a point $\underline{t}_2$ between $\underline{t}_1$ and $\underline{t}_0$ and a point $\underline{t}_3$ such that $\underline{t}_2 < \underline{t}_3$, such that $\underline{t}_2 \in \underline{P}_i$, $\underline{t}_3 \in \underline{P}_k$ and for all $\underline{t}$ between $\underline{t}_2$ and $\underline{t}_3$, $\underline{t} \in \underline{P}_j$. If $\underline{t}_3 < \underline{t}_0$ then (15) holds. So suppose $\underline{t}_0 \leq \underline{t}_3$. Then for all $\underline{t}$ between $\underline{t}_2$ and $\underline{t}_0$, $\underline{t} \in \underline{P}_j$. Further, since $\underline{t}_0$ satisfies the second conjunct of (14) there is a $\underline{t}_4$ between $\underline{t}_2$ and $\underline{t}_0$ such that $\underline{t}_4 \in \underline{P}_k$. Since for all $\underline{t}$ between $\underline{t}_2$ and $\underline{t}_4$, $\underline{t} \in \underline{P}_j$, (15) holds again. Since this is the case for arbitrary $\mathfrak{M}$, (14) implies Φ.

Both conjuncts of (14) are expressible, the first in view of R 3, and the second because of the assertion above. Thus by R 4, (14) is expressible. So Φ is expressible.

We shall now show, for $n>0$, the expressibility of the members of Lp_{n+1}, under the assumption that all members of Lp_n are expressible. The argument will be essentially the same as the one we just used for Lp_1.

Let $\Phi \in Lp_{n+1}$. Thus Φ is

$$(\forall x)(x<y \rightarrow (\exists u)(x<u<y \wedge (\exists v)(u<v<y \wedge \psi)))$$

for some variables x,y,u,v and some $\psi \in Bf_{n+1}(u,v)$. Let $\langle Q_1,\ldots,Q_{2n+3}\rangle = S(\psi)$. Let $\chi \in Bf_n(u,v)$ such that $S(\chi) = \langle Q_3,\ldots,Q_{2n+3}\rangle$; and let ρ be the formula $(\forall x)(x<y \rightarrow (\exists u)(x<u<y \wedge (\exists v)(u<v<y \wedge \chi)))$.
Then $\rho \in Lp_n(y)$ and therefore ρ is expressible, by assumption. Also $P'((\exists v > u)\psi)$ is expressible, as $P'((\exists v > u)\psi) \in Bf$. We claim that ϕ is equivalent to

(16)... $P'((\exists v > u)\psi) \wedge \rho$.

Again it is obvious that ϕ implies (16). To see that the converse holds, we consider again an arbitrary linear structure $\mathfrak{M} = \langle \underline{M}, <, \underline{P} \rangle$ and a point $\underline{t}_0$ of $\underline{M}$ which satisfies (16). Let $\underline{t}_1$ be a point of $\underline{M}$ such that $\underline{t}_1 < \underline{t}_0$. We have to show that there exist $\underline{t}_2, \underline{t}_3$ such that

(17) ... $\underline{t}_1 < \underline{t}_2 < \underline{t}_3 < \underline{t}_0$

and $\underline{t}_2, \underline{t}_3$ satisfy ψ in $\mathfrak{M}$. Since $\underline{t}_0$ satisfies (16) it also satisfies $P'((\exists v > u)\psi)$. One easily sees that $\underline{M}$ must then contain a sequence $\langle \underline{u}_1,\ldots,\underline{u}_{n+2}\rangle$ of points of $\underline{M}$ which satisfies $(\exists v > u)\psi$ where $\underline{u}_1$ lies between $\underline{t}_1$ and $\underline{t}_0$. Again, if $\underline{u}_{n+2} < \underline{t}_0$, then (17) holds if we replace $\underline{t}_2$ by $\underline{u}_1$ and $\underline{t}_3$ by $\underline{u}_{n+2}$. Suppose that $\underline{u}_{n+2} \geqslant \underline{t}_0$. Then there is a $k < n$ such that $\underline{u}_k < \underline{t}_0$ and $\underline{u}_{k+1} \geqslant \underline{t}_0$. Since $\underline{t}_0$ satisfies ρ in $\mathfrak{M}$, there are $\underline{w}$, $\underline{w}'$ in $\underline{M}$ such that $\underline{u}_k < \underline{w} < \underline{w}' < \underline{t}_0$ and $\underline{w}, \underline{w}'$ satisfy χ in $\mathfrak{M}$. Then, as before, we can assert the existence of a sequence $\langle \underline{v}_2, \ldots, \underline{v}_{n+2} \rangle$ which satisfies' χ in $\mathfrak{M}$, where $\underline{v}_2 = \underline{w}$ and $\underline{v}_{n+2} = \underline{w}'$. One easily verifies that the sequence $\langle \underline{u}_1, \ldots, \underline{u}_k, \underline{v}_{k+1}, \ldots, \underline{v}_{n+2} \rangle$ satisfies' ψ in $\mathfrak{M}$. Therefore (17) holds with $\underline{u}_1$ instead of $\underline{t}_2$ and $\underline{v}_{n+2}$ instead of $\underline{t}_3$. We conclude that (16) implies ϕ. It follows that ϕ is expressible.

Lemma 3: All members of E are expressible.

Proof: By lemmas 1,2 all members of $\mathrm{At} \cup \mathrm{Bp} \cup \mathrm{Bf} \cup \mathrm{Lp} \cup \mathrm{Lf}$ are expressible. Then by R 4, all members of E are expressible.

Convention. From now on, to the end of this chapter, 'valid,' 'equivalent,' 'expressible,' etc., will stand for 'Com-valid,' 'Com-equivalent,' 'Com-expressive,' etc., rather than 'Lin-valid,'....

Lemma 4: If ϕ, ψ have the property I(x,y) then $\phi \wedge \psi$ has the property I(x,y).

Proof: Let ϕ, ψ have the property I(x,y).

So there are $\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n \in \mathrm{Ep}(x,y)$ such that $\mathrm{Lin} \models x < y \rightarrow (\phi \leftrightarrow \bigvee_i^m \phi_i)$ and $\mathrm{Lin} \models x < y \rightarrow (\psi \leftrightarrow \bigvee_j^n \psi_j)$. $\therefore$ $\mathrm{Lin} \models x < y \rightarrow (\phi \wedge \psi \leftrightarrow \bigvee_{i,j} (\phi_i \wedge \psi_j))$. So it suffices to show that if $\phi, \psi \in \mathrm{Ep}(x,y)$ then there are $\chi_1, \ldots, \chi_p \in \mathrm{Ep}(x,y)$ such that $\mathrm{Lin} \models x < y \rightarrow (\phi \wedge \psi \leftrightarrow \bigvee_l^p \chi_l)$

Assume that $\phi, \psi \in \mathrm{Ep}(x,y)$; then, for some numbers m, n, there are $\phi', \psi' \in \mathrm{Bp}(x,y)$ such that

$S(\phi') = \langle P_1, P_2, \ldots, P_{2m+1} \rangle$ and $S(\psi') = \langle P_{2m+2}, \ldots, P_{2m+2n+2} \rangle$ and

$\phi_1, \ldots, \phi_{2m+1}, \psi_1, \ldots, \psi_{2n+1} \in E$, such that

$\phi = [\phi'] \frac{\phi_1}{P_1} \cdots \frac{\phi_{2m+1}}{P_{2m+1}}$

and $\psi = [\psi'] \frac{\psi_1}{P_{2m+2}} \cdots \frac{\psi_{2n+1}}{P_{2m+2n+2}}$

Suppose that

(18) ... $\mathrm{Lin} \models x < y\ (\phi' \wedge \psi' \leftrightarrow \bigvee_{K}^{p} \chi'_K)$, for some $\chi'_1, \ldots, \chi'_p \in Ep(x,y)$.

Then, if we put, for $k = 1, \ldots, p$,

$\chi_K = [\chi'_K] \frac{\phi_1}{P_1} \cdots \frac{\phi_{2m+1}}{P_{2m+1}} \frac{\psi_1}{P_{2m+2}} \cdots \frac{\psi_{2n+1}}{P_{2n+2m+1}}$, $\mathrm{Lin} \models x < y \to (\phi \wedge \psi \leftrightarrow \bigvee_{K}^{p} \chi_K)$

So it suffices to show (18).

We introduce the following terminology:

Let us call a binary relation R a <u>quasi-ordering</u> iff R is reflexive and transitive and for all a, b $\in$ Fld (R) $\langle a,b \rangle \in R$ or $\langle b,a \rangle \in R$.

For a quasi-ordering R we define: $a \equiv b \pmod{R}$ iff $\langle a,b \rangle \in R$ and $\langle b,a \rangle \in R$.

If R and R' are quasi-orderings, then we call F <u>a fusion of R and R'</u> if $F \subseteq (\mathrm{Fld}(R) \cup \mathrm{Fld}(R'))^2$, F is a quasi-ordering, $F \cap (\mathrm{Fld}(R) \times \mathrm{Fld}(R)) = R$ and $F \cap (\mathrm{Fld}(R') \times \mathrm{Fld}(R')) = R'$.

Let us call a finite sequence all members of which are different a <u>pure</u> sequence. Of course one may regard a pure sequence S as a quasi-ordering, i.e., we may identify

S with the reflexive binary relation R on the set of its members such that for different members a and b of S $\langle a,b \rangle \in R$ iff a precedes b in S. Thus we can also speak about <u>fusions</u> of two pure sequences. If S and S' are pure and have no common members, then they <u>form a pure pair</u>. Let S, S' form a pure pair and let F be a fusion of S and S'. Then we call a member b of S' <u>independent relative to</u> F if for no member a of S a = b (mod F). If U is the set consisting of the members of S and the independent members of S', then F establishes a linear order on U. We will call the cardinality of U the <u>length</u> of F.

According to this terminology Se(ϕ') and Se(ψ') form a pure pair.

Let $F_1, \ldots, F_p$ be all those fusions of Se(ϕ') and Se(ψ') such that for $i = 1, \ldots, p$

$$(19)\ldots \qquad P_1 = P_{2m+2} \text{ (mod } F_i\text{), and}$$

$$P_{2m+1} = P_{2m+2n+2} \text{ (mod } F_i\text{).}$$

Let $U = \{P_1, \ldots, P_{2m+2n+2}\}$.

With each F_i we associate a formula ϕ_i as follows:

Let n_i be the length of F_i. Let $\langle Q_1, \ldots, Q_{n_i} \rangle$ list all the members of Se(ϕ') and those members of Se(ψ') which are independent relative to F_i, in the order that F_i establishes between them. Let σ_i be a formula of

$Bp_{n-1}(x,y)$ in which no 1-place predicate letter occurs more than once. Let $S(\sigma_i') = \langle Q_1', \ldots, Q_{2n_i-1} \rangle$.

We define a function f_i from U to subsets of $\{1, \ldots, 2n_i-1\}$ as follows:

1) if Q is a member of $Se(\phi')$ or $Se(\psi')$, then $f_i(Q) = \{2k-1\}$, where k is the unique number such that $Q = Q_k (\mathrm{mod}\ F_i)$.

2) if Q is P_{2j} for some $j \leqslant m$, then $f_i(Q) =$
$$\{k \leqslant 2n_i-1 : \cup f_i(P_{2j-1}) < k < \cup f_i(P_{2j+1})\}.$$

2') if Q is P_{2j+1} for some j such that $m+1 \leqslant j \leqslant m+n$, then
$$f_i(Q) = \{k \leqslant 2n_i-1 : \cup f_i(P_{2j}) < k < \cup f_i(P_{2j+2})\}$$

One easily sees that 1), 2) and 2') define f_i for all members of U, and that for each $k \leqslant 2n_i-1$ there is a Q in U such that $k \in f_i(Q)$. Further we put, for $k = 1, \ldots, 2n_i-1$, $\rho_k = Q_{k_1}(x) \wedge \ldots \wedge Q_{k_r}(x)$, where $Q_{k_1} \ldots, Q_{k_r}$ are all those Q's such that $k \in F_i(Q)$.

Finally, let $\chi_i = [\sigma_i] \frac{\rho_1}{Q_1'} \cdots \frac{\rho_{2n_i-1}}{Q_{2n_i-1}'}$

Clearly, for $i = 1, \ldots, p$, $\chi_i \in Ep(x,y)$.

We claim that $Lin \models x < y \rightarrow (\phi' \wedge \psi' \leftrightarrow \chi_i)$

In the first place we have to show that each χ_i implies both ϕ' and ψ'. Let $\mathfrak{M} = \langle \underline{M}, <, \underline{P} \rangle$ be an arbitrary linear structure and $\underline{t}_1, t_2$ be points of $\underline{M}$ such that $\underline{t}_1 < \underline{t}_2$ and $\underline{t}_1, \underline{t}_2$ satisfy χ_i in $\mathfrak{M}$. Then there are points $\underline{u}_1, \ldots, \underline{u}_n$ in $\underline{M}$ such that $\underline{u}_1 = \underline{t}_1$, $\underline{u}_{n_i} = \underline{t}_2$ and for $k = 1, \ldots, n_i$, and any predicate letter Q, $\underline{u}_k \in \underline{Q}$[2] iff $2k-1 \in f_i(Q)$ and for $k = 1, \ldots, n_i - 1$ all $\underline{t}$ between $\underline{u}_k$ and $\underline{u}_{k+1}$ belong to $\underline{Q}$ iff $2k \in f_i(Q)$.

Let $k(1), \ldots, k(m+1)$ be those numbers $\leqslant 2n_i - 1$ such that for $j = 1, \ldots, m+1$ $f_i(P_{2j-1}) = k(j)$. Then one easily verifies that $\langle \underline{u}_{k(1)}, \ldots, \underline{u}_{k(m+1)} \rangle$ satisfies' ϕ'. Further, since F_i satisfies (23), $k(1) = 1$ and $k(m+1) = n_i$. Thus $\underline{u}_{k(1)} = \underline{t}_1$ and $\underline{u}_{k(m+1)} = \underline{t}_2$, and $\underline{t}_1, \underline{t}_2$ satisfy ϕ' in $\mathfrak{M}$. It follows that χ_i implies ϕ'. In the same way one shows that χ_i implies ψ'.

In the second place we have to show that $\phi' \wedge \psi'$ implies $\bigvee_{i=1}^{p} \chi_i$.

Let $\mathfrak{M} = \langle \underline{M}, <, \underline{P} \rangle$ be an arbitrary linear structure and $\underline{t}_1, \underline{t}_2$ be points of $\underline{M}$ which satisfy $\phi' \wedge \psi'$ in $\mathfrak{M}$. Then, since $\underline{t}_1, \underline{t}_2$ satisfy ϕ', there is a sequence $\langle \underline{u}_1, \ldots, \underline{u}_{m+1} \rangle$ of points of $\underline{M}$ which satisfies' ϕ' in $\mathfrak{M}$ such that $\underline{u}_1 = \underline{t}_1$, $\underline{u}_{m+1} = \underline{t}_2$.
Also since $\underline{t}_1$, $\underline{t}_2$ satisfy ψ' in $\mathfrak{M}$, there is a sequence $\langle \underline{v}_1, \ldots, \underline{v}_{n+1} \rangle$ of points of $\underline{M}$ such that $\underline{v}_1 = \underline{t}_1$, $\underline{v}_{n+1} = \underline{t}_2$, and $\langle \underline{v}_1, \ldots, \underline{v}_{n+1} \rangle$ satisfies' ψ'. $<$ induces a fusion F of $Se(\phi')$ and $Se(\psi')$ so that

[2]By $\underline{Q}$ we understand here the interpretation of Q in $\mathfrak{M}$.

$\langle P_{2h-1}, P_{2m+2k} \rangle \in F$ iff $\underline{u}_h \leqslant \underline{v}_k$ and $\langle P_{2m+2k},$ $P_{2h-1} \rangle \in F$ iff $\underline{v}_k \leqslant \underline{u}_h$. Clearly, F satisfies (19) and is thus one of the F_i listed above. One easily verifies that $\underline{t}_1, \underline{t}_2$ satisfy χ_i in $\mathfrak{M}$.

This completes the proof of Lemma 4.

Lemma 5. If x,y,z are distinct variables, $\chi_1, \ldots, \chi_m$ belong to E(z), $\Phi_1, \ldots, \Phi_n$ have the property I(x,z) and $\Psi_1, \ldots, \Psi_p$ have the property I(z,y) then

(20)... $(\exists z)\,(x<z<y \wedge \bigwedge_i^m \chi_i \wedge \bigwedge_j^n \phi_j \wedge \bigwedge_k^p \psi_k)$

has the property I(x,y).

Proof: Let χ_i ϕ_j, ψ_k, be as in the hypothesis of the lemma. By the definition of E, $\chi = \bigwedge_i^m \chi_i \in E$. Therefore it suffices to prove the lemma for m = 1. Moreover, we may restrict ourselves to the case where χ_1 is $Q_0(z)$ where Q_0 is a predicate letter which occurs in none of the ϕ_j and ψ_k. Then the lemma follows for arbitrary $\chi_1 \in E(z)$ by R 8. So assume m = 1, $\chi_1 = Q_0(z)$ and Q_0 occurs in none of the $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_p$.

By lemma 4 $\bigwedge_j^n \phi_j$ (call it 'ϕ') has the property I(x,z) and $\bigwedge_k^p \psi_k$ (call it 'ψ') has the property I(z,y). So there are $\rho_1, \ldots, \rho_r \in Ep(x,z)$ such that Com $\models x < z \rightarrow (\phi \leftrightarrow \bigvee_i^r \rho_i)$ and $\sigma_1, \ldots, \sigma_s \in Ep(z,y)$ such that

Com $\models z < y \rightarrow (\psi \leftrightarrow \bigvee_j^s \sigma_j)$

Then (20) is equivalent to $\bigvee_i^r \bigvee_j^s (\exists z)(x < z < y \wedge Q (z) \wedge$

$\rho_i \wedge \sigma_i$).

So it suffices to show that if $\rho \in Rp(x,z)$ and $\sigma \in Ep(z,y)$, then

(21)... $(\exists z)\ (x < z < y \wedge Q_0(z) \wedge \rho \wedge \sigma\ )$

has the property $I(x,y)$.

Again in view of R 8 we may restrict attention to the case where $\rho \in Bp(x,z)$, $\sigma \in Bp(z,y)$ and no predicate letter occurs more than once in $\rho \wedge \sigma$. Let Q' be the last member of $S(\rho)$ and Q'' the first member of $S(\sigma)$. Let S be the sequence such that $\langle Q'' \rangle \frown S = S(\sigma)$. Let π be a member of $Bp(x,y)$ such that $S(\pi) = S(\rho)\frown S$, and let $\pi' = [\pi]\ \dfrac{Q_0(z)\ \wedge\ Q'(z)\ \wedge\ Q''\ (z)}{Q'}$ Then $\pi' \in$

Ep(x,y) and, as is easily verified, (21) is equivalent to π'.

<u>Lemma 6</u>. Let $\phi \in Bp_n(x,y)$. Let $\mathfrak{M} = \langle \underline{M}, <\ ,\ \underline{P} \rangle$ be a structure such that $\langle \underline{M}, < \rangle$ is a linear ordering. Let $\underline{t}_1$, $\underline{t}_2$, $\underline{t}_3$, $\underline{t}_4$ $\in$ M such that $\underline{t}_1 < \underline{t}_2 < \underline{t}_3 < \underline{t}_4$. Assume that $\underline{t}_1, \underline{t}_2$ satisfy ϕ (x,y) in $\mathfrak{M}$ and that $\underline{t}_1$, $\underline{t}_4$ satisfy ϕ (x,y) in $\mathfrak{M}$. Then $\underline{t}_1, \underline{t}_3$ satisfy ϕ (x,y) in $\mathfrak{M}$ if and only if $\underline{t}_3$ satisfies $(\exists x < y)\phi(x,y)$ in $\mathfrak{M}$.

<u>Proof</u>: Obviously if $\underline{t}_1, \underline{t}_3$ satisfy ϕ (x,y) in $\mathfrak{M}$ then $\underline{t}_3$ satisfies $(\exists x < y)\ \phi\ (x,y)$ in $\mathfrak{M}$.

To prove the lemma in the other direction, assume that $\underline{t}_3$

satisfies $(\exists x < y)\phi(x,y)$. Then there is a sequence $\langle \underline{v}_1,\dots,\underline{v}_{n+1}\rangle$ such that $\underline{v}_{n+1} = \underline{t}$ and $\langle \underline{v},\dots,\underline{v}_{n+1}\rangle$ satisfies' ϕ.

Suppose $\underline{t}_1 \leqslant \underline{v}_1$. Since $\underline{t}_1, \underline{t}_4$ satisfy $\phi(x,y)$ there is a sequence $\langle \underline{w}_1,\dots,\underline{w}_{n+1}\rangle$ which satisfies' $\phi(x,y)$ such that $\underline{w}_1 = \underline{t}_1$ and $\underline{w}_{n+1} = \underline{t}_4$. Since $\underline{w}_1 \leqslant \underline{v}_1$ and $\underline{v}_{n+1} < \underline{w}_{n+1}$ there is a $k \leqslant n$ such that $\underline{w}_k \leqslant \underline{v}_k < \underline{w}_{k+1}$. It follows that $\langle \underline{w}_1,\dots,\underline{w}_k, \underline{v}_{k+1},\dots,\underline{v}_{n+1}\rangle$ satisfies' $\phi(x,z)$. Thus $\underline{t}_1$ $(=\underline{w}_1)$ and $\underline{t}_3$ $(=\underline{v}_{n+1})$ satisfy $\phi(x,y)$. Suppose $\underline{v}_1 \leqslant \underline{t}_1$. Since $\underline{t}_1, \underline{t}_2$ satisfy $\phi(x,y)$, there is a sequence $\langle \underline{w}_1,\dots,\underline{w}_{n+1}\rangle$ which satisfies' $\phi(x,y)$ such that $\underline{w}_1 = \underline{t}_1$ and $\underline{w}_{n+1} = \underline{t}$. Since $\underline{v}_1 \leqslant \underline{w}_1$ and $\underline{w}_{n+1} < \underline{v}_{n+1}$, there is a $k \leqslant n$ such that $\underline{v}_k \leqslant \underline{w}_k < v_{k+1}$. Thus $\langle \underline{w}_1,\dots, \underline{w}_k, \underline{v}_{k+1},\dots,\underline{v}_{n+1}\rangle$ satisfies' $\phi(x,y)$. So $\underline{t}$ $(=\underline{w}_1)$ and $\underline{t}_3$ $(=\underline{v}_{n+1})$ satisfy $\phi(x,y)$.

Lemma 7. Let x,y,u,v be distinct variables and ψ_1, ψ_2 formulae of L.

a) Let $\phi \in Bp(v,y)$. Then there are a natural number p, conjunctions $\sigma_0,\dots,\sigma_p$ of members of $Bp(u,y)$ and disjunctions $\rho_0,\dots,\rho_p$ of members of $Bp(v,u)$ such that

(i)

(22) ... $(\exists u)(x<u<y \wedge \psi_1 \wedge (\forall v)(x<v<u \rightarrow \psi_2 \vee \phi))$

is Lin-equivalent to

$$\bigvee_{i}^{p} (\exists u)(x<u<y \wedge \psi_1 \wedge \sigma_1 \wedge (\forall v)(x<v<u \rightarrow \psi_2 \vee \rho_i)),$$

and

(ii) each ρ_i $(i \leqslant p)$ contains at most one disjunct with the same length as ϕ, and no disjunct with greater length.

b) Let $\phi \in Bp(x,v)$. Then there are a natural number p and formulae $\sigma_0,\dots,\sigma_p \in Bp(x,u)$, $\rho_0,\dots,\rho_p \in Bp(u,v)$

such that

(i) $(\exists u)(x<u<y \wedge \psi_1 \wedge (\forall v)(u<v<y \rightarrow \psi_2 \vee \phi))$

is Lin-equivalent to

$$\bigvee_{i}^{p} (\exists u)(x<u<y \wedge \psi_1 \wedge \sigma_i \wedge (\forall v)(u<v<y \rightarrow (\neg\psi_2 \vee \rho_i))$$

and

(ii) ρ_i $(i \leqslant p)$ satisfies a (ii).

Proof: We prove a). The proof of b) is similar. Assume $\phi \in Bp_n(v,y)$. Let $(R_0, S_0),\dots, (R_r, S_r)$ be all the different pairs of sequences of predicate letters such that $R_i{}^\frown S_i = S(\phi)$ and S_i has length $\geqslant 2$. For $i = 0,\dots, r$ let $Z(i)$ abbreviate the statement: 'S_i begins with a member of $Se(\phi)$'. Let for $i=0,\dots,r$

$$S_i^* = \begin{cases} S_i & \text{if } Z(i) \\ \langle Q \rangle \frown S_i & \text{if not } Z(i) \text{ and } Q \text{ is the first member of } S_i \end{cases}$$

$$R_i^* = \begin{cases} R_i \frown \langle Q \rangle & \text{if } Z(i) \text{ and } Q \text{ is the first member of } S_i \\ R_i \frown \langle Q,Q \rangle & \text{if not } Z(i) \text{ and } Q \text{ is the first member of } S_i \end{cases}$$

Let 2^{r+1} be the set of all functions from $\{0,\ldots,r\}$ into $\{0,1\}$. With each $\alpha \in 2^{r+1}$ we associate σ'_α, σ_α, ρ_α as follows:

Let for $i = 0,\ldots, r$

$$\omega_\alpha(i) = \begin{cases} \sigma_i & \text{if } \alpha(i) = 1 \\ \neg\sigma_i & \text{if } \alpha(i) = 0 \end{cases}$$

Put $\sigma'_\alpha = \bigwedge_i^r \omega_\alpha(i)$;

σ_α = the conjunction of the formulae σ_i such that $\alpha(i) = 1$;

ρ_α = the disjunction of the formulae ρ_i such that $\alpha(i) = 1$.

Clearly $\models \bigvee_{\alpha \in 2^{r+1}} \sigma'_\alpha$

Thus $(\exists u)(x<u<y \wedge \psi_1 \wedge (\forall v)(x<v<u \rightarrow \psi_2 \vee \Phi))$

is Lin-equivalent to

$$(\exists u)(x<u<y \wedge \psi_1 \wedge \bigvee_{\alpha \in 2^{r+1}} \sigma'_\alpha \wedge (\forall v)(x<v<u \rightarrow \psi_2 \vee \Phi))$$

which in turn is equivalent to

$$\bigvee_{\alpha\in 2^{r+1}} (\exists u)\ (x<u<y \wedge \psi_1 \wedge \sigma'_\alpha \wedge (\forall v)\ (x<v<u \rightarrow \psi_2 \vee \phi\ ))$$

This last formula Lin-implies

(23).... $\bigvee_{\alpha\in 2^{r+1}} (\exists u)\ (x<u<y \wedge \psi_1 \wedge \sigma_\alpha \wedge (\forall v)(x<v<u \rightarrow$
$\psi_2 \vee \rho_\alpha\ ))$

Furthermore each disjunct of (23) Lin-implies (22). It follows that (22) and (23) are Lin-equivalent.

<u>Lemma 8</u>. If x,y,z are distinct variables and F is a finite subset of $At(z) \cup Bp(x,z) \cup Bp(z,y)$, then

(24).... $(\forall z)(x<z<y \rightarrow \bigvee_{\psi\in F}\psi)$ has the property I(x,y).

<u>Proof</u>. Let us understand by the <u>length</u> of a formula $\phi \in Bp \cup Bf$ the length of $S(\phi)$. For $\phi \in At$ we put the <u>length</u> of ϕ equal to 1. For finite subsets F, F' of $At \cup Bp$ we define: $F' \prec F$ iff for some number n all formulae in F' have length $\leq n$ and F contains more formulae of length n than F'. Clearly $\prec$ is well-founded. We prove the lemma by induction on $\prec$. So let x, y, z be distinct variables, and F a finite non-empty subset of $At(z) \cup Bp(x,z) \cup Bp(z,y)$, and let us assume the lemma for all triples of distinct variables u,v,w and all $F' \subseteq Bp(u,w) \cup At(w) \cup Bp(w,v)$ such that $F' \prec F$. We may assume that all predicate letters that occur in any member of F occur only once (i.e. only once in that member of F and in no

other member of F). For suppose that the lemma holds for all $G \subseteq At(z) \cup Bet(x,z) \cup Bet(z,y)$ which satisfy this extra condition and are such that for all F' $F' \prec F$ iff $F' \prec G$. Among these G there is a G_0 which is the range of a one-one function with domain F that preserves length. Since the lemma holds for G_0, it also holds for F, by R 8. If $F \subseteq At(z)$ then (24) is clearly equivalent to a member of Ep(x,y) and thus has the property I(x,y). So we may assume that $F \notin At(z)$. We may also assume that F contains at least one member of the form Q(z), where Q occurs in no other member of F. For let $Q_0(z)$ be such a formula and let $G_0 = F \cup \{Q_0(z)\}$. Then for all F', $F' \prec G$ iff $F' \prec F$ and so, since the lemma holds for G_0, it holds for F.

Since $(\forall z)(x<z<y \to \bigvee_{\psi \in F} \psi)$ is equivalent to $[(\forall z)\ (x<z<y \to \bigvee_{\psi \in G} \psi)]$

$$\frac{P_0(z) \lor \neg P_0(z)}{Q_0}$$

it follows from R8 that the lemma holds for F if it holds for G. Let ϕ be a member of F of maximal length. Let $F' = F - \{\phi\}$.

During the remainder of this proof we will for any variable v and any $\psi \in F'$, understand by $\psi(v)$ the formula which we obtain when we replace z everywhere in ψ by v. For ϕ we will exhibit both free variables.

Let $\chi = (\forall z)(x<z<y \to \bigvee_{\psi \in F} \psi)$. Show that χ has I(x,y).

<u>Case I</u>: $\phi \in Bp_n(x,z)$. Let v, w, u_1, u_2 be distinct variables, different from x, y,z.

Consider the following formulae:

$$\alpha_0 = \neg(\exists v)(x<v<y \wedge \phi(x,v))$$

$$\alpha_1 = (\forall v)(x<v \rightarrow (\exists w)(x<w<v \wedge \phi(x,w)))$$

$$\alpha_2 = x<u_1<y \wedge \phi(x,u_1) \wedge \neg(\exists v)(x<v<u_1 \wedge \phi(x,v))$$

$$\alpha_3 = x<u_1<y \wedge (\forall v)(u_1<v<y \rightarrow (\exists w)(u_1<w<v \wedge \phi(x,w))) \wedge \neg\phi(x,u_1) \wedge \neg(\exists v)(x<v<u_1 \wedge \phi(x,v))$$

$$\beta_1 = (\forall v)(v<y \rightarrow (\exists w)(v<w<y \wedge \phi(x,w)))$$

$$\beta_2 = x<u_2<y \wedge \phi(x,u_2) \wedge \neg(\exists v)(u_2<v<y \wedge \phi(x,v))$$

$$\beta_3 = x<u_2<y \wedge (\forall v)(x \leq v<u_2 \rightarrow (\exists w)(v<w<u_2 \wedge \phi(x,w))) \wedge \neg\phi(x,u_2) \wedge \neg(\exists v)(u_2<v<y \wedge \phi(x,v))$$

$$\gamma_1 = \alpha_0$$

$$\gamma_2 = \alpha_1 \wedge \beta_1$$

$$\gamma_3 = \alpha_1 \wedge (\exists u_2)\beta_2$$

$$\gamma_4 = \alpha_1 \wedge (\exists u_2)\beta_3$$

$$\gamma_5 = (\exists u_1)(\alpha_2 \wedge [\alpha_0]\tfrac{u_1}{x})$$

$$\gamma_6 = (\exists u_1)\alpha_2 \wedge \beta_1$$

$$\gamma_7 = (\exists u_1)(\alpha_2 \wedge (\exists u_2)[\beta_2]\tfrac{u_1}{x})$$

$$\gamma_8 = (\exists u_1)(\alpha_2 \wedge (\exists u_2)[\beta_3]\tfrac{u_1}{x})$$

$$\gamma_9 = (\exists u_1)\alpha_3 \wedge \beta_1$$

$$\gamma_{10} = (\exists u_1)(\alpha_3 \wedge (\exists u_2)[\beta_2]\tfrac{u_1}{x})$$

$$\gamma_{11} = (\exists u_1)(\alpha_3 \wedge (\exists u_2)[\beta_3]\tfrac{u_1}{x})$$

One easily verifies that Com $\models \bigvee_{i}^{11} \gamma_i$. So χ is Com-equivalent to $\bigvee_{i=1}^{11}(\gamma_i \wedge \chi)$. We will now introduce for $i = 1,\ldots,11$ a formu[illegible] χ_i, such that Lin $\models \gamma_i \wedge \chi \to \chi_i$ and Lin $\models \chi_i \to \chi$. Then we have Com $\models \chi \leftrightarrow \bigvee_{i}^{11} \chi_i$; moreover the χ_i will be chosen in such a way that we can prove that they have the property $I(x,y)$.

Let $\chi_1 = (\forall z)(x<z<y \to \bigvee_{\psi \in F'} \psi)$

Then clearly Lin $\models \chi_1 \to \chi$ and Lin $\models \gamma_1 \wedge \chi \to \chi_1$.

Let $\chi_2 = \gamma_2 \wedge (\forall z)\ (x<z<y \to \ (\exists x<z)\ \phi\,(x,z) \vee \bigvee_{\psi \in F'} \psi\ )$

Then clearly Lin $\models \gamma_2 \wedge \chi \to \chi_2$. To show that χ_2 implies χ we argue as follows: Let $\mathfrak{M} = \langle M, <, \underline{P}\rangle$ be an arbitrary linear structure and $\underline{t}_1, \underline{t}_2$ points of M such that $\underline{t}_1 < \underline{t}_2$ which satisfy χ_2. We have to show for each point $\underline{t}$ between $\underline{t}_1$ and $\underline{t}_2$ that there is a $\psi \in F$ such that $\underline{t}_1$, $\underline{t}$ or $\underline{t}, \underline{t}_2$ satisfy ψ (according as $\psi \in Bp(x,z)$ or $\psi \in Bp(z,y)$)

So let $\underline{t}$ be a point between $\underline{t}_1$ and $\underline{t}_2$. If $\underline{t}_1, \underline{t}$ satisfy some $\psi \in F' \cap Bp(x,z)$ or $\underline{t}, \underline{t}_2$ satisfy some $\psi \in F' \cap Bp(z,y)$, then we are done.

So let us assume that neither is the case. Then $\underline{t}$ must satisfy $(\exists x<z)\ \phi(x,z)$.

We want to show that $\underline{t}_1, \underline{t}$ satisfy $\phi(x,z)$ in $\mathfrak{M}$. In view of lemma 6 it suffices to show that there is a $\underline{t}_3 \in M$ such that $\underline{t}_1 < \underline{t}_3 < \underline{t}$ and $\underline{t}_1$, $\underline{t}_3$ satisfy $\phi(x,z)$ in $\mathfrak{M}$, and a $\underline{t}_4 \in M$ such that $\underline{t} < \underline{t}_4$ and $\underline{t}_1$, $\underline{t}_4$ satisfy $\phi\,(x,z)$ in $\mathfrak{M}$. But this is clearly the case since $\underline{t}_1$ satisfies α_1 in $\mathfrak{M}$, and $\underline{t}_2$ satisfies β_1 in $\mathfrak{M}$.

Let $\chi_3 = \alpha_1 \wedge (\exists u_2)(x < u_2 < y \wedge \phi(x,u_2) \wedge$

$(\forall v)(x<v<u_2 \to (\exists x<v)\,\phi(x,v) \vee \bigvee_{\psi\in F'} \psi(v)) \wedge$

$(\forall v)(u_2 < v < y \to \bigvee_{\psi\in F''} \psi(v)))$

It is fairly obvious that $\gamma_3 \wedge \chi$ implies χ_3. For consider the formula:

(25).... $\alpha_1 \wedge (\exists u_2)\ (x<u_2<y \wedge \phi(x,u_2) \wedge \bigvee_{\psi\in F} \psi(u_2) \wedge$

$(\forall v)(x<v<u_2 \to \bigvee_{\psi\in F} \psi(v)) \wedge (\forall v)(u_2 < v < y \to \neg\phi(x,v)$

$\wedge \bigvee_{\psi\in F} \psi(v)))$

Clearly (25) is equivalent to $\gamma_3 \wedge \chi$. One easily sees that (25) implies χ_3.

To see that χ_3 implies χ we argue as follows:

Let $\mathfrak{M} = \langle M, <, \underline{P}\rangle$ be an arbitrary linear structure and $\underline{t}_1,\underline{t}_2$ points in M such that $\underline{t}_1 < \underline{t}_2$ and $\underline{t}_1,\underline{t}_2$ satisfy χ_3. Let $\underline{t}_0$ be a point between $\underline{t}_1$ and $\underline{t}_2$ such that $\underline{t}_1$, $\underline{t}_0$ satisfy ϕ and for all $\underline{t}$ between $\underline{t}_0$ and $\underline{t}_2$ there is a $\psi \in F'$ such that either $\underline{t}_1,\underline{t}$ or $\underline{t}$, $\underline{t}_2$ satisfy ψ (according as $\psi \in Bp(x,z)$ or $\psi \in Bp(z,y)$. Clearly if $\underline{t}_0 < \underline{t}$, then there is a $\psi \in F$ such that either $\underline{t}_1$, $\underline{t}$ satisfy ψ or $\underline{t},\underline{t}_2$ satisfy ψ (according as $\psi \in Bp(x,z)$ or $\psi \in Bp(z,y)$). If $\underline{t} = \underline{t}_0$ then $\underline{t}_1,\underline{t}$ satisfy ϕ. If $\underline{t} < \underline{t}_0$ then we use lemma 6.

Let $\chi_4 = \alpha_1 \wedge (\exists u_2)(x<u_2<y \wedge (\forall v)(x<v<u_2 \to (\exists x<v)\ \phi(x,v)$

$$\vee \bigvee_{\psi \in F'} \psi(v)) \wedge \bigvee_{\psi \in F'} \psi(u_2) \wedge (\forall v)(u_2 < v < y \rightarrow \bigvee_{\psi \in F'} \psi(v)));$$

$$\chi_5 = (\exists u_1)(x < u_1 < y \wedge (\forall v)(x < v < u_1 \rightarrow \bigvee_{\psi \in F'} \psi(v)) \wedge \phi(x, u_1) \wedge$$

$$(\forall v)(u_1 < v < y \rightarrow \bigvee_{\psi \in F'} \psi(v)));$$

$$\chi_6 = (\exists u_1)(x < u_1 < y \wedge \phi(x, u_1) \wedge (\forall v)(x < v < u_1 \rightarrow \bigvee_{\psi \in F'} \psi(v))$$

$$\wedge (\forall v)(u_1 < v < y \rightarrow (\exists x < v) \phi(x, v) \vee \bigvee_{\psi \in F'} \psi(v)));$$

$$\chi_7 = (\exists u_1)(x < u_1 < y \wedge \phi(x, u_1) \wedge (\forall v)(x < v < u_1 \rightarrow \bigvee_{\psi \in F'} \psi(v))$$

$$\wedge (\exists u_2)(u_1 < u_2 < y \wedge (\forall v)(u_1 < v < u_2 \rightarrow$$

$$(\exists x < v) \phi(x, v) \vee \bigvee_{\psi \in F'} \psi(v))) \wedge \phi(x, u_2)$$

$$\wedge (\forall v)(u_2 < v < y \rightarrow \bigvee_{\psi \in F'} \psi(v))));$$

$$\chi_8 = (\exists u_1)(x < u_1 < y \wedge (\forall v)(x < v < u_1 \rightarrow \bigvee_{\psi \in F'} \psi(v) \wedge$$

$$\phi(x, u_1) \wedge (\exists u_2)(u_1 < u_2 < y \wedge \bigvee_{\psi \in F'} \psi(u_2) \wedge$$

$$(\forall v)(u_1 < v < u_2 \rightarrow (\exists x < v) \phi(x, v) \vee \bigvee_{\psi \in F'} \psi(v)$$

$$(\forall v)(u_2 < v < y \rightarrow \bigvee_{\psi \in F'} \psi(v))));$$

$$\chi_9 = (\exists u_1)(x < u_1 < y \quad (\forall v)(x < v < u_1 \rightarrow \bigvee_{\psi \in F'} \psi(v)) \wedge \bigvee_{\psi \in F'} \psi(u_1)$$

$$\wedge (\forall v)(u_1 < v < y \rightarrow (\exists x < v) \phi(x, v) \vee \bigvee_{\psi \in F'} \psi(v)))$$

$$\chi_{10} = (\exists u_1)\ (x < u_1 < y \wedge (\forall v)(x < v < u_1 \to \bigvee_{\psi \in F'} \psi(v)) \wedge \bigvee_{\psi \in F'} \psi(u_1)$$

$$\wedge\ (\exists u_2)(u_1 < u_2 < y \wedge (\forall v)\ (\ u_1 < v < u_2 \to (\exists x < v)\ \phi(x, v)$$

$$\vee \bigvee_{\psi \in F'} \psi(v)) \wedge (\forall v)\ (u_2 < v < y \to \bigvee_{\psi \in F'} \psi(v)))) \wedge \phi(x, u_2)))$$

$$\chi_{11} = (\exists u_1)(x < u_1 < y \wedge (\forall v)\ (x < v < u_1 \to \bigvee_{\psi \in F'} \psi(v)) \wedge \bigvee_{\psi \in F'} \psi(u_1)$$

$$\wedge\ (\exists u_2)(u_1 < u_2 < y \wedge (\forall v)(u_1 < v < u_2 \to (\exists x < v)\phi(x, v)$$

$$\vee \bigvee_{\psi \in F'} \psi(v)) \qquad \wedge \bigvee_{\psi \in F'} \psi(u_2)$$

$$\wedge \quad (\forall v)\ (u_2 < v < y \to \bigvee_{\psi \in F'} \psi(v))))$$

For $i = 4,\dots,11$ the proofs that Lin $\models \gamma_i \wedge \chi \to \chi_i$ and Lin $\models \chi_i \to \chi$ are similar to those for $i = 1,2,3$. We omit them. It remains to be shown for $i = 1,\dots,11$ that χ_i has the property $I(x,y)$.

χ_1 has the property $I(x,y)$ by induction hypothesis.

$i = 2$: Let $\rho = (\forall z)(x < z < y \to \bigvee_{\psi \in F} \psi)$. The third conjunct of χ_2 is equivalent to $[\rho]\dfrac{Q_0(z) \vee (\exists x < z)\ \phi(x,z)}{Q_0}$

Moreover ρ has $I(x,y)$ by induction hypothesis. So the third conjunct of χ_2 has $I(x,y)$ in view of R 8.

To see that α_1 has $I(x,y)$ we argue as follows: Let $S(\phi) = \langle Q_1,\dots,Q_{2n+1}\rangle$. Let $\rho' =$

$$Q_1(x) \wedge (\exists u)(x < u < y \wedge (\forall v)(x < v < u \to \qquad Q_2(v)))$$

If $n = 1$ then let $\rho'' = (\forall v)(x < v \to (\exists w)(x < w < v \wedge Q_3(w)))$, otherwise let $\rho'' = (\forall v)(x < v \to (\exists w_2)(x < w_2 < v \wedge (\exists w_1)(w_2 < w_1 < v_1 \wedge \phi')))$, where ϕ' is a member of $Bp(w_2,w_1)$ such that $S(\phi') = \langle Q_3,\ldots,Q_{2n+1}\rangle$. One easily verifies that $Lin \models x < v \to (\alpha_1 \leftrightarrow \rho' \wedge \rho'')$. Further both ρ' and ρ'' have the property $I(x,y)$.

For ρ' is equivalent to a member of $Ep(x,y)$. If $n = 1$ then ρ'' is equivalent to a member of $Ep(x,y)$. If $n > 1$ then ρ'' belongs to $Lf(x)$ and thus is equivalent to a member of $Ep(x,y)$. In the same way we show that β_1 has $I(x,y)$, Thus all conjuncts of χ_2 have $I(x,y)$. It follows from lemma 5 that χ_2 has $I(x,y)$.

$i = 3$. We just showed that the first conjunct of χ_3, i.e. α_1, has the property $I(x,y)$. So let us consider the second conjunct of χ_3. If $F' \subseteq At(z)$ then this disjunct belongs to $Ep(x,y)$ and thus has $I(x,y)$. So let us assume that $F' \nsubseteq At(z)$. Suppose that $\rho \in Bp(z,y) \cap F'$. Let $F'' = F' - \{\rho\}$.

We can then write the second conjunct of χ_2 equivalently as

$$(\exists u_2)(x < u_2 < y \wedge \phi(x,u_2) \wedge (\forall v)(u_2 < v < y \to \bigvee_{\psi \in F''} \psi(v))$$

$$(\forall v)(x < v < u_2 \to ((\exists x < v)\ \phi(x,v) \vee \bigvee_{\psi \in F''} \psi(v)) \vee \rho))$$

By lemma 7 this last formula is equivalent to a disjunction

$$(25) \;\ldots\; \bigvee_{\alpha} (\exists u_2)\,(x < u_2 < y \;\wedge\; \phi(x,u_2) \wedge (\forall v)\,(u_2 < v < y \rightarrow \bigvee_{\psi \in F''} \psi(v)) \;\wedge\; \sigma_\alpha \wedge (\forall v)\,(x < v < y_2 \rightarrow ((\exists x < v)\,\phi(x,v) \;\vee\; \bigvee_{\psi \in F''} \psi(v) \;\vee\; \rho_\alpha))$$

where each of the σ_α is a conjunction of members of $Bp(u_2,y)$ and each ρ_α is a disjunction of members $\rho_{\alpha,0}, \ldots, \rho_{\alpha,r(\alpha)}$ of $Bp(x,u_2)$ at most one of which has the same length as ρ and none of which has greater length. It follows that $F'' \cup \{\rho_{\alpha,0}, \ldots, \rho_{\alpha,r(\alpha)}\} \prec F$.

If $F'' \subseteq At(z)$ then we can conclude, using the induction hypothesis, that for each disjunct of (25), its conjuncts beginning with a universal quantifier have $I(u_2, y)$ and $I(x,u_2)$ respectively.

It follows from lemma 5 that every disjunct of (25) has $I(x,y)$. If $F'' \not\subseteq At(z)$, then we apply 7 again, this time to each disjunct of (25). This process will eventually exhaust $F' - At(z)$ completely and leave us with a disjunction each disjunct of which has $I(x,y)$. This completes the proof that the second conjunct of χ_3 has $I(x,y)$. It follows that χ_3 has $I(x,y)$. In a similar fashion one shows that $\chi_4, \ldots, \chi_{11}$ have the property $I(x,y)$. This completes the proof that χ has

$I(x,y)$ for the case that $\phi \in Bp(x,z)$. If $\phi \in Bp(z,y)$ we only need to remember that there is a $\phi' \in Bf(z,y)$ such that $S(\phi') = S(\phi)$. If we put $F^* = (F - \{\phi\}) \cup \{\phi'\}$ an argument entirely symmetrical to the one given above will show that $(\forall z)(x < z < y \rightarrow \bigvee_{\psi \in F'^*} \psi)$ has $I(x,y)$. However, the last formula is equivalent to $(\forall z)(x < z < y \rightarrow \bigvee_{\psi \in F} \psi)$. Thus again χ has $I(x,y)$. This completes the induction step and thus the proof of lemma 8.

Lemma 9. If x,y,z are distinct variables, $\chi_0, \ldots, \chi_m$ belong to $E(z)$, $\phi_0, \ldots, \phi_n$ have the property $I(x,z)$ and $\psi_0, \ldots, \psi_p$ have the property $I(z,y)$ then

(26)... $(\forall z)(x < z < y \rightarrow \bigvee_k^m \chi_k \vee \bigvee_i^n \phi_i \vee \bigvee_j \psi_j$

has the property $I(x,y)$.

Proof: Let $\chi_0, \ldots, \chi_m$, $\phi_0, \ldots, \phi_n$, $\psi_0, \ldots, \psi_p$ be as in the hypothesis. Put $\chi = \bigvee_i \chi_i$; $\phi = \bigvee_j^n \phi_j$; $\psi = \bigvee_k^p \psi_k$. Clearly $\chi \in E(z)$, ϕ has $I(x,z)$ and ψ has $I(z,y)$. Let $\rho_0, \ldots, \rho_r \in Ep(x,z)$ such that $Com \models x < z \rightarrow (\phi \leftrightarrow \bigvee_i^r \rho_i)$.

Similarly let $\sigma_0, \ldots, \sigma_s \in Ep(z,y)$ such that $Com \models z < y \rightarrow (\psi \leftrightarrow \bigvee_j^s \sigma_j)$. Then (26) is equivalent to

(27) ... $(\forall z)(x < z < y \rightarrow \chi \vee \bigvee_i^r \rho_i \vee \bigvee_j^s \sigma_j$

So it suffices to show that (27) has the property $I(x,y)$.

But this can be inferred from Lemma 8 by the same reasoning that was used in the proof of Lemma 5.

Lemma 10. All conjunctions of members of Bet $(x,y) \cup$ At $(x) \cup$ At (y) have the property I (x,y).

Proof: Let us first remark that all members of Ep (x,y) have I (x,y). This follows from Lemmas 1 and 3 and R 8. Further

(i) All members of Bet_1 (x,y) have the property I (x,y), since every member of Bet_1 (x,y) is equivalent to a member of Ep (x,y).

(ii) Suppose that for all variables u, v all members of Bet_n (u,v) have I (u,v). Then it follows from lemmas 5 and 9 that all members of Bet_{n+1} (x,y) have I (x,y) for all x and y. Thus all members of Bet (x,y) have the property I (x,y). It follows from lemma 4 that all conjunctions of members of Bet $(x,y) \cup$ At $(x) \cup$ At(y) have I (x,y).

We can now prove theorem 2. As we observed it suffices to show that every $\phi \in \mathcal{D}$ is Com-expressible. So let $\phi \in \mathcal{D}$. Suppose that $\phi \in$ Bef (y). Then ϕ is of the form

(28)... $(\exists x)\,(x < y \wedge \phi_0 \wedge \ldots \wedge \phi_n)$, where $\phi_0, \ldots, \phi_n \in$ Bet $(x,y) \cup$ At $(x) \cup$ At(y). By Lemma 10 there are $\psi_0, \ldots, \psi_p \in$ Ep (x,y) such that (28) is Com-equivalent to $(\exists x)\,(x < y \wedge \bigvee_j^p \psi_j)$, and thus to

$\bigvee_{j}^{p} (\exists x < y)\ \psi_j$, all disjuncts of which are members of E; therefore, by Lemma 3, ϕ is Com-expressible. If $\Phi \in$ Aft (y) the argument is the same.

CHAPTER IV

OTHER RESULTS ON EXPRESSIBILITY

In this chapter we will consider weaker notions of expressibility (such as $\mathcal{R}$e-expressibility and $\mathcal{I}$n-expressibility). We first prove a theorem which is of crucial importance in connection with Theorem I.1 and which we announced already in the first chapter.

THEOREM 1. SINCE and UNTIL are not $\mathcal{R}$e-expressible in terms of sentential connectives and monadic $\mathcal{R}$e-tenses.

In the proof of this theorem we will make use of Theorem I.1; this is not really necessary but simplifies the argument. The idea of the proof is the following:

We choose two particular $\mathcal{R}$e-propositions (i.e. subsets of Re) P_0 and P_1, and real numbers t, t' and show that if $\mathcal{S}$ is the set consisting of all monadic $\mathcal{R}$e-tenses and the $\mathcal{R}$e-connectives NOT and AND and η is any formula of a sentential language TL for S, which contains no variables other than q_0 and q_1, then η is true at t in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(\mathrm{TL})\rangle$ if η is true at t' in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(\mathrm{TL})\rangle$. On the other hand it will be immediate from the choice of P_0, P_1, t, t' that the formula

$S(q_0, q_1)$ of TL_1 is true at t in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TL_1)\rangle$ but not at t' in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TL_1)\rangle$. Thus the $\mathcal{R}e$-tense SINCE is expressed by no formula η of TL.

Convention: In this chapter we use the following notation: If $\langle I, < \rangle$ is a linear ordering and, $i, i' \in I$, $i < i'$ then (i,i') and $[i,i']$ will denote the 'open' and 'closed' intervals with i and i' as endpoints, respectively (i.e., the sets $\{ j \in I: i < j \wedge j < i' \}$ and $\{ j \in I: i \leq j \wedge j \leq i' \}$).

Definition 1. a) A $\mathcal{R}e$-proposition P is discrete if and only if for all $t \in Re$ there is a $y \in Re$ such that $t <_{\mathcal{R}e} y$ and $(t,y) \cap P = \emptyset$ and there is a $z \in Re$ such that $z <_{\mathcal{R}e} t$ and $(z,t) \cap P = \emptyset$.

b) P is unbounded if and only if either $P = \emptyset$ or for all $t \in Re$ there is a $y \in Re$ such that $t <_{\mathcal{R}e} y$ and $y \in P$ and there is a $z \in Re$ such that $z <_{\mathcal{R}e} t$ and $z \in P$.

c) DU(P) if and only if either P is discrete and unbounded or Re - P is discrete and unbounded.

LEMMA 1. Let $P \subseteq Re$ and DU(P). Then for any formula η of TL_1

(1) $\{t \in Re: \eta$ is true at t in $\langle\langle P, \ldots\rangle, \mathcal{R}(TL)\rangle\}$ is one of the four sets $Re, \emptyset, P, Re - P$.

Proof. By induction on η. Let $P \subseteq Re$ and DU(P). For formulae η of TL_1 put

$\bar{\eta} = \{t \in Re: \eta$ is true at t in $\langle\langle P, \ldots \rangle$, $\mathcal{R}(TL_1)\rangle$. We will assume that P is discrete and unbounded. In case that Re - P is discrete and unbounded the argument is essentially the same.

1) $\eta = q_1 : \bar{\eta} = P$.

2) a) Let $\eta = \neg\zeta$, and assume (1) for ζ ; then (1) holds for η .

b) Similarly if $\eta = \zeta \wedge \theta$ and (1) holds for ζ and θ then (1) holds for η .

3) a) Let $\eta = S(\zeta, \theta)$, and assume (1) for ζ and θ. If $\bar{\zeta} = \emptyset$ or $\bar{\theta} = \emptyset$ or $\bar{\theta} = P$ then $\bar{\eta} = \emptyset$. If $\bar{\theta} = Re$ or $\bar{\theta} = Re - P$ and $\bar{\zeta} \neq \emptyset$, then $\bar{\eta} = Re$.

b) Similarly if $\eta = U(\zeta, \theta)$ then $\bar{\eta} = \emptyset$ or $\bar{\eta} = Re$.

<u>COROLLARY</u>1. Let O be a monadic tense, P a $\mathcal{R}$e-proposition such that DU(P). Then O(P) is one of the sets P, Re - P, Re, $\emptyset$.

<u>Proof</u>: In view of Corollary I.1, there is a formula η of TL_1 with no other variables than q_0 such that $O(P) = \{t \in Re: \eta$ is true at t in $\langle\langle P, \ldots \rangle, \mathcal{R}(TL_1)\rangle\}$. By the Lemma 1 the right hand side is equal to one of the four above mentioned sets.

<u>LEMMA 2</u>. Let $n \in \omega$ and let $P_1, \ldots, P_n$ be discrete unbounded subsets of Re such that for $i, j \leq n$, $i \neq j$ $P_i \cap P_j = \emptyset$. Then for each set P in the field of subsets

of Re generated by $P_1, \ldots, P_n$ there is a subset $\mathcal{P}(P)$ of $\{P_1, \ldots, P_n\}$ such that either $P = \bigcup \mathcal{P}(P)$ or $Re - P = \bigcup \mathcal{P}(P)$.

<u>Proof</u>: By induction on the principle of generation.

1) if $P = P_i$, then put $\mathcal{P}(P) = \{P_i\}$.

2) If $P = Re - P'$, put $\mathcal{P}(P) = \mathcal{P}(P')$.

3) Let $P = P' \cap P''$. If $P' = \bigcup \mathcal{P}(P')$ and $P'' = \bigcup \mathcal{P}(P'')$ then put $\mathcal{P}(P) = \mathcal{P}(P') \cap \mathcal{P}(P'')$.

If $Re - P' = \bigcup \mathcal{P}(P')$ and $Re - P'' = \bigcup \mathcal{P}(P'')$ then put $\mathcal{P}(P) = \mathcal{P}(P') \cup \mathcal{P}(P'')$.

If $Re - P' = \bigcup \mathcal{P}(P')$ and $P'' = \bigcup \mathcal{P}(P'')$ then put $\mathcal{P}(P) = \mathcal{P}(P'') - \mathcal{P}(P')$.

If $P' = \bigcup \mathcal{P}(P)$ and $Re - P'' = \bigcup \mathcal{P}(P)$ then put $\mathcal{P}(P) = \mathcal{P}(P') - \mathcal{P}(P'')$.

Let $\mathfrak{M}$ be the set of all monadic $\mathcal{R}$e-tenses together with the $\mathcal{R}$e-connectives AND and NOT. Let TM be a sentential language for $\mathfrak{M}$.

<u>LEMMA</u> 3. Let $P_1, \ldots, P_n$ be discrete unbounded Re-subsets of Re such that for $i, j \leq n$, $i \neq j$, $P_i \cap P_j = \emptyset$. Then for every formula η of TM $\{ t \in Re: \eta$ is true at t in $\langle\langle P_1, \ldots, P_n, \ldots\rangle, \mathcal{R}(TM)\rangle\}$ belongs to the field of subsets of Re generated by $P_1, \ldots, P_n$.

<u>Proof</u>: Immediate from Corollary 1 and Lemma 2.

Now let P_0 be the set of all even numbers, and P_1 the set of all odd numbers. Then P_0 and P_1 are discrete and unbounded and $P_0 \cap P_1 = \emptyset$. Since every member of the field of subsets of Re generated by P_0 and P_1 either contains all real numbers which are not integers, or none such, it follows from lemma 3 that for every formula η of TM and real numbers t, t' which are not integers

(1) η is true at t in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TM)\rangle$ iff η is true at t' in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TM)\rangle$.

On the other hand the formula

(2) $S(q_0, \neg q_1)$ of TL_2 is true at +1/2 in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TL_2)\rangle$ but not at - 1/2 in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TL_2)\rangle$.

If SINCE were expressible in terms of $\mathfrak{M}$ then there would be a formula η of TM such that for all $t \in$ Re η is true at t in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TM)\rangle$ iff $S(q_0, \neg q_1)$ is true at t in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TL_2)\rangle$. But this is impossible in view of (1) and (2). Therefore SINCE is not expressible in terms of monadic operators. Similarly since the formula $U(q_0, \neg q_1)$ of TL_2 is true at - 1/2 in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TL_2)\rangle$ but not at + 1/2 in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TL_2)\rangle$, UNTIL is not $\mathcal{R}$e-expressible in terms of $\mathfrak{M}$.

Theorem 1 states the inexpressibility of SINCE and UNTIL in terms of monadic tenses and sentential connectives, on the assumption that time is like the real numbers. However, from the proof of Theorem 1 one easily sees that the same fact is true if time is like any other particular dense linear order with more than one element. The situation is different however if time is discrete.

Definition 2. PAST_1 is the $\mathcal{Y}n$-tense defined by the formula

$$(\exists t_1)(t_1 < t_0 \wedge P_0(t_1) \wedge (\forall t_2)(t_1 < t_2 < t_0 \rightarrow P_0(t_2) \wedge \neg P_0(t_2))).$$

FUTURE_1 is the $\mathcal{Y}n$-tense defined by

$$(\exists t_1)(t_0 < t_1 \wedge P_0(t_1) \wedge (\forall t_2)(t_0 < t_2 < t_1 \rightarrow P_0(t_2) \wedge \neg P_0(t_2))).$$

ISOLP is the $\mathcal{Y}n$-tense defined by

$$(\exists t_1)(t_1 < t_0 \wedge P_0(t_1) \wedge (\exists t_2)(t_2 < t_1 \wedge \neg P_0(t_2) \wedge$$
$$(\forall t_3)(t_2 < t_3 < t_1 \rightarrow P_0(t_3) \wedge \neg P_0(t_3))) \wedge (\forall t_4)(t_1 < t_4 < t_0 \rightarrow \neg P_0(t_4))) .$$

ISOLF is the $\mathcal{Y}n$-tense defined by

$$(\exists t_1)(t_0 < t_1 \wedge P_0(t_1) \wedge (\exists t_2)(t_1 < t_2 \wedge \neg P_0(t_2) \wedge$$
$$(\forall t_3)(t_1 < t_3 < t_2 \rightarrow P_0(t_3) \wedge \neg P_0(t_3))) \wedge (\forall t_4)(t_0 < t_4 < t_1 \rightarrow \neg P_0(t_4))).$$

Let $\mathcal{S}$ be the set of $\mathcal{Y}n$-tenses $\{$ ISOLP, ISOLF, PAST_1, FUTURE_1 NOT, AND $\}$.

THEOREM 2. The $\mathcal{Y}n$-tenses SINCE and UNTIL are $\mathcal{Y}n$-expressible in terms of $\mathcal{S}$.

Proof: Let TS be the sentential language for $\mathcal{S}$ such that $TS(S_1)$ = ISOLP, $TS(U_1)$ = ISOLF, $TS(P_1) = \mathrm{PAST}_1$, $TS(F_1) = \mathrm{FUTURE}_1$, $TS(\neg)$ = NOT, $TS(\wedge)$ = AND.

Let η_1 be the formula

$(q_0 \wedge F_1 \neg q_0 \wedge F_1 q_1) \vee (\neg q_0 \wedge \neg q_1 \wedge F_1 \neg q_0 \wedge F_1 F_1 \neg q_0) \vee$

$P_1(\neg q_0 \wedge \neg q_1 \wedge F_1 \neg q_0 \wedge F_1 F_1 \neg q_0)$ of TS,

and η_2 the formula

$(q_0 \wedge P_1 \neg q_0 \wedge P_1 q_1) \vee (\neg q_0 \wedge \neg q_1 \wedge P_1 \neg q_0 \wedge P_1 P_1 \neg q_0) \vee$

$F_1(\neg q_0 \wedge \neg q_1 \wedge P_1 \neg q_0 \wedge P_1 P_1 \neg q_0)$.

Let $\zeta_1 = [F_1 q_0 \vee (F_1 q_1 \wedge (F_1 F_1 q_0 \vee F_1 F_1 q_1))] \wedge [(F_1 q_0 \vee$

$(F_1 q_1 \wedge F_1 F_1 q_0)) \vee (F_1(\neg q_0 \wedge q_1) \wedge F_1 F_1(\neg q_0 \wedge q_1) \wedge U_1 \eta_1)]$.

Let $\zeta_2 = [P_1 q_0 \vee (P_1 q_1 \wedge (P_1 P_1 q_0 \vee P_1 P_1 q_1))] \wedge [(P_1 q_0 \vee$

$(P_1 q_1 \wedge P_1 P_1 q_0)) \vee (P_1(\neg q_0 \wedge q_1) \wedge P_1 P_1(\neg q_0 \wedge q_1) \wedge S_1 \eta_2)]$.

Then

(1) The $\mathcal{O}$n-tense UNTIL is the $\mathcal{O}$n-tense expressed by ζ_1,

and

(2) The $\mathcal{O}$n-tense SINCE is the $\mathcal{O}$n-tense expressed by ζ_2.

We will prove only (1). The proof of (2) is the same.

Let us state explicitly properties of the operators P_1, F_1, S_1, U_1 which should be obvious from the formulae that characterize the corresponding $\mathcal{O}$n-tenses:

For any interpretation $\mathfrak{A}$ for TS relative to $\mathcal{O}$n, η a formula of TS and $i \in In$

$P_1\eta$ is true at i in $\mathfrak{A}$ iff η is true at $i-1$ in $\mathfrak{A}$.

$F_1\eta$ is true at i in $\mathfrak{A}$ iff η is true at $i+1$ in $\mathfrak{A}$.

$S_1\eta$ is true at i in $\mathfrak{A}$ iff there is a $j < i$[1] such that η is true at j in $\mathfrak{A}$ and η is false at $j-1$ in $\mathfrak{A}$ and for all k such that $j < k < i$, η is false at k in $\mathfrak{A}$.

$U_1\eta$ is true at i in $\mathfrak{A}$ iff there is a $j > i$ such that η is true at j in $\mathfrak{A}$ and η is false at $j+1$ in $\mathfrak{A}$ and for all k such that $i < k < j$, η is false at k in $\mathfrak{A}$.

To show (1) it suffices to demonstrate that if $P_0, P_1 \subseteq$ In and $t \in$ In, then

(3) ζ_1 is true at t in $\langle\langle P_0, P_1, \ldots\rangle, \mathcal{R}(TS)\rangle$ iff there is a $t' > t$ such that $t' \in P_0$ and for all t'' such that $t < t'' < t'$, $t'' \in P_1$. We will show this for $t = 0$. The argument will be seen to apply to arbitrary t.

First assume the right hand side of (3). It is clear that the first conjunct of ζ_1 is true at 0. To show that the second conjunct (call it ζ_1') is true at 0, we argue as follows: Let i be the smallest integer > 0

[1]Henceforth we will write '$<$' for '$<_{\mathfrak{I}n}$' and '$j > i$' for '$i < j$.'

such that $i \in P_0$ and for all j such that $0 < j < i$, $j \in P_1$. If $i = 1$ or $i = 2$, then clearly ζ_1' is true at 0 in $\langle\langle P_0, P_1, \ldots \rangle, \mathcal{R}(TS)\rangle$ since the first or second disjunct is true at 0 in $\langle\langle P_0, P_1, \ldots \rangle, \mathcal{R}(TS)\rangle$, respectively.

So suppose that $i \geqslant 3$. We show that the third disjunct of ζ_1 is true at 0 (in $\langle\langle P_0, P_1 \ldots \rangle, \mathcal{R}(TS)\rangle$). Certainly its first two conjuncts are true at 0; for i is the smallest integer > 0 such that $i \in P_0$ and for all j such that $0 < j < i$, $j \in P_1$.

We further show that i is the smallest integer > 0 at which η_1 is true, and that η_1 is false at $i + 1$. η_1 is true at i since its first disjunct is true at i. Further each disjunct of η_1 is false at $i + 1$; for the first two contain the conjunct $F_1 \neg q_0$ and so could be true at $i + 1$ only if q_0 were false at i. The third could be true at $i + 1$ only if the second were true at $i + 2$, which is again impossible for it contains the conjunct $F_1 F_1 \neg q_0$. Further for all j such that $0 < j < i$ η_1 is false at j. For the first disjunct of η_1 is false at such j since it contains the conjunct q_0; the second is false because of the conjunct $\neg q_1$; the third is false at $j < i - 1$ because of the conjunct $\neg q_1$ and at $i - 1$ because of the conjunct $\neg q_0$.

Now assume that the right hand side of (3) fails with 0 for t. We want to show that ζ_1 is not true at 0.

Let us assume that the first conjunct of ζ_1 is true at 0. Then since the right hand side of (3) fails, $1 \in P_1$, $2 \in P_1$, $1 \notin P_0$, $2 \notin P_0$. We want to show that ζ_1', the second conjunct of ζ_1, is false at 0. Clearly the first two disjuncts of ζ_1' are false at 0. We show that $U_1 \eta_1$ is false at 0, so that also the third disjunct is false at 0. Therefor we argue as follows: Since the right hand side of (3) fails, either

a) there is an $i > 0$ such that $i \notin P_1$, $i \notin P_0$ and for all j such that $0 < j < i$, $j \in P_1$, $j \notin P_0$ or else

b) there is no $i > 0$ such that $i \in P_0$.

a) Since $i \notin P_1$, $i \geqslant 3$; we will show that η_1 is true at $i - 1$ and at i and at no j such that $0 < j < i - 1$. η_1 is true at i since its second disjunct is true at i. Similarly the third disjunct of η_1 is true at $i - 1$, and so η_1 is true at $i - 1$. Now let j be a positive integer $< i - 1$. The first disjunct of η_1 is false at j since it contains q_0; the second is false at j since it contains $\neg q_1$ and the third since it contains $\neg q_1$--whereas q_1 is true at $j + 1$. It follows that $U_1 \eta_1$ is false at 0.

b) The first disjunct of η_1 is false at all positive integers. Further the second and third are both false at 1. If η_1 is true at no positive integer then $U_1 \eta_1$ is false at 0. So suppose that η_1 is true at some positive integer. Let k be the smallest such. Then the second disjunct of η_1 is not true at k, for in that case the third would be true at k - 1, -- which is positive since $k > 1$. So the third disjunct is true at k but then the second disjunct is true at k + 1. So η_1 is true at k and at k + 1 and at no integer between 0 and k. Therefore $U_1 \eta_1$ is false at 0. Q.E.D.

The $\mathfrak{A}$n-tenses ISOLP and ISOLF are not very simple. And so it would be nice if in Theorem 2 we could replace the set $\mathcal{S}^0$ by a set of simpler or more familiar tenses, e.g., by the set $\mathcal{S}' = \{$ PAST$_1$, FUTURE$_1$, PAST, FUTURE, NOT, AND $\}$. That this is not the case is shown by the following

<u>THEOREM 3</u>. ISOLP and ISOLF are not $\mathfrak{A}$n-expressible in terms of $\mathcal{S}'$.

<u>Proof</u>: Let TS' be the sentential language for the set mentioned in the theorem, such that TS'(P) = PAST, TS'(F) = FUTURE, TS'(P_1) = PAST$_1$, TS'(F_1) = FUTURE$_1$, TS'($\neg$) = NOT, TS'($\wedge$) = AND.

Let $P_0 = \{n^2 : n \in \omega\} \cup \{(2n)^2 + 1 : n \in \omega\}$

Then $\text{ISOLF}(P_0) = \{ i \in \omega : \bigvee_{n \in \omega} ((2n)^2 \leq i < (2n+1)^2\}$ and $\text{ISOLP}(P_0) = \{1\} \cup \{ i \in \omega : \bigvee_{n \in \omega} ((2n+3)^2 < i \leq (2n+4)^2 + 1)\}$

On the other hand we will show that for each formula η of TS' there are natural numbers n_η, m_η such that if $n, n' > n_\eta$ and $n^2 + m_\eta < i < (n+1)^2 - m_\eta$ and $(n')^2 + m_\eta < i' < (n'+1)^2 - m_\eta$, then η is true at i in $\langle\langle P_0, \ldots \rangle, \mathcal{R}(TS')\rangle$ iff η is true at i' in

$\langle\langle P_0, \ldots \rangle, \mathcal{R}(TS') \rangle$. For arbitrary formula η of TS' let $\bar{\eta} = \{ j \in In: \eta$ is true at j in $\langle\langle P_0, \ldots \rangle, \mathcal{R}(TS') \rangle \}$. We show by induction on η :

(1) There are integers n_η , m_η , such that $m_\eta < n_\eta^2$ and

(a) if $n, n' > n_\eta$ and $n^2 + m_\eta < i < (n+1)^2 - m_\eta$ and $n'^2 + m_\eta < i' < (n'+1)^2 - m_\eta$ then $i \in \bar{\eta}$ iff $i' \in \bar{\eta}$.

(b) if $n, n' > n_\eta$ and $-m_\eta \leqslant k \leqslant m_\eta$ then $n^2 + k \in \bar{\eta}$ iff $n'^2 + k \in \bar{\eta}$.

(i) Let $\eta = q_0$. Take $n_\eta = 2$, $m_\eta = 2$. Then (1) clearly holds.

(ii) Let $\eta = \neg \zeta$, and assume (1) for ζ . Clearly if ζ , n_ζ , m_ζ satisfy (1.a) and (1.b), then η, n_ζ , m_ζ satisfy (1.a) and (1.b). It follows that (1) holds for η .

(iii) Let $\eta = \zeta \wedge \theta$ and assume (1) for ζ and for θ . Let $n_\eta = \max(n_\zeta, n_\theta)$ and $m_\eta = \max(m_\zeta, m_\theta)$. We observe that for any formula ρ of TS'

(2) if ρ, n_ρ, m_ρ satisfy (1.a) ((1.b)) and $n \geqslant n_\rho$; $m \geqslant m_\rho$ then ρ , n, m satisfy (1.a) ((1.b)).

So ζ , n_η , m_η satisfy (1.a) and (1.b) and θ , n_η, m_η satisfy (1.a) and (1.b). One then easily concludes that $\zeta \wedge \theta$, n_η , m_η satisfy (1.a) and (1.b).

(iv) Let $\eta = P_1 \zeta$ and assume (1) for ζ . Put $n_\eta = n_\zeta + 1$, $m_\eta = m_\zeta + 1$. Clearly $m_\eta < (n_\eta)^2$, and by (2) ζ , n_η ; m_ζ satisfy (1.a) and (1.b).

(a) Let $k, k' > n_\eta$ and $k^2 + m_\eta < i < (k+1)^2 - m_\eta$, $k'^2 + m_\eta < i' < (k'+1)^2 - m_\eta$. Then $k^2 + m_\zeta < i - 1 < (k+1)^2 - m_\zeta$ and $k'^2 + m_\zeta < i' - 1 < (k'+1)^2 - m_\zeta$. Therefore $i - 1 \in \bar{\zeta}$ iff $i' - 1 \in \bar{\zeta}$; therefore $i \in \bar{\eta}$ iff $i' \in \bar{\eta}$.

(b) Let $0 < j < m_\eta$; then $k^2 \pm j \in \bar{\eta}$ iff $(k^2 \pm j) - 1 \in \bar{\zeta}$ iff $(k'^2 \pm j) - 1 \in \bar{\zeta}$ iff $k'^2 \pm j \in \bar{\eta}$.

(v) Let $\eta = F_1 \zeta$ and assume (1) for ζ. Similar to (iv).

(vi) Let $\eta = P \zeta$ and assume (1) for ζ. Let $n_\eta = (n_\zeta + 2)$, $m_\eta = 0$.

(a) Suppose there is an $i \leq n^2_\eta$ such that $i \in \bar{\zeta}$. Then for all $j \geqslant n^2_\eta$, $j \in \bar{\eta}$ and (1.a) and (1.b) follow immediately for η, n_η, m_η.

(b) Suppose there is no $i \leq n^2_\eta$ such that $i \in \bar{\zeta}$. Then in particular for all i such that $(n_\zeta + 1)^2 \leqslant i < (n_\zeta + 2)^2$, $i \notin \bar{\zeta}$. Then in view of the fact that ζ, n_ζ, m_ζ satisfy (1.a) and (1.b), for all $i \geqslant (n_\zeta + 2)^2$ $i \notin \zeta$. Thus for all $i \geqslant n_\eta$ $i \notin \bar{\eta}$ and so η, n_η, m_η satisfy (1.a) and (1.b).

(vii) $\eta = F \zeta$ and assume (1) for ζ. Put $n_\eta = (n_\zeta + 2)$, $m_\eta = 0$.

(a) Suppose there is no $i \leq n^2_\eta$, such that $i \in \bar{\zeta}$; then, since ζ, n_ζ, m_ζ satisfy (1.a) and (1.b), there will be no $i \geqslant n^2_\eta$ either, such that $i \in \bar{\zeta}$;

and so for all $i \geqslant n_\eta^2$, $i \notin \bar{\eta}$; it follows that η, n_η, m_η satisfy (1.a) and (1.b).

(b) Suppose there is an i such that $(n_\zeta + 1)^2 < i < (n_\zeta + 2)^2$, and $i \in \bar{\zeta}$. Let $(n_\zeta + 1)^2 + j$ be such an i. Then for all $k \geqslant n_\eta$, $k^2 + j \in \bar{\zeta}$. Therefore for all $i > n_\eta^2$, $i \in \overline{F\zeta}$. Consequently η, n_η, m_η satisfy (1.a) and (1.b). Q. E. D.

It is also impossible to replace in Theorem 2 the set $\mathcal{S}$ by the set $\mathcal{S}'' = \{$ ISOLP, ISOLF, PAST, FUTURE, NOT, AND $\}$. This follows from

<u>THEOREM 4</u>. $PAST_1$ and $FUTURE_1$ are not $\mathcal{S}$n-expressible in terms of $\mathcal{S}''$.

<u>Proof</u>. Let $P_0 = \{ i \in In: i = 2 (\text{mod } 4) \}$. Then $PAST_1(P_0) = \{ i \in In: i = 3 (\text{mod } 4) \}$ and $FUTURE_1(P_0) = \{ i \in In: i = 1 (\text{mod } 4) \}$. On the other hand we have

(1) for every formula η of a sentential language TL for $\mathcal{S}''$ the set $\{ i \in In: \eta$ is true at i in $\langle\langle P_0, \ldots\rangle, \mathcal{R}(TL)\rangle \}$ is equal to one of the sets:

(i) In

(ii) ϕ

(iii) $\{ i \in In: i = 2 (\text{mod } 4) \}$

(iv) $\{ i \in In: i \neq 2 (\text{mod } 4) \}$

(v) $\{ i \in In: i = 0 (\text{mod } 4) \}$

(vi) $\{ i \in In: i \neq 0 (\text{mod } 2) \}$

(vii) $\{ i \in In: i = 0 (\text{mod } 2) \}$

(viii) $\{ i \in In: i \neq 0 (\text{mod } 2) \}$

One easily verifies that:

(a) (i) - (viii) form a field of subsets of In.

(b) ISOLF ((i)) = ISOLP ((i)) = (ii)

ISOLF ((ii)) = ISOLP ((ii)) = (ii)

ISOLF ((iii)) = ISOLP ((iii)) = (i)

ISOLF ((iv)) = ISOLP ((iv)) = (v)

ISOLF ((v)) = ISOLP ((v)) = (i)

ISOLF ((vi)) = ISOLP ((vi)) = (iii)

ISOLF ((vii)) = ISOLP ((vii)) = (i)

ISOLF ((viii)) = ISOLP ((viii)) = (i)

(where the roman numerals stand for the sets indexed by them above.)

(c) FUTURE ($\emptyset$) = PAST ($\emptyset$) = $\emptyset$; and for any other set Q in the field consisting of the sets (i) - (viii), FUTURE (Q) = PAST (Q) = In. (a), (b) and (c) imply (1). Q.E.D.

We will now show that

THEOREM 5. There is a $\mathcal{R}$a - tense which is not $\mathcal{R}$a - expressible in terms of the $\mathcal{R}$a - tenses NOT, AND, SINCE and UNTIL.

Proof: Let us understand, in this proof, by (r,s), where r and s are real numbers, the set of rational numbers between r and s. We consider the following two interpretations for TL_1:

$$\mathfrak{A} = \langle\langle Q, \ldots\rangle, \mathcal{R}(TL_1)\rangle \text{ and}$$

$$\mathfrak{A}' = \langle\langle Q', \ldots\rangle, \mathcal{R}(TL_1)\rangle ,$$

where Q and Q' are the sets of rational numbers:

$$Q = \bigcup_{i \in In} (2i + \sqrt{2},\ 2i + 1 + \sqrt{2})$$

$$Q' = \bigcup_{\substack{i \in In \\ i \neq 0}} (2i + \sqrt{2},\ 2i + 1 + \sqrt{2}) \cup \bigcup_{\substack{i \in In \\ i \geqslant 0}} \left(\sqrt{2} + \frac{1}{2i + 1},\ \sqrt{2} + \frac{1}{2i + 2}\right)$$

Then the following tense defining formula is satisfied by the rational number 3 in $\mathfrak{A}$, but not by 3 in $\mathfrak{A}'$:

(1)..$(\exists t_1)(t_1 < t_o \wedge P_o(t_1) \wedge (\forall t_2)(t_1 < t_2 \wedge t_2 < t_o \rightarrow \{(P_o(t_2) \wedge (\exists t_3)(t_2 < t_3 < t_o \wedge (\forall t_4)(t_2 < t_4 < t_3 \rightarrow P_o(t_4)))\} \vee \{\neg P_o(t_2) \wedge (\exists t_3)(t_1 < t_3 < t_2 \wedge (\forall t_4)(t_3 < t_4 < t_3 \rightarrow \neg P_o\ t_4)))\}))$.

On the other hand it is the case for every formula η of TL_1 which contains no other sentential constants than q_o that

(2)..For any two rational numbers r and s

(a) If ($r \in Q$ iff $s \in Q$) then η is true at r in $\mathfrak{A}$ iff η is true at s in $\mathfrak{A}$.

(b) If ($r \in Q'$ iff $s \in Q'$) then η is true at r in $\mathfrak{A}'$ iff η is true at s in $\mathfrak{A}'$.

(c) If ($r \in Q$ iff $s \in Q'$) then η is true at r in $\mathfrak{A}$ iff η is true at s in $\mathfrak{A}'$.

The proof of (2) is by induction on η. Let us put for arbitrary formulae of TL_1:

$$\overline{\eta_{\mathfrak{A}}} = \{ r \in Ra: \eta \text{ is true at } r \text{ in } \mathfrak{A} \}$$

$$\overline{\eta_{\mathfrak{A}'}} = \{ r \in Ra: \eta \text{ is true at } r \text{ in } \mathfrak{A}' \}$$

The induction goes as follows:

(i) If $\eta = q_0$ then (a), (b) and (c) are obvious.

(ii) If $\eta = \neg \zeta$ or $\eta = \zeta \wedge \theta$ then the induction step is just as in previous proofs of this chapter; we leave the argument to the reader.

(iii) $\eta = S(\zeta, \theta)$.

Show (a). By induction hypothesis ζ and θ are are among the sets Q, Ra - Q, Ra and $\emptyset$. Let r and s be rational numbers such that $r \in Q$ iff $s \in Q$. First suppose that $r \in Q$ and $s \in Q$. If $\overline{\theta}_{\mathfrak{A}} = Ra - Q$ or $\overline{\theta}_{\mathfrak{A}} = \emptyset$, then η is not true at r in $\mathfrak{A}$ and η is not true at s in $\mathfrak{A}$. Suppose that $\overline{\theta}_{\mathfrak{A}} = Ra$. If $\overline{\zeta}_{\mathfrak{A}} = \emptyset$, then η is not true at r in $\mathfrak{A}$ and η is not true at s in $\mathfrak{A}$. Otherwise η is true at both r and s in $\mathfrak{A}$. Now suppose that $\overline{\theta}_{\mathfrak{A}} = Q$. Then, if $\overline{\zeta}_{\mathfrak{A}} = \emptyset$ or $\overline{\zeta}_{\mathfrak{A}} = Ra - Q$, then η is true neither at r nor at s in $\mathfrak{A}$; if $\overline{\zeta}_{\mathfrak{A}} = Ra$ or $\overline{\zeta}_{\mathfrak{A}} = Q$ then η is true at both r and s in $\mathfrak{A}$. This completes the proof of (a) for the case where r and s belong to Q. The case in which they do not belong to Q is handled in the same way.

Show (b). The argument is analogous to the argument given under (a).

Show (c). From the induction hypothesis it follows that

$$\overline{\zeta}_{\mathfrak{A}} = Q \text{ iff } \overline{\zeta}_{\mathfrak{A}'} = Q'$$
$$\overline{\zeta}_{\mathfrak{A}} = Ra - Q \text{ iff } \overline{\zeta}_{\mathfrak{A}'} = Ra - Q'$$

$$\overline{\zeta}_{\mathfrak{A}} = Ra \text{ iff } \overline{\zeta}_{\mathfrak{A}'} = Ra$$
$$\overline{\zeta}_{\mathfrak{A}} = \emptyset \text{ iff } \overline{\zeta}_{\mathfrak{A}'} = \emptyset$$

and that the same conditions hold with θ instead of ζ. Using these facts we can show, by an argument similar to the one given under (a), that if $r \in Q$ iff $s \in Q$ then η is true at r in $\mathfrak{A}$ iff η is true at s in $\mathfrak{A}'$.

(iv) $\eta = U(\zeta, \theta)$. The argument is the same as under (iii). This concludes the proof of (2) for all formulae of TL_1 with no other sentential constants than q_0. Thus for any 1 - place schema η of TL_1 the 1 - place $\mathcal{R}a$ - tense f $\mathcal{R}a$ - expressed by η is such that $3 \in f_{\mathcal{R}a}(Q)$ iff $3 \in f_{\mathcal{R}a}(Q')$ and so f is not the $\mathcal{R}a$ - tense defined by (1). Q.E.D.

We conclude this chapter with the proof of the assertion, made in Chapter II, that if there are infinitely many moments of time then the present progressive tense is not expressible in terms of PAST, FUTURE and sentential connectives (see pp. 18,19). Let us first remark that the question of how to define the present progressive is not completely unambiguous. At first sight the most natural definition seems to be by means of the tense defining formula:

(1) $$P_0(t_0) \wedge (\exists t_1)(t_1 < t_0 \wedge (\forall t_2)(t_1 < t_2 < t_0 \rightarrow P_0(t_2))) \wedge (\exists t_3)(t_0 < t_3 \wedge (\forall t_4)(t_0 < t_4 < t_3 \rightarrow P_0(t_4)))$$

The plausibility of this definition depends, however, strongly on the assumption that time is dense. (Indeed, the existence of the progressive tense in certain natural languages may be evidence that the speakers of those languages think of time as being dense). But if time is discrete then, according to the definition suggested above, the present progressive tense will coincide with the simple present, and thus be expressible in terms of any set of tenses, namely by the formula 'q_0'. Therefore it is better to adopt

$$(2)\quad P_0(t_0) \wedge (\exists t_1)(t_1 < t_0 \wedge P_0(t_1) \wedge (\forall t_2)(t_1 < t_2 < t_0 \rightarrow P_0(t_2))) \wedge (\exists t_3)(t_0 < t_3 \wedge P_0(t_3) \wedge (\forall t_4)(t_0 < t_4 < t_3 \rightarrow P_0(t_4)))$$

as the tense defining formula for the present progressive. The tense defined by (2) will coincide with the one defined by (1) when time is dense and will not reduce to the simple present when time is discrete. On the basis of this latter definition we can indeed show that the present progressive is not expressible in terms of PAST and FUTURE and sentential connectives if only time has infinitely many points. As before we represent the structure of time as $\mathcal{T} = \langle T, < \rangle$.

We first assume that time has no endpoints. Let $\{t_i : i \in \operatorname{In}\}$ be a subset of T such that for $i < j$, $t_i < t_j$. Let

$P_0 = [t_{-2}, t_2] \cup \{t_{2i} : i \in In\}$. Then, if PR is the tense defined by (2), $PR(P_0) = (t_{-2}, t_2)$. On the other hand the field of subsets of T generated by P_0 consists of the sets T, ϕ, P_0, $T-P_0$; and further PAST (P_0) = PAST $(T-P_0)$ = FUTURE (P_0) = FUTURE $(T-P_0)$ = PAST (T) = FUTURE (T) = T and PAST (ϕ) = FUTURE (ϕ) = ϕ. It follows that PR is not expressible in terms of the tenses PAST, FUTURE and sentential connectives.

Let us now assume that time has a first moment but no last moment. Let t_0 be the first moment of $\mathcal{T}$ (i.e. the element of T such that for all $t \in T$ $t = t_0$ or $t_0 < t$.) Let $\{t_i : i \in \omega\}$ be a subset of T such that for $i < j$, $t_i < t_j$. For $i \in \omega$ let $P_i = \{t_{2j} : j \in \omega\} \cup [t_{i+2}, t_{i+4}]$: Let TL be the sentential language $\{\langle P, \text{PAST}\rangle, \langle F, \text{FUTURE}\rangle, \langle \neg, \text{NOT}\rangle, \langle \wedge, \text{AND}\rangle\}$ for the set $\mathcal{S} = \{$PAST, FUTURE, NOT, AND$\}$.

We assign to each formula η of TL which contains no other variables than q_0 a depth d (η) as follows:

$$d(q_0) = 0$$
$$d(\neg\eta) = d(\eta)$$
$$d(\eta \wedge \zeta) = \max(d(\eta), d(\zeta))$$
$$d(P\eta) = d(\eta) + 2$$
$$d(F\eta) = d(\eta)$$

For $i \in \omega$ and η a formula of TL let $\bar{\eta}_i = \{t \in T : \eta$ is true at t in $\langle\langle P_i, \ldots\rangle, \mathcal{R}(TL)\rangle\}$.

For $t \in T$ let $F(t) = \{t' \in T: t < t'\}$. We show by induction on η:

(1) If $d(\eta) \leq i$ then $F(t_{d(\eta)}) \cap \bar{\eta}_i$ is one of the following sets:

$$F(t_{d(\eta)}) \cap P_1,\ F(t_{d(\eta)}) - P_1,\ F(t_{d(\eta)}),\ \phi.$$

Proof:

(i) $\eta = q_0$; then $\bar{\eta}_i = P_i$ for all $i \geqslant d(\eta)$ (which is 0 in this case) and (1) follows immediately for η.

(ii) $\eta = \neg\zeta$; assume (1) for ζ. Then (1) clearly holds for η.

(iii) $\eta = \zeta \wedge \theta$; assume (1) for ζ and for θ. Since $d(\eta) \geqslant d(\zeta)$ and $d(\eta) \geqslant d(\theta)$ it follows from the induction hypothesis that for $i \geqslant d(\eta)$ $F(t_{d(\eta)}) \cap \bar{\eta}_i$ is one of the four sets mentioned in (1).

(iv) $\eta = F\zeta$; assume (1) for ζ. Let $i \geqslant d(\eta)$. If there is a $j > d(\zeta)$ such that $t_j \in \bar{\zeta}_i$, then for every k there is an $m > k$ such that $m \in \bar{\zeta}_i$, and so $\bar{\eta}_i = T$. If there is no such j then $F(t_{d(\eta)}) \cap \bar{\eta}_i = \phi$.

(v) $\eta = P\zeta$; assume (1) for ζ. Let $i \geqslant d(\eta)$. If there is a $j \geqslant d(\zeta)$ such that $t_j \in \bar{\zeta}_i$ then there is a $j < d(\zeta) + 2$ such that $t_j \in \bar{\zeta}_i$. Therefore $F(t_{d(\eta)}) \cap \bar{\eta}_i = F(t_{d(\eta)})$. If there is no $j \geqslant d(\zeta)$ such that $t_j \in \bar{\zeta}_i$ then $F(t_{d(\eta)}) \cap \bar{\eta}_i$ will be either $F(t_{d(\eta)})$ or ϕ

depending on whether there is a $j < d(\zeta)$ such that $t_j \in \bar{\zeta}_i$, or not. From (1) it is clear that PR is not expressible in terms of $\mathscr{S}$. For let η be any formula of TL with no other variables than q_0. Let $i \geqslant d(\eta)$. Then $PR(P_i) = (t_{i+2}, t_{i+4})$. This set is included in $F(t_{d(\eta)})$. So $PR(P_i) \cap F(t_{d(\eta)}) = (t_{i+2}, t_{i+4})$. But by (1) this set does not coincide with $F(t_{d(\eta)}) \cap \{t: \eta \text{ is true at } t \text{ in } \langle\langle P_i, \ldots\rangle, \mathcal{R}(TL)\rangle\}$.

A similar proof shows the inexpressibility of PR in terms of $\mathscr{S}$ in case time has a last, but not a first point. Indeed in this case there will be a subset $\{t_i: i \in \omega\}$ of T such that for $i < j$, $t_j < t_i$. Putting for $i \in \omega$ $P_i = \{t_{2i}: i \in \omega\} \cup [t_{i+4}, t_{i+2}]$; for η of TL, $\bar{\eta}_i$ as before; letting $P(t) = \{t' \in T: t' < t\}$ and defining the depth' of η by:

$$d'(q_0) = 0$$
$$d'(\neg\eta) = d'(\eta)$$
$$d'(\eta \wedge \zeta) = \max(d'(\eta), d'(\zeta))$$
$$d'(P\eta) = d'(\eta)$$
$$d'(F\eta) = d'(\eta) + 2;$$

we can show that:

(2) If $i \geqslant d'(\eta)$ then $P(t_{d'(\eta)}) \cap \bar{\eta}_i$ is one of the sets:

$$P(t_{d'(\eta)}) \cap P_i,\ P(t_{d'(\eta)}) - P_i,\ P(t_{d'(\eta)}),\ \phi .$$

Now assume that time has both a first and a last moment. In that case there will be subsets $\{t_i: i \in \omega\}$ and $\{t_i': i \in \omega\}$ of T such that for $i, j \in \omega$ $t_i <_T t_j'$ and if $i < j$ then $t_i <_T t_j$ and $t_j' <_T t_i'$. Let $P_{ij} = \{t_{2k}: k \in \omega\} \cup \{t_{2k}': k \in \omega\} \cup [t_{i+2}, t_{i+4}] \cup [t_{j+4}', t_{j+2}']$. Then $PR(P_{ij}) = (t_{i+2}, t_{i+4}) \cup (t_{j+4}', t_{j+2}')$, and so if $m \geqslant i$ and $n \geqslant j$, then $F(t_m) \cap P(t_n') \cap PR(P_{ij}) = (t_{i+2}, t_{i+4}) \cup (t_{j+4}', t_{j+2}')$. On the other hand, if η is a formula of TL with no other variables than q_0, $d(\eta) \geqslant i$ and $d'(\eta) \geqslant j$, then, putting $\overline{\eta}_{ij} = \{t \in T: \eta$ is true at t in $\langle\langle P_{ij}, \ldots\rangle, \mathcal{R}(TL)\rangle\}$, $F(t_{d(\eta)}) \cap P(t_{d'(\eta)}') \cap \overline{\eta}_{ij}$ is one of the four sets: $F(t_{d(\eta)}) \cap P(t_{d(\eta)}') \cap P_{ij}$, $F(t_{d(\eta)}) \cap P(t_{d'(\eta)}') - P_{ij}$, $F(t_{d(\eta)}) \cap P(t_{d'(\eta)}')$, ϕ. Thus again PR is not expressible in terms of $\mathcal{S}''$.

BIBLIOGRAPHY

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Cocchiarella, Nino B. "Tense and Modal Logic: A Study in the Topology of Temporal Reference." Unpublished Doctoral dissertation, The University of California, Los Angeles, 1966.

____________. "A Completeness Theorem for Tense Logic," The Journal of Symbolic Logic, Vol. 31, No. 4, December 1966, p. 689.

Ehrenfeucht, Andrzej. "Decidability of the Theory of the Linear Order Relation," A.M.S. Notices, Vol. 6, No. 3, Issue 38, June 1959, pp. 556.

Läuchli, Hans, and J. Leonard. "On the Elementary Theory of Linear Order," Fundamenta Mathematica, Vol. 53, 1966, pp. 109-115.

Leonard, J. See Läuchli and Leonard.

Prior, Arthur N. Past, Present and Future. Oxford: The Clarendon Press, 1967. 217 pp.