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A Calculus for First Order Discourse Representation Structures

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Abstract. This paper presents a sound and complete proof system for the first order fragment of Discourse Representation Theory. Since the inferences that human language users draw from the verbal input they receive for the most transcend the capacities of such a system, it can be no more than a basis on which more powerful systems, which are capable of producing those inferences, may then be built. Nevertheless, even within the general setting of first order logic the structure of the "formulas" of DRS-languages, i.e. of the Discourse Representation Structures suggest for the components of such a system inference rules that differ somewhat from those usually found in proof systems for the first order predicate calculus and which are, we believe, more in keeping with inference patterns that are actually employed in common sense reasoning.

This is why we have decided to publish the present exercise, in spite of the fact that it is not one for which a great deal of originality could be claimed. In fact, it could be argued that the problem addressed in this paper was solved when Gödel first established the completeness of the system of Principia Mathematica for first order logic. For the DRS-languages we consider here are straightforwardly intertranslatable with standard formulations of the predicate calculus; in fact the translations are so straightforward that any sound and complete proof system for first order logic can be used as a sound and complete proof system for DRSs: simply translate the DRSs into formulas of predicate logic and then proceed as usual. As a matter of fact, this is how one has chosen to proceed in some implementations of DRT, which involve inferencing as well as semantic representation; an example is the Lex system developed jointly by IBM and the University of Tübingen (see in particular (Guenthner et al. 1986)).

In the light of the close and simple connections between DRT and standard predicate logic, publication of what will be presented in this paper can be justified only in terms of the special mash we have tried to achieve between the general form and the particular rules of our proof system on the one hand and on the other the distinctive architecture of DRS-like semantic representation. Some additional justification is necessary, however, as there exist a number of other proof systems for first order DRT, some of which have pursued more or less the same aims that have motivated the system presented here. We are explicitly aware of those developed by (Koons 1988), (Saurer 1990), (Sedogbo and Eytan 1987), (Reinhart 1989), (Gabbay and Reyle 1994); perhaps there are others. (Sedogbo and Eytan 1987) is a tableau system, and (Reinhart 1989) and (Gabbay and Reyle 1994) are resolution based, goal directed. These systems may promise particular advantages when it comes to implementing inference engines operating on DRS-like premises. But they do not aim to conform to certain canons of actual inferencing by human interpreters of natural language; and indeed the proof procedures they propose depart quite drastically from what one could plausibly assume to go in the head of such an interpreter. Only (Koons 1988) and (Saurer 1990) are, like our system, inspired by the methods of natural deduction. But there are some differences in the choice of basic rules. In particular both (Koons 1988) and (Saurer 1990) have among their primitive rules the Rule of Reiteration, which permits the copying of a DRS condition from a DRS to any of its sub-DRSs. In our system this is a derived rule (see Section 4 below).

We will develop our system in several stages. The necessary intuitions and the formal background are provided in Sections 1 and 2. (The formal definitions can be found also in the first two chapters of (Kamp and Reyle 1993). The first system we present is for a sublanguage of the one defined in Section 2, which differs from the full language in that it lacks identity and disjunction. The core of the paper consists of Section 3, where the proof system for this sublanguage is presented, and Section 5, which extends the system for the full language, including disjunctions (Section 5.1) and identity (Section 5.2) and then establishes soundness and completeness for the full system. Section 4 deals with certain derived inference principles.

1. Introduction

Discourse Representation Theory is a theory of the semantic content of linguistic expressions – in particular, of natural language sentences, discourses and texts, as well as, more recently, of the content and structure of thought. As a theory of the content of sentences and texts DRT has the following basic structure. Each text T consisting of sentences $S_1, \ldots, S_n$ determines a semantic representation K(T) (Such representations are called "Discourse Representation Structures", or "DRSs"). K(T) is constructed stepwise, by a procedure – the so-called "DRS-construction algorithm" – which processes the sentences S_i one by one, in the order in which they occur in the text. The algorithm incorporates the content of S_i into the DRS K_{i-1}, which it has already constructed for the sentences $S_1, \ldots, S_{i-1}$. The process of incorporating S_i into K_{i-1} makes use on the one hand of the syntactic structure of S_i and on the other of the form of K_{i-1}, and results in a new DRS which represents the integral content of the sentences. The process is designed to identify and encode the semantic connections between the successive sentences of the text – such as, for instance, those produced by pronouns whose anaphoric antecedents occur in earlier sentences – which are largely responsible for the cohesion that distinguishes genuine texts from mere successions of (unconnected) sentences. As a result, the final DRS K(T) represents the semantic content of the text as a whole, and does not just act as a compendium of the separate contents of the sentences $S_1, \ldots, S_n$.

According to what we have just said, DRSs emerge through application of the DRS-construction algorithm to texts belonging to the given natural language, and so the class of DRSs could be defined as consisting of just those structures which can be constructed in this way from an independently specified class of natural language inputs. It is also possible, however, to define the class of "well-formed" DRSs directly, in the manner familiar from mathematical logic. Such a definition, which we will give below, presents DRSs as formulas belonging to some formal language – the "DRS language" – for which we can then specify a suitable semantics, logic and proof theory. This is the perspective we adopt here.

To understand the motivation behind the particular way in which we set up the concept of logical consequence between DRSs – for which we will then develop a sound and complete inference system – it is nevertheless useful to keep the linguistic motivations of DRT in mind. When questions of logical consequence arise within natural language, it is often the case that the (putative) conclusion receives its intended interpretation only with reference to the given premises, viz. by being linked to them by the kind of intersentential connections to which we alluded above. Consider, for instance, the following variation on a wellworn example from Russell:

> A barber from Williamsburgh shaves everyone from that town who does not shave himself. (1)
> Conclusion: The barber shaves himself.

Here the conclusion follows logically from the premis, but only on the assumption that the phrase **the barber** refers to the barber mentioned in the premis. A similar situation, but one that is if anything more common, arises in the context of questions. When I tell you

> Bill met a woman last week with whom he has fallen head over heel in love. (2)

and you ask:

> Have they met since then? (3)

it is only by relating your question (in particular, the pronoun **they** and the adverbial phrase **since then** which it contains) to my own utterance that I can interpret it as you intend (that is, as: "Have Bill and the woman you mentioned met since that time last week when he fell in love with her?"). It has been pointed out repeatedly in the literature on automated question answering that to answer a yes-no question Q one must prove, from premises contained in one's knowledge base, either the queried proposition Q′, in which case the answer is "yes", or its negation not-Q′, with the answer "no". But how precisely is Q′ related to Q? Roughly speaking it is the proposition expressed by the sentence which we get when replacing Q's question mode by the declarative mode – in the case of (3) this would be the sentence (4).

> They have met since then. (4)

However, what proposition is expressed by (4) can only be understood in the context of (2). A systematic method for representing the content of such queried propositions must therefore make use of just those principles for encoding cross-sentential connections which DRT is designed to capture. The natural way to proceed in a case such as this one is therefore to form (i) the DRS K for (2); (ii) the result K_Q of incorporating (4) into K; (iii) the result $K_{\neg Q}$ of incorporating the negation of (4) into K; and then prove either K_Q or $K_{\neg Q}$ from K.

This is the form in which we will define the concept of logical consequence. The relation of logical consequence is one which in first instance obtains between a DRS K and some extension K′ of K. This does not mean that we prevent ourselves from considering questions of inference and logical consequence in all other situations, where the conclusion is not an extension of the premise DRS. But these we reduce to the basic case, by stipulating that K′ is a consequence of K just in case the result $K_{K'}$ of incorporating K′ into K is a consequence of K.

2. The DRS Language

As we said in the introduction, we will treat DRSs as formulas of some formal language $\mathcal{L}$. The vocabulary of $\mathcal{L}$ consists of

1. an infinite set R of discourse referents, and
2. for each number n an infinite set P^n of n-place predicates; let $V = \bigcup_n P^n$;
3. the identity symbol '='.

A DRS is a pair consisting of

(i) a set of discourse referents (the *universe* of the DRS), and

(ii) a set of DRS-*conditions*.

As DRS-conditions may contain DRSs as constituents DRSs and DRS-conditions must be defined by simultaneous recursion. For the purpose of the present paper we will restrict ourselves to *finite* DRSs, i.e. DRSs built up with finite universes and finite condition sets. The definition is as follows.

DEFINITION 1

(i) A *finite DRS* K *confined to* V *and* R is a pair, consisting of a finite subset U_K (possibly empty) of R and a finite set Con_K of finite DRS-conditions confined to V and R;

(ii) A *finite DRS-condition confined to* V *and* R is an expression of one of the following forms:

(a) $\mathbf{x} = \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ belong to R

(b) $P(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, where $\mathbf{x}_1, \ldots, \mathbf{x}_n$ belong to R and P to P^n

(c) $\neg K$, where K is a finite DRS confined to V and R

(d) $K_1 \Rightarrow K_2$, where K_1 and K_2 are finite DRSs confined to V and R.

(e) $K_1 \vee \ldots \vee K_n$, where for some $n \geq 2$, $K_1, \ldots, K_n$ are finite DRSs confined to V and R.

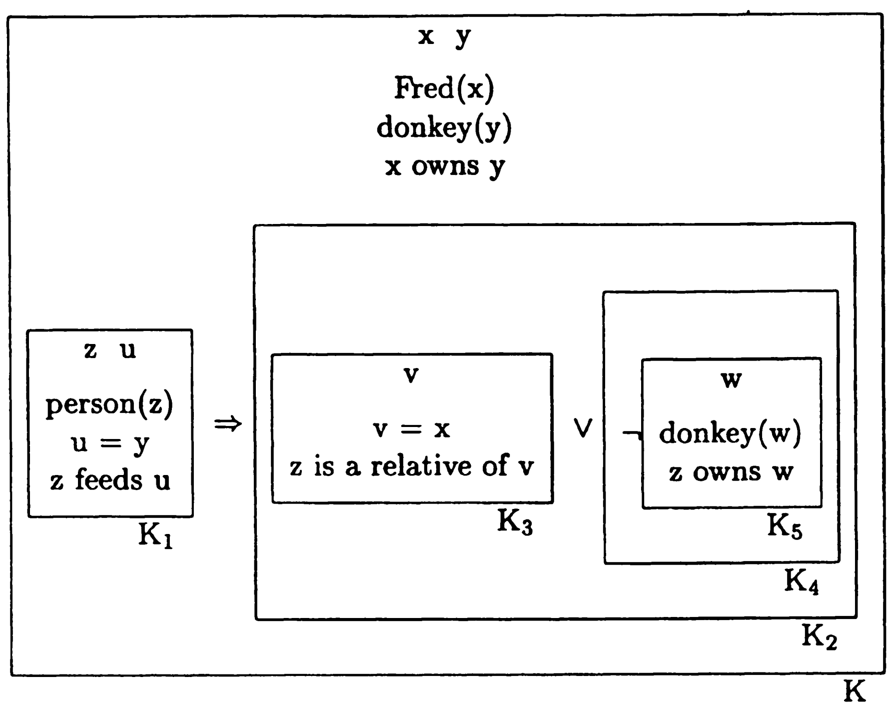
Since there is no danger of confusion in this paper we will often omit the qualification "finite" and simply talk of DRSs. It is useful to abbreviate a DRS $\langle \{\}, \{\neg K\} \rangle$ as $\overset{\circ}{\neg} K$.

It has become common practice to represent DRSs graphically as boxes; complex conditions then lead to nesting of boxes. For instance the two-sentence text

Fred owns a donkey. Every person who feeds it is either a relative of his or doesn't own a donkey himself. (5)

can be represented by a DRS which in box notation looks as follows:

(6)



We assume familiarity with this notation and will use it throughout.

Where K is a DRS we write U_K for the universe of K and Con_K for the set of conditions of K. K′ is called an *immediate sub*-DRS of K if

(i) $\neg K' \in Con_K$, or

(ii) there is a DRS K″ such that one of $K' \Rightarrow K''$, $K'' \Rightarrow K'$ belongs to Con_K, or

(iii) there are DRSs $K_1, \ldots, K_n$ such that for some i with $0 \leq i \leq n$
$K_1 \vee K_2 \vee \ldots \vee K_{i-1} \vee K' \vee K_{i+1} \vee \ldots \vee K_n \in Con_K$.

The relation "K′ *is a sub-DRS of* K" is the reflexive and transitive closure of "K′ is an immediate sub-DRS of K". Thus in (6) K, $K_1, \ldots, K_5$ are all sub-DRSs of K; K_1 and K_2 are the immediate sub-DRSs of K; K_3 and K_4 the immediate sub-DRSs of K_2; and K_5 is the immediate sub-DRS of K_4. "K′ is a sub-DRS of K" is abbreviated as "$K' \leq K$".

DEFINITION 2 Let K be a DRS, **x** a discourse referent and γ a DRS-condition. We say that **x** is *accessible from* γ *in* K if **x** belongs to U_{K_1}, where

(i) $K_1 \leq K$, and

(ii) for some K_2, γ occurs in Con_{K_2}, and either

(a) $K_2 \leq K_1$; or

(b) there is a DRS K_3 such that for some $K_4 \leq K$ $K_1 \Rightarrow K_3$ is in Con_{K_4} and $K_2 \leq K_3$.

Just as for predicate logic, the meaning of DRSs and DRS-conditions can be characterized model-theoretically. We take the models for $\mathcal{L}$ to be precisely the models for the language of predicate logic whose non-logical vocabulary consists of the predicates of $\mathcal{L}$. Thus, a *model* is any pair $\langle U_M, Pred_M \rangle$ where (i) U_M is a non-empty set and (ii) $Pred_M$ maps each n-place predicate of $\mathcal{L}$ onto an n-place relation over U_M. A DRS K is *correct* with respect to a model M if it is possible to embed U_K truthfully into U_M, i.e. if there is a function $f{:}U_K \to U_M$ which verifies in M each of the conditions in Con_K. For atomic conditions $P^n(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ verification simply means that the individuals which f correlates with the discourse referents in U_K stand in the relation $Pred_M(P^n)$. For complex conditions verification is defined via correct embeddings of their constituent DRSs, so that, again, the notions of condition verification and of correct DRS embedding must be defined by simultaneous recursion. For the sake of uniformity we use the term *verification* in relation to DRSs as well as DRS-conditions. So we simultaneously define "function f verifies the DRS K in M" and "function f verifies the DRS-condition γ in M".

DEFINITION 3 Let K be a DRS confined to V and R, γ a DRS-condition, and let f be an *embedding from* some subset of V *into* M, i.e. a function whose domain is included in R and whose range is included in U_M.

(i) f *verifies* the DRS K in M iff f verifies each of the conditions belonging to Con_K in M.
(ii) f *verifies* the condition γ in M iff
  (a) γ is of the form $\mathbf{x} = \mathbf{y}$ and f is defined on $\mathbf{x}$ and $\mathbf{y}$ and maps them onto the same element of U_M.
  (b) γ is of the form $P(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, f is defined on $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and $\langle f(\mathbf{x}_1), \ldots, f(\mathbf{x}_n)\rangle \in Pred_M(P)$.
  (c) γ is of the form $\neg K'$ and there is no embedding g from R into M such that g extends f, $Dom(g) = Dom(f) \cup U_{K'}$ and g verifies K′ in M.
  (d) γ is of the form $K_1 \Rightarrow K_2$ and for every extension g of f such that $Dom(g) = Dom(f) \cup U_{K_1}$ which verifies K_1 in M there is an extension h of g such that $Dom(h) = Dom(g) \cup U_{K_2}$ and h verifies K_2 in M.
  (e) γ is of the form $K_1 \vee K_2 \vee \ldots \vee K_n$ and for some i ($i = 1, \ldots, n$) f verifies K_i in M.

We often write '$M \models_f K$' (or '$M \models_f \gamma$') for 'f verifies K (or γ) in M'.

In the sequel we will have to make use of a number of syntactic notions connected with DRSs and DRS-conditions. First, we must specify for each DRS or DRS-condition the set of its *declared discourse referents*, the set of its *free discourse referents*, and set of its *discourse referents* simpliciter (the last set is the union of the two previous sets). The declared discourse referents of K (or γ) are those which occur in the universe of K or in that of some sub-DRS. Thus we can define the set of *declared discourse referents of* a given DRS K – in symbols $\underline{U}(K)$ – or *of* a DRS-condition γ – in symbols $\underline{U}(\gamma)$ – as follows.

DEFINITION 4

(i) if γ is an atomic condition, then $\underline{U}(\gamma) = \{\}$.
(ii) if γ is of the form $\neg K$, then $\underline{U}(\gamma) = \underline{U}(K)$.
(iii) if γ is of the form $K_1 \Rightarrow K_2$, then $\underline{U}(\gamma) = \underline{U}(K_1) \cup \underline{U}(K_2)$.
(iv) if γ is of the form $K_1 \vee \ldots \vee K_n$, then $\underline{U}(\gamma) = \underline{U}(K_1) \cup \ldots \cup \underline{U}(K_n)$.
(v) $\underline{U}(K) = U_K \cup \bigcup_{\gamma \in Con_K} \underline{U}(\gamma)$.

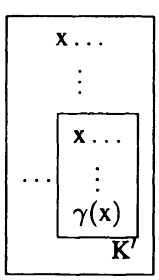
The *free discourse referents of* K are those which occur in K without being declared in K (similarly for γ):

DEFINITION 5

(i) If γ is a condition of the form $P(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, then $Fr(\gamma) = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$.
(ii) If γ is a condition of the form $\mathbf{x} = \mathbf{y}$, then $Fr(\gamma) = \{\mathbf{x}, \mathbf{y}\}$.
(iii) If γ is a condition of the form $\neg K$, then $Fr(\gamma) = Fr(K)$.
(iv) If γ is a condition of the form $K_1 \Rightarrow K_2$, then $Fr(\gamma) = Fr(K_1) \cup (Fr(K_2) \setminus U_{K_1})$.
(v) If γ is a condition of the form $K_1 \vee \ldots \vee K_n$, then $Fr(\gamma) = Fr(K_1) \cup \ldots \cup Fr(K_n)$.
(vi) $Fr(K) = (\bigcup_{\gamma \in Con_K} Fr(\gamma)) \setminus U_K$

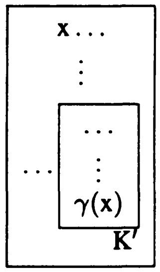
The set of discourse referents of K, dr(K), consists, as we said, of the declared and the free discourse referents together: $dr(K) = \underline{U}(K) \cup Fr(K)$ (similarly $dr(\gamma) = \underline{U}(\gamma) \cup Fr(\gamma)$ for any DRS-condition γ). A DRS K is called *proper* iff $Fr(K) = \phi$.

Inspection of the definition of verification reveals that when a discourse referent $\mathbf{x}$ occurs both in the universe U_K of some DRS K and also in $U_{K'}$, where K′ is a proper sub-DRS of K, then the "declaration" of $\mathbf{x}$ in $U_{K'}$ is otiose: Its presence there does not affect verification in any way, and so it can be eliminated without any semantic effect. To put it more graphically, a DRS of the form (7)



(7)

and the result (8)



(8)

of eliminating **x** from $U_{K'}$ are equivalent. For technical reasons configurations like the one in (7), where the same discourse referent occurs in two distinct DRS universes, one subordinate to the other, turn out to be a nuisance, and so we will set them aside. We will, that is, restrict our attention to *pure* DRSs; these are DRSs in which otiose declarations do not occur.

DEFINITION 6 A DRS K is *pure* if for every two distinct DRSs K_1 and K_2 such that K_1 is a sub-DRS of K_2 and K_2 a sub-DRS of K $U_{K_1} \cap U_{K_2} = \phi$.

We are now ready to define what it is for a DRS to be *true in* a model M and what it means for one DRS to be a *logical consequence* of another DRS.

The first notion, that of truth, is confined to proper DRSs.

DEFINITION 7 Let K be a proper DRS, M a model. K is *true in* M (in symbols $M \models K$) iff there exists a function $f{:}U_K \rightarrow U_M$ which verifies K in M.

The second notion, that of one DRS K′ being a logical consequence of another DRS K, will be defined for improper as well as proper DRSs. We recall the remarks made in the introduction that the typical situation in which questions of valid inference (and thus also of logical consequence) arise is that where the premise DRS K is included in K′. However, our definition will apply equally to situations where $K \not\subseteq K'$.

Intuitively K′ is a logical consequence of K if whenever we have M and f such that $M \models_f K$ then this should give a way of verifying K′ in M as well. If we are to capture this idea in the right way, however, we must be careful that f does not block verification of K′ for irrelevant reasons, viz because it assigns the wrong values to discourse referents that do not occur in K but do occur in K′. Therefore we must restrict attention to embeddings f which are "minimal" in the relevant sense, i.e. are defined only for the discourse referents in U_K and those in $Fr(K) \cup Fr(K')$. (The reason why we also include the discourse referents from Fr(K′) which are neither in Fr(K) nor in U_K is not obvious from what has just been said. Actually the cases where there are such discourse referents have no clearly perspicuous relevance to the intuitive concept of logical consequence as semantic validity of inference. The choice to include them relates to technical considerations which it would carry us too far to explain here.)

In the light of this preamble Definition 8 should be clear

DEFINITION 8 Let K, K′ be pure (but not necessarily proper) DRSs. Thus K′ is a logical consequence of K (in symbols $K \models K'$) iff the following condition holds:

Suppose M is a model and that f is a function from $U_K \cup Fr(K) \cup Fr(K')$ into U_M, such that $M \models_f K$. Then there is a function $g \supseteq_{U_{K'}} f$ such that $M \models_g K'$.

When K′ is a logical consequence of K and K is a logical consequence of K′ we say that K and K′ are *logically equivalent*. Note that if K′ is a logical consequence of K and $K \subseteq K'$, then K and K′ are logically equivalent.

It is a familiar fact about predicate logic that when we rename all bound variables of a given formula in a 1–1 fashion, the resulting formula is equivalent to the original one. Formulas that are related to each other by such renamings are called *alphabetic variants*. Renaming of declared discourse referents similarly leads to equivalent DRSs. As this is an operation of which we will make heavy use, it deserves a formally exact definition.

DEFINITION 9

(a) Let f be a function from discourse referents to discourse referents. Let f^+ be the function defined by: $f^+(\mathbf{x}) = f(\mathbf{x})$ if $\mathbf{x} \in Dom(f)$ and $f^+(\mathbf{x}) = \mathbf{x}$ if $\mathbf{x} \notin Dom(f)$. We define by recursion on DRSs K and DRS-conditions γ the *variant of* K (or γ) *according to* f, f(K) (or $f(\gamma)$):
  (i) If γ is a condition of the form $P(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, then $f(\gamma) = P(f^+(\mathbf{x}_1), \ldots, f^+(\mathbf{x}_n))$.
  (ii) If γ is a condition of the form $\mathbf{x} = \mathbf{y}$, then $f(\gamma)$ is $f^+(\mathbf{x}) = f^+(\mathbf{y})$.
  (iii) If γ is a condition of the form $\neg K$, then $f(\gamma) = \neg f(K)$.
  (iv) If γ is a condition of the form $K_1 \Rightarrow K_2$, then $f(\gamma) = f(K_1) \Rightarrow f(K_2)$.
  (v) If γ is a condition of the form $K_1 \vee \ldots \vee K_n$, then $f(\gamma) = f(K_1) \vee \ldots \vee f(K_n)$.
  (vi) $f(K) = \langle \{f^+(\mathbf{x})\}_{\mathbf{x} \in U_K}, \{f(\gamma)\}_{\gamma \in Con_K} \rangle$.

(b) Suppose that f is a 1–1 function from $\underline{U}(K)$ into the set of discourse referents V such that $Ran(f) \cap Fr(K) = \phi$. Then f(K) is called the *alphabetic variant of* K *according to* f.

It can be shown without too much difficulty that if K′ is an alphabetic variant of K according to f then, for any model M and embedding g of U_K in M, $M \models_g K$ iff $M \models_{g'} K'$, where g′ is the function defined by: $g'(\mathbf{y}) = g(f^{-1}(\mathbf{y}))$ for $\mathbf{y} \in U_{K'}$.

Another notion we will need is that of one DRS K having an homomorphic copy that is part of another DRS K′ – or, as we will put it, of K being homomorphically embeddable within K′. The formal definition is as follows.

DEFINITION 10 Let f be a function from $\underline{U}(K)$ in $\underline{U}(K')$ such that f is 1–1 on $\underline{U}(K) \backslash U_K$. If $f(K) \subseteq K'$, then f is called an *embedding of* K in K′.

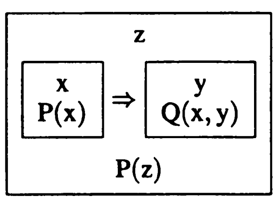
3. Deduction Rules

3.1. Inference Rules and Direct Proof

The general architecture of the proof system we will present is borrowed from Kalish and Montague.* The main distinguishing feature of this architecture is that each 'goal' – that is, proposition one wants to prove – and each of the lemmata one wants to establish as intermediate steps in that proof – is written down explicitly as a "show-line" **Show: A**, where **A** is the goal in question. When **A** has been proved the show-line is cancelled. Usually this is done by crossing out the word 'Show': **~~Show~~: A**. (Because of this feature the Kalish & Montague system deserves to be considered as the first deduction system that explicitly formalizes the strategy of 'backward chaining', well before this concept became common currency in the theorem proving community.)

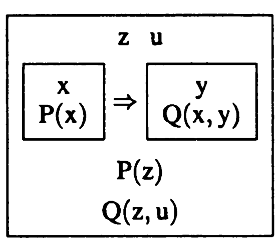
We will develop our proof system in several stages. In this and the next two sections we will be concerned with a sub-language of the DRS language given in Definition 1, which differs from the full language in having neither disjunction nor identity. We will introduce disjunctions in Section 6.1 and identity in 6.2.

The basic concept of inference, we said, applies to a premise DRS K such that $K \subseteq K'$. Nevertheless we will allow proof goals that do not contain the premise DRS as part, the idea being that once the show-line has been cancelled it is the union of premise DRS and the DRS in the show-line that is established as proved. Let us consider a very simple example to see what this means. Let K be the DRS (9).



(9)

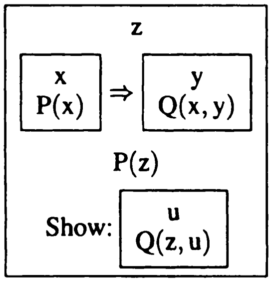
Let K′ be the extension (10) of (9),



(10)

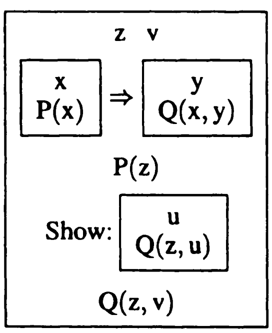
and suppose we want to prove (10) from (9). Then we start by adding a show-line **Show: K″** where K″ is the part of (10) that is disjoint from (9):

* See (Kalish and Montague 1964); and see also (Bonevac 1986), which contains a particularly smooth and elegant deduction system for *classical predicate logic*.



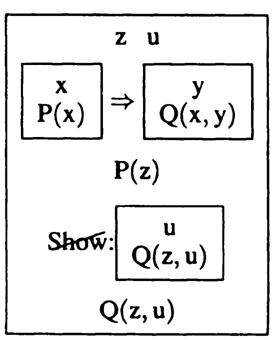
(11)

To prove the show-line we try to introduce, through the application of certain inference rules, an alphabetic variant of the show-line DRS. In the present instance the rule to be applied is that of *Detachment* (**DET**), which permits us to add a copy of the right hand side K_2 of an implicative condition $K_1 \Rightarrow K_2$ (belonging to a DRS K_0) to K_0, provided that the left hand side K_1 can be 'matched' with a part of K_0. In the example $\boxed{\begin{array}{c} x \\ P(x) \end{array}}$ can be matched by 'unifying' **x** with **z**; because of this a corresponding copy of $\boxed{\begin{array}{c} y \\ Q(x,y) \end{array}}$ – one in which **z** takes the place of **x** and some new discourse referent **v** that of **y** – may be added to the DRS (9). This transforms (11) into (12):



(12)

At this point the DRS of the show-line is embedded in the DRS outside the show-line (and a fortiori, has a copy that is included in that DRS), which entitles us to cancel the show-line:



(13)

Formally **DET** can be characterized as follows.

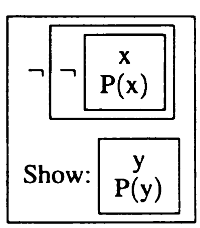
> **DET**(achment), or **G**(eneralized) **M**(odus) **P**(onens):
>
> Suppose $K_1 \Rightarrow K_2 \in Con_K$ and suppose there is an embedding f of K_1 in K. Let g be an extension of f to U_{K_2}, such that $g \backslash f$ is 1–1 and such that g maps U_{K_2} to a set of discourse referents that are new to the (extended[a]) DRS K. Then we may add $g(K_2)$ to K.
>
> [a] An extended DRS is a DRS with show-lines. For a rigorous definition see Def. 11 below.

Proofs like the one just presented, in which the goal is written down as a show-line and then matched by some additions that inference rules permit us to make to the premise DRS, but which do not involve any "sub-goals", are called *direct proofs*. They involve one *rule of proof*, according to which a show-line may be cancelled if its DRS is a copy of a part of the DRS containing the show-line. In the next section we will encounter other rules of proof, which will allow us to prove goals by establishing intermediate goals.

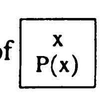
> **R**(ule of) **D**(irect) **P**(roof):
>
> Suppose K contains a show-line **Show: K** (or **Show:** γ), and that there is an alphabetic variant K'' of K' (γ' of γ) in K, i.e. $Con_{K''} \subseteq Con_K$ and $U_{K''} \subseteq U_K$ (or $\gamma' \in Con_K$), then the show-line may be cancelled.

Besides the Rule of Direct Proof direct proofs require the use of *inference rules*. The rule of Detachment that we used in our example is one of these. In the remainder of this section we will introduce two additional inference rules, and give an impression of what kinds of deductions they enable us to carry out by the method of direct proof.

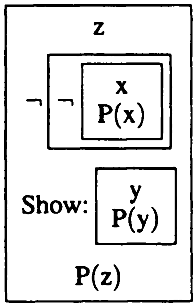
The first of these is the rule of *Double Negation Elimination*, **DN**. In the simplest instances this rule amounts to the cancellation of two negation signs, as in the following example.



(14)

DN permits us to introduce a variant of 

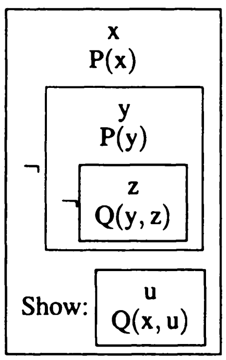
as a new condition, giving



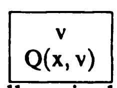
(15)

which then allows an application of the Rule of Direct Proof.

In the general case, where the DRS to which the outer negation sign applies contains material besides the DRS to which the inner negation applies, **DN** is more complicated. An application of **DN** in its more general setting is given in the next example.



(16)

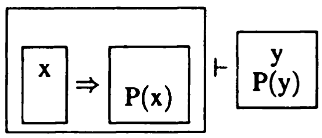
In (16) **DN** permits the addition of the DRS 

, where **v** is a new discourse referent. The intuitive justification is essentially as in the case of **DET**: the condition of (16) beginning with the outer negation sign says that there cannot be an object y such that **P(y)** without it also being the case that for some **z Q(y,z)**. Since the first condition says that **P(x)**, it follows that for some **v Q(x,v)**.

Formally the rule of Double Negation (Elimination) can be characterized as follows:

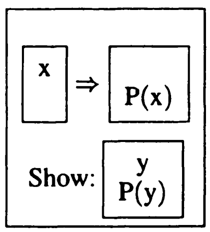
> Double Negation (Elimination):
>
> Suppose K contains a condition of the form $\neg K_1$ such that for some K_2 $\neg K_2 \in Con_{K_1}$ and suppose there is an embedding f of $K_1 \backslash \langle \{\}, \{\neg K_2\} \rangle$ into K. Let g be an extension of f to U_{K_2}, such that g\f is 1–1 and such that g maps $U(K_2)$ to a set of discourse referents that are new to the (extended[a]) DRS K. Then we may add $g(K_2)$ to K.
>
> [a] An extended DRS is a DRS with show-lines. For a rigorous definition see Def. 11 below.

The second and last rule of this section is the so-called Rule of Non-Empty Universe (**NEU**). This rule reflects the assumption – standardly made in mathematical logic – that there is of necessity at least one thing. Formally this means that we consider only models with non-empty universes. The proof-theoretic implication of this semantic assumption is that it is always permissible to introduce a discourse referent at the highest level of the DRS. The need for a rule which forsees the introduction of new discourse referents becomes apparent in the following example. Under the assumption that there exists at least one thing, the proposition that everything has the property P entails that something does:



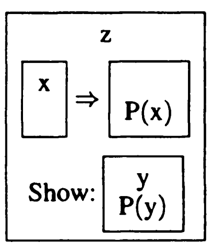
(17)

Or, written in the form of our proof system



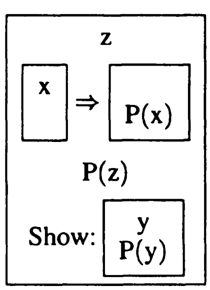
(18)

To prove (18) we must make use of **NEU**:



(19)

The presence of **z** enables us to apply **DET**, which yields



(20)

Formally we characterize **NEU** as follows.

N(on)-**E**(mpty) **U**(niverse):

Any (finite) collection of new discourse referents may be added to U_K .

We are now ready to give a formal definition of direct proofs. A proof of the argument $K \vdash K'$ is a sequence of syntactic objects the first of which is the DRS K with the added show-line **Show: K′** and the last of which includes (in the appropriate sense, to be defined below) an alphabetic variant of K′. As our examples show, the syntactic objects that occur in proofs are "DRSs" containing cancelled or uncancelled show-lines. (In the next section, where we will introduce additional rules of proof, the objects that make up proofs will become a great deal more complex yet.) In order to be precise we must first define the new extended notion of a DRS (in which discourse referents and conditions may occur as well as cancelled or uncancelled show-lines).

DEFINITION 11

(a) An *uncancelled show-line* is a pair consisting of
  (i) the sign **Show:** and
  (ii) either a DRS or a DRS-condition.
(b) A *cancelled show-line* is a pair consisting of
  (i) the sign ~~**Show**~~**:** and
  (ii) either a DRS or a DRS-condition.
(c) An *extended* DRS is a triple consisting of
  (i) a set of discourse referents,
  (ii) a set of DRS-conditions,
  (iii) a set of cancelled and uncancelled show-lines.

Earlier we decided to restrict our attention to pure DRSs. But what exactly is a pure extended DRS? To apply Definition 6 (of a *pure* DRS) we need to decide what status the DRSs have that occur in cancelled show-lines. The additional stipulation needed is this: An extended DRS $\langle U_K, Con_K, \{\mathbf{Show: K_1}, \ldots, \mathbf{Show: K_m}, \cancel{\mathbf{Show}}\mathbf{: K_{m+1}}, \ldots, \cancel{\mathbf{Show}}\mathbf{: K_{m+n}}, \mathbf{Show:} \gamma_1, \ldots, \mathbf{Show:} \gamma_r, \cancel{\mathbf{Show}}\mathbf{:} \gamma_{r+1}, \ldots, \cancel{\mathbf{Show}}\mathbf{:} \gamma_{r+s}\}\rangle$ is pure if the DRS $\langle U_K, Con_K\rangle \cup K_{m+1} \cup \ldots \cup K_{m+n} \cup \langle\{\}, \{\gamma_{r+1}\}\rangle \cup \ldots \cup \langle\{\}, \{\gamma_{r+s}\}\rangle$ is pure.*

In the sequel we will usually refer to extended DRSs simply as DRSs. No confusion should arise from this. Before we can define the notion of Proof we must first give a formal definition of the Rule of Direct Proof.

DFINITION 12 A *Direct Proof* of $K \vdash K'$ is a sequence of DRSs $K_1, \ldots, K_n$ such that

(i) K_1 is the DRS we get by adding to K the show-line which has as its second member the DRS K″, which is an alphabetic variant of K′ such that K_1 is pure.

* We will see the effect of this stipulation, when we discuss the rule of conditional proof below.

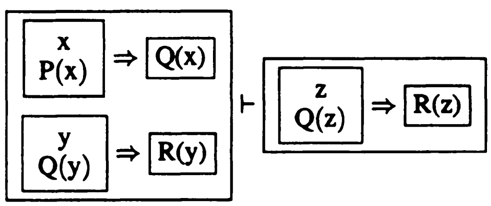
(ii) for $i = 1, \ldots, n-2$, K_{i+1} is obtained from K_i by
 (a) application of one of the rules of inference
 (b) adding a new uncancelled show-line **Show: P**, where P is either a DRS or a DRS-condition;

(iii) K_n results from K_{n-1} by an application of the Rule of Direct Proof which cancels the show-line **Show:** $\mathbf{K''}$ mentioned in (i).

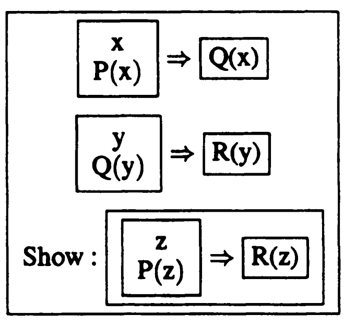
3.2. Rules of Proof

Suppose we want to prove from a given premise DRS K_0 a DRS-condition which has the form of a conditional, say $K_1 \Rightarrow K_2$. The natural way to proceed would seem to be this: Assume K_1 as an additional premise and prove K_2 from the thus enlarged premise set. This method of proof is familiar enough from formal logic. It is known as the *Rule of Conditional Proof.*

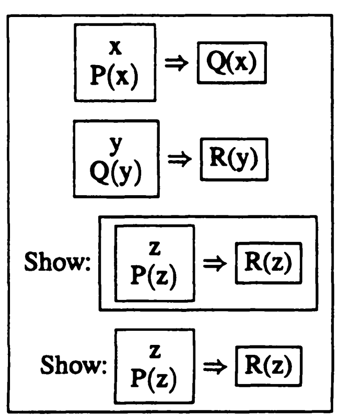
Consider (22), which is the first step of a proof, ⟨(22), . . . , (27)⟩, for (21).



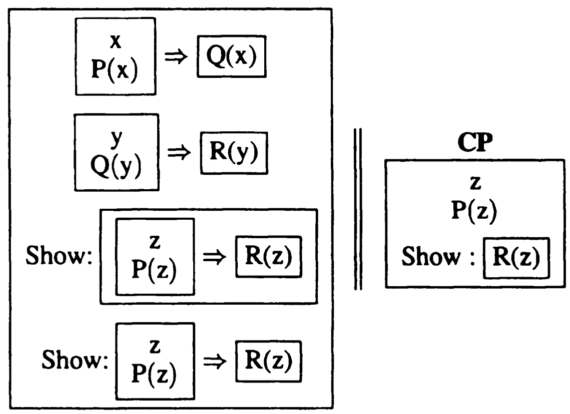
(21)



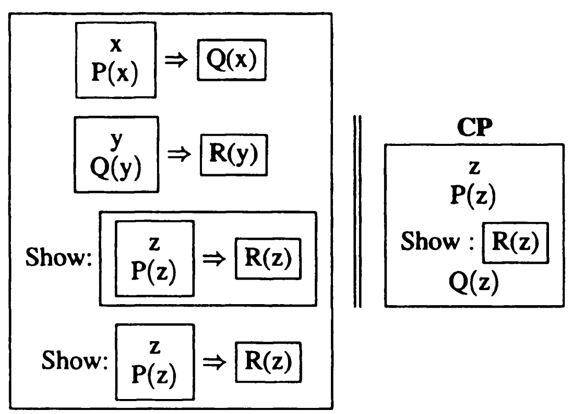
(22)



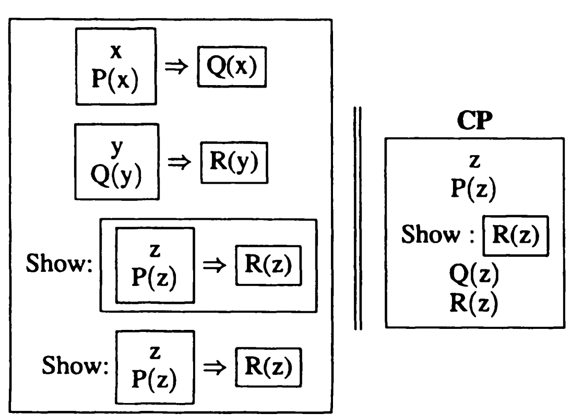
(23)


x
P(x)
⇒ Q(x)
y
Q(y)
⇒ R(y)
Show: z P(z) ⇒ R(z)
Show: z P(z) ⇒ R(z)
CP
z
P(z)
Show : R(z)

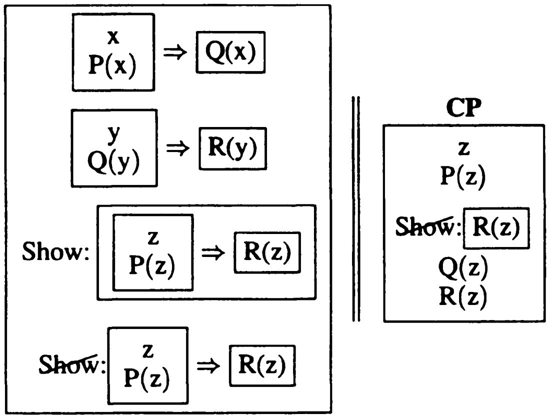
(24)


x
P(x)
⇒ Q(x)
y
Q(y)
⇒ R(y)
Show: z P(z) ⇒ R(z)
Show: z P(z) ⇒ R(z)
CP
z
P(z)
Show : R(z)
Q(z)

(25)


x
P(x)
⇒ Q(x)
y
Q(y)
⇒ R(y)
Show: z P(z) ⇒ R(z)
Show: z P(z) ⇒ R(z)
CP
z
P(z)
Show : R(z)
Q(z)
R(z)

(26)



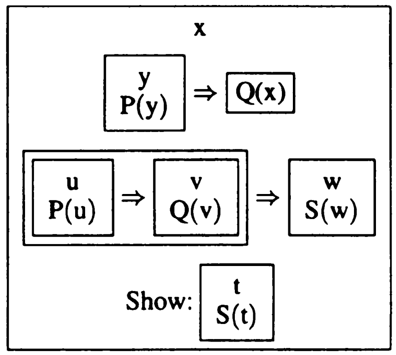
(27)

The parts of this derivation which have to do with the principle of Conditional Proof are the transitions from (23) to (24) and from (26) to (27). In the transition from (23) to (24) we *open up* a new *subderivation*, one in which K_1 is among the premises that may be used. This subderivation is located to the right of the vertical double bar in (24) (as well as in the two following diagrams (26) and (27), the understanding being that the premises available in this subderivation consist of the asserted part of the DRS to the right of the double bar together with the asserted part of the DRS to its left. To indicate that the subderivation is being set up for the sake of proving **R(z)** using [z | P(z)] as additional premise we start the subderivation with the new premise [z | P(z)] together with the "goal" **Show: R(z)**.

The Rule of Conditional Proof comes to function in the transition from (26) to (27). In (26) the show-line on the right of the '||' may be cancelled as it matches the last condition **R(z)** of the DRS on the right. Since cancellation of this show-line is the demonstration that **R(z)** has been derived with the additional premise [z | P(z)], we may conclude – this *is* the Rule of Conditional Proof – that the condition of the show-line on the left has been proved from the remaining premises. So the cancellation of the show-line on the right carries with it cancellation of the show-line on the left.

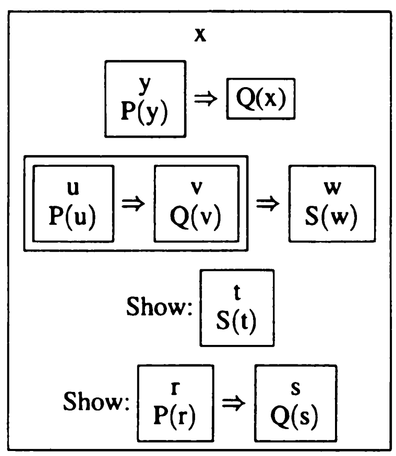
Once stage (27) has been reached, the subderivation has done its work and no longer counts in relation to the further proof steps that may still be taken. In the present case the stipulation that the subderivation no longer counts is immaterial; since an application of the rule of direct proof to stage (27) will complete the derivation. But the next example, in which a conditional condition is proved as a lemma, should make the need for such a stipulation clearer.

Consider the following derivation problem:



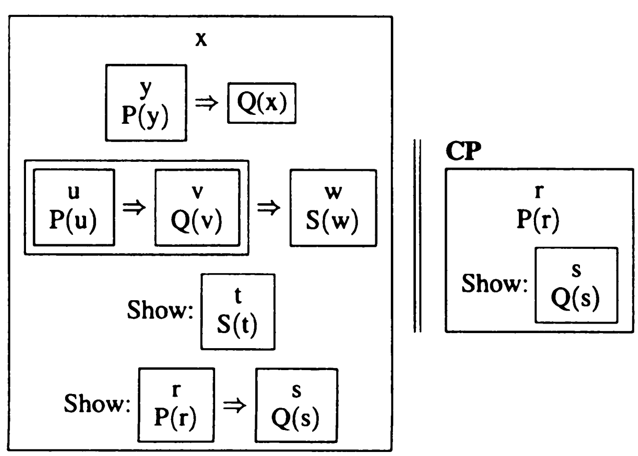
(28)

The obvious strategy for carrying through this derivation is to first show the left-hand side of the last conditional. In other words, we add a new show-line, as in (29):



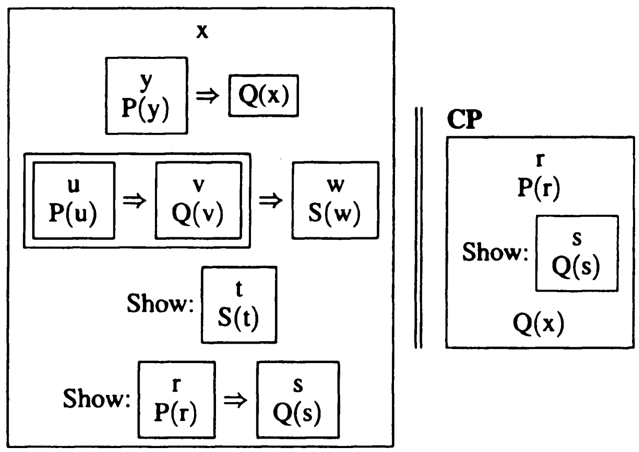
(29)

To establish this last show-line we resort once more to the Rule of Conditional Proof, setting up a subderivation as in (30):



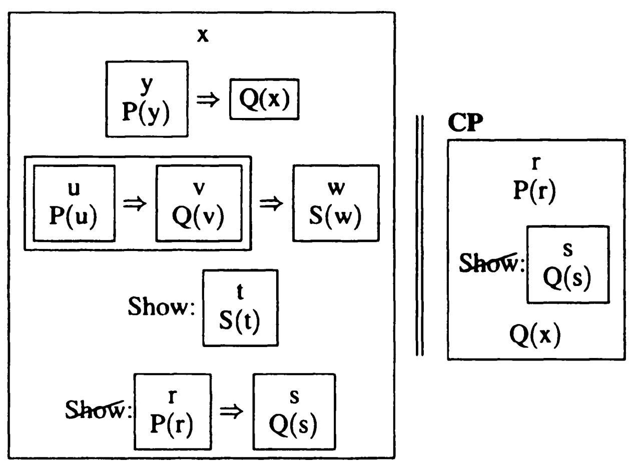
(30)

The additional assumption $\boxed{\begin{matrix} r \\ P(r) \end{matrix}}$ of this subderivation allows us to apply the rule of Detachment with respect to the first implication of the original DRS. The result of this application is (31):



(31)

Now, the conclusion of this subderivation is immediate since the content of the show-line is matched by the part $\boxed{\begin{matrix} x \\ Q(x) \end{matrix}}$ of the premise DRS. Cancellation produces the structure in (32).



(32)

The derivation now continues on the left-hand side, using the condition $\boxed{\begin{matrix} r \\ P(r) \end{matrix}} \Rightarrow \boxed{\begin{matrix} s \\ Q(s) \end{matrix}}$ of the cancelled show-line as part of the premise DRS. The completed subderivation on the right plays no further role in the remainder of the derivation.

In general it will be possible to use the Rule of Conditional Proof also inside subderivations that arose through an earlier application of the rule. Because of this recursive aspect one has to be careful when trying to define the rule. The proper way to proceed is to describe how proof structures can be transformed through the application of rules, both those rules which we introduced earlier and the new Rule of Conditional Proof. To this end we define by simultaneous recursion the notion of a *proof stage* S together with two derived notions, that of the *active part of*) a proof stage, AP(S), and of the *assertions of* a proof stage, K(S). While we are at it, we will define, as part of the same simultaneous recursion, also a fourth concept, that of the *premise* DRS of a proof stage, PR(S). This notion will be used in Section 6.1.

Informally, the active part AP(S) of a stage S is that subderivation whose show-line one is currently trying to prove or, if no sub-goal is being pursued, the main derivation. The premise DRS of S, PR(S) consists of (i) the initial DRS, except for its show-line and (ii) all assumptions that have been introduced in the setting up of subderivations that are not yet completed. The assertion of S, K(S), is a DRS which includes the premise DRS, but which contains also all discourse referents and conditions which have been introduced through the application of inference rules and completed applications of rules of proof, and which do not belong to subderivations that have already been completed and closed.

DEFINITION 13

(i) (a) A pure DRS is a proof stage.
(b) The union of a proof stage S and a show-line **Show: K′**, or **Show:** γ, is a proof stage S′, provided S′ is pure. AP(S′) = PR(S′) = K(S′) = S.

(ii) Application of inference rules: If S is a proof stage, K′ can be obtained by applying one of the inference rules to K(S), and S′ results from S by adding K′ to AP(S), then S′ is a proof stage,
AP(S′) = AP(S) ∪ K, PR(S′) = PR(S) and K(S′) = K(S) ∪ K′.

(iii) Direct Proof
(a) *Introduction:* 1. Suppose S is a proof stage, such that AP(S) contains a show-line of the form **Show:** γ. If PR(S) ∪ ⟨{}, {γ}⟩ is pure, then S′ is a proof stage, and

AP(S′) = AP(S) ∪ ⟨{},{**Show:** γ}⟩,
PR(S′) = PR(S) and K(S′) = K(S).

2. Similarly if S is a proof stage, such that AP(S) contains a show-line of the form **Show: K**, where K is a DRS, and if PR(S) ∪ K is pure, then S′ is a proof stage,

AP(S′) = AP(S) ∪ ⟨{},{**Show: K**}⟩,
PR(S′) = PR(S) and K(S′) = K(S).

(b) *Cancellation:*
1. If S is a proof stage, the last introduced show-line is **Show:** γ, an alphabetic variant of γ belongs to $\mathrm{Con}_{K(S)}$ and S′ results from cancelling the **Show:** of this show-line, then S′ is a proof stage,

$AP(S') = AP(S)$, $PR(S') = PR(S)$ and
$K(S') = K(S) \cup \langle\{\},\{\gamma\}\rangle$.

2. If S is a proof stage, the last introduced show-line is **Show: K**, an alphabetic variant of K is included in K(S) and S′ results from cancelling the **Show:** of this show-line, then S′ is a proof stage,

$AP(S') = AP(S)$, $PR(S') = PR(S)$ and $K(S') = K(S) \cup K$.

(iv) Conditional Proof:

(a) *Introduction:*

Suppose S is a proof stage such that AP(S) contains a show-line of the form **Show:** $\mathbf{K_1 \Rightarrow K_2}$. And S′ results from S by adding immediately to the right of AP(S) a structure of the form*

CP

K_1
Show : K_2

Then AP(S′) is the structure

CP

K_1'
Show : K_2'

,

$PR(S') = PR(S) \cup K_1$ and $K(S') = K(S) \cup K_1$

(b) *Cancellation:*

Suppose that S is a proof stage, that AP(S) has the form

CP

K_1'
Show : K_2'
⋮

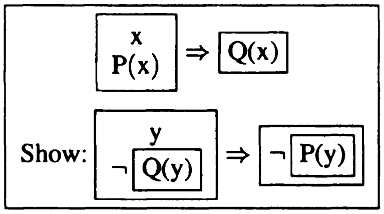
,

containing the show-line **Show:** $\mathbf{K_2}$, and that **Show:** $\mathbf{K_1 \Rightarrow K_2}$ is the show-line that justified the introduction of **Show:** $\mathbf{K_2}$, through an application of (iv.a). We obtain a proof stage S′ by cancelling this show-line. Let S″ be the first stage containing the show-line **Show:** $\mathbf{K_1 \Rightarrow K_2}$. Thus $AP(S') = AP(S'')$, $PR(S') = PR(S'')$ and $K(S') = K(S'') \cup \langle\{\}, K_1 \Rightarrow K_2\rangle$.

Clause (iv.a) describes in detail how the proof structures resulting from applications of the Rule of Conditional Proof are set up; and clause (iv.b) how the show-lines which give rise to such structures may be cancelled.

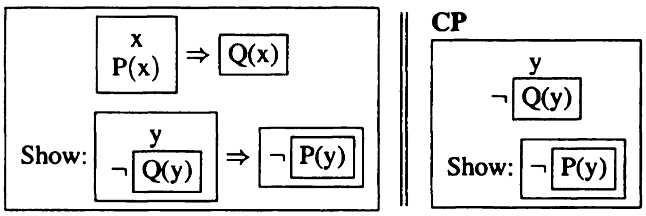
Even now that we have the Rule of Conditional Proof on board, there are still certain valid inferences that we cannot derive. A simple example is

* As $PR(S) \cup K_1 \cup K_2$ is pure in virtue of (i) there is not need to introduce alphabetic variants, K_1' and K_2' here.



(33)

Since what we need to prove has the form of a conditional it seems reasonable to try the Rule of Conditional Proof and transform (33) into (34):

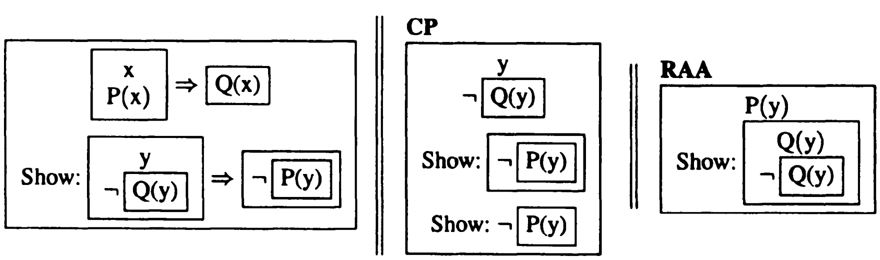


(34)

But this does not see us through. Nevertheless it is intuitively clear that $\neg$**P(y)** must hold. For if not, i.e. if we had **P(y)**, then we would get **Q(y)** with the help of [x P(x)] ⇒ [Q(x)], which would directly contradict the condition ¬[**Q(y)**] below 'CP', showing the impossibility of the assumption that **P(y)**.

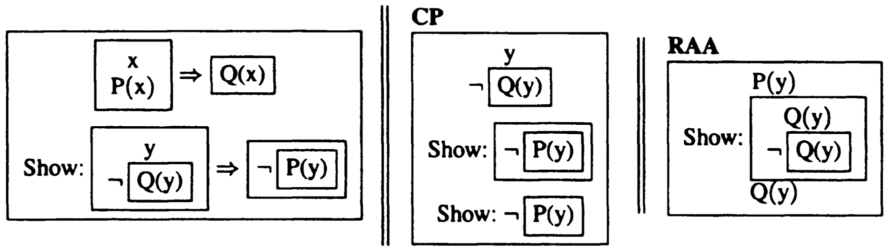
This form of reasoning – A, for if not A then we would get an explicit contradiction – is known as reasoning by *Reductio ad Absurdum*. In many inference systems the method of R(eductio) A(d) A(bsurdum) has been formalized as a separate Rule of Proof. We will do the same.

To get an idea of the form that the Rule of RAA takes in the present proof system let us complete the derivation of the problem we were discussing. As with Conditional Proof, using the rule means setting up a new subderivation. Again we start this subderivation immediately to the right of the current active part of the proof. Thus the next step (after having added the show-line ¬[**P(y)**], which invokes the single condition of the DRS of the right-hand side show-line in (34)), leads to the following proof stage:



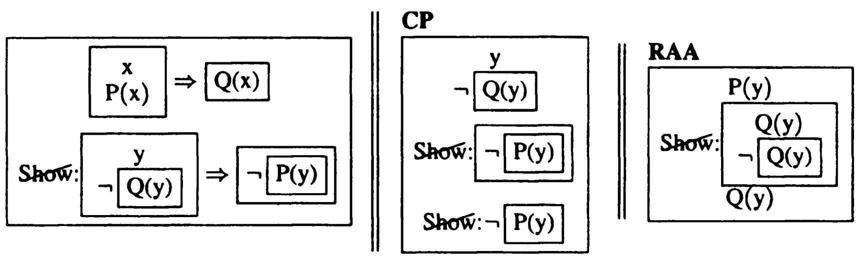
(35)

The next step is to apply Detachment using [x P(x)] ⇒ [Q(x)], **y** and **P(y)** on the right. This leads to the addition of **Q(y)**:



(36)

Since $\boxed{\begin{array}{c} Q(y) \\ \neg\boxed{Q(y)} \end{array}}$ is now included in the premise DRS, we may cancel the right-most show-line. Therewith we have also shown the impossibility of the assumption **P(y)**, thus that $\neg\boxed{\mathbf{P(y)}}$ holds. In other words, we may cancel the show-line **Show:** $\neg\boxed{\mathbf{P(y)}}$ as well. In the present case this then triggers also the cancellation of the show-line **Show:** $\boxed{\neg\boxed{\mathbf{P(y)}}}$ (since its DRS is now included in the premise DRS) and with that – this is part of the Rule of Conditional Proof – also the show-line on the extreme left. So we end up with:



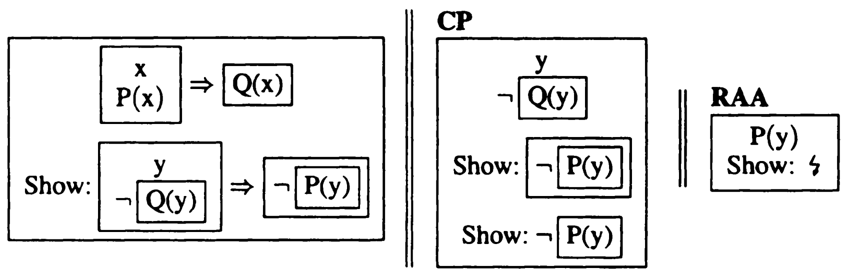
(37)

Note that for the success of the method of RAA it is immaterial what contradiction we derive. It is enough that it be some contradiction, which can be formally recognized as such – in the sense in which $\boxed{\begin{array}{c} Q(y) \\ \neg\boxed{Q(y)} \end{array}}$ can be so recognized since it involves two conditions, one of which is the negation of the other. More generally, we can define the notion of an explicitly contradictory DRS as follows.

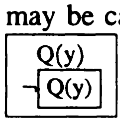
DEFINITION 14 A DRS K is called *contradictory* iff there are $\mu \subseteq U_K$, $\Gamma \subseteq Con_K$, and K' such that K' is an alphabetic variant of $\langle\mu,\Gamma\rangle$ and $\neg K' \in Con_K$.

We will adopt the policy that in order to prove a DRS or DRS-condition by means of RAA it suffices to derive from its contrary some DRS that is contradictory in the sense of this definition. In practice it is often not so easy to predict in advance which particular contradictory DRS it will be possible to derive. It will of course always be possible to see how the proof goes and then fill in the contradictory DRS in the show-line that RAA introduces once one can see which one it should be. But in practice it is more convenient to have a notational device that basically means "some contradictory DRS or other". We will use the symbol '↯' for this purpose.

Thus '**Show:** ϟ' means that one has to prove some contradictory DRS. This task has been completed, and the show-line may be cancelled as soon as the premise DRS includes some contradictory DRS. Using this device stage (35) of the proof we have just gone through will look like this:

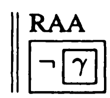


Cancellation happens in the transition from (36) to (37) just as before: **Show:** ϟ may be cancelled because the premise DRS contains a contradictory DRS (viz.



).

It is sufficient to formulate RAA only with respect to show-lines of the form **Show: K**. For suppose we had a proof stage S with a show-line **Show:** γ for some DRS-condition γ in AP(S), which we could only prove by a subproof of the form



(such as e.g. in the example above). Then we can also achieve this by introducing a new show-line **Show:** $\boxed{\gamma}$ into AP(S) to which we may then apply this restricted version of RAA. Notice that the two AP(S)s have exactly the same form. So if they succeed we may cancel **Show:** $\boxed{\gamma}$ which leads to an addition of γ to K(S). But this immediately allows us to cancel **Show:** γ by the Rule of Direct Proof.

It should be intuitively clear from the example we have discussed how the two parts of the Rule of RAA – its introduction and its cancellation – work in general. The rule's description in the box below should be clear enough for practical purposes. Nevertheless we extend Definition 13 with a pair of clauses corresponding to the introduction and cancellation step of the Rule of RAA.

DEFINITION 13 (continued)

(v) RAA

(a) *Introduction:*

Suppose S is a proof stage such that AP(S) contains a show-line of the form **Show: K**. And S′ results from S by adding immediately to the right of AP(S) a structure of the form

RAA

$\neg K$ Show: ↯

Then AP(S′) is the structure

RAA

$\neg K$ Show: ↯

,

PR(S′) = PR(S) ∪ $\overset{\circ}{\neg}$K and K(S′) = K(S) ∪ $\overset{\circ}{\neg}$K

(b) *Cancellation:* Suppose that S is a proof stage, that AP(S) has the form

RAA

$\neg K$ Show: ↯ ⋮

,

containing the show-line **Show:** ↯, and that **Show: K** is the show-line that justified the introduction of **Show:** ↯, through an application of (v.a). We obtain a proof stage S′ by cancelling this show-line. Let S″ be the first stage containing the show-line **Show: K**. Thus AP(S′) = AP(S″), PR(S′) = PR(S″) and K(S′) = K(S″) ∪ K.

We conclude with concise statements of the rules of CP and RAA, which lack the explicitness achieved by the relevant clauses of Definition 13, but which are easier to memorize.

Conditional Proof:

If Con_K contains the show-line **Show:** $\mathbf{K_1} \Rightarrow \mathbf{K_2}$, then we may introduce the structure

CP

K_1 Show: K_2

immediately to the right. When the show-line of this structure is cancelled, the show-line **Show:** $\mathbf{K_1} \Rightarrow \mathbf{K_2}$ may be cancelled as well.

Reductio Ad Absurdum:

Suppose Con_K contains the show-line **Show: $\mathbf{K_1}$**, then we may introduce introduce immediately to the right the structure

RAA

$$\boxed{\begin{array}{c} \neg K_1 \\ \text{Show: ↯} \end{array}}$$

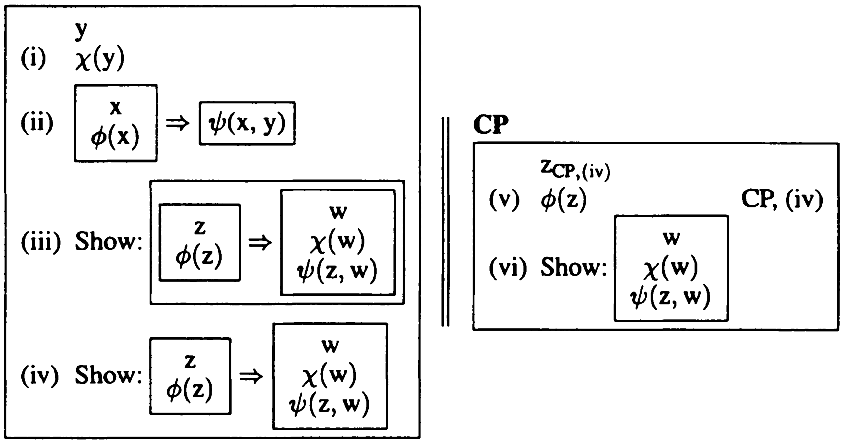
when the show-line of this structure is cancelled, the show-line **Show: $\mathbf{K_1}$** may be cancelled as well.

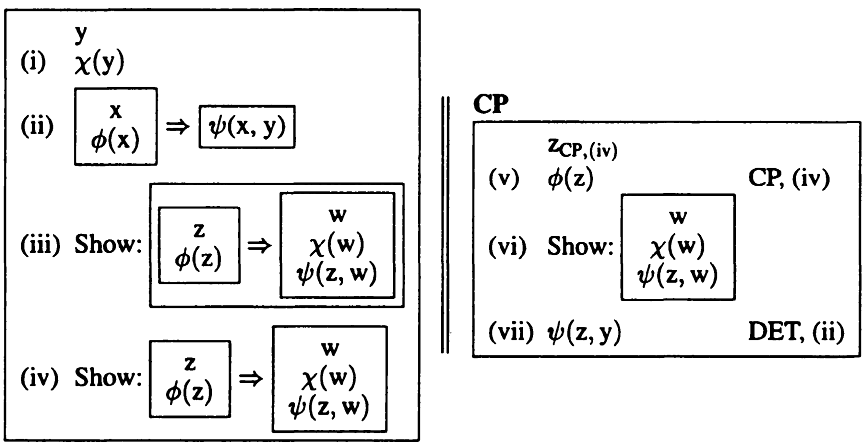
3.3. KEEPING SCORE

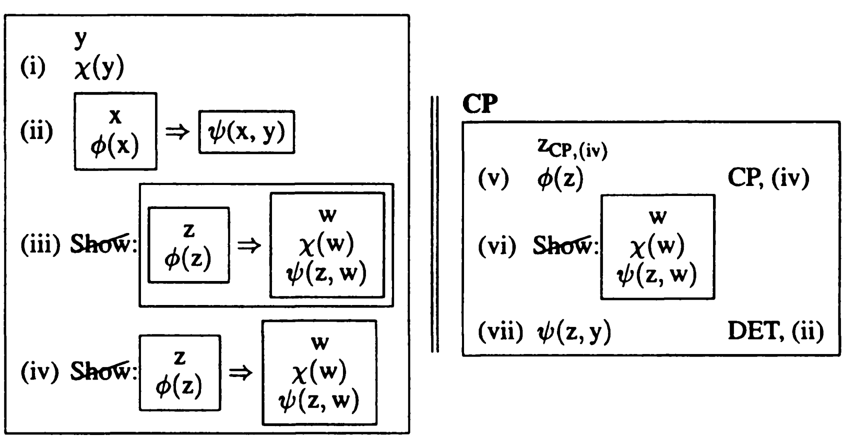
Many of the existing proof systems for predicate logic include notational devices to record which rule each of the steps in a proof employs and the exploited premises by this rule. This is so in particular for the natural deduction system of (Kalish and Montague 1964) and (Bonevac 1986) that have been the principal inspiration for the system developed here. For the present system, where the applications of certain rules are sometimes quite difficult to recognize for what they are because of the multiple term substitutions they involve, the need for such a device seems especially urgent, so we now proceed to introduce one.

First, we number all the DRS-conditions (including show-lines) as they appear in the proof beginning with those of the premise DRS in the order in which they are listed. We do this by writing the number of each condition in parentheses to its left. On the right hand side of those conditions which are obtained by application of an inference rule we indicate which rule this is, as well as the condition to which the rule has been applied. This same information – name of the rule plus number of the condition used as premise – is also attached to each of the discourse referents (if any) which the rule application introduces. To identify applications of the rules of proof we have been using a notational device already, that of writing the name of the rule – 'CP' or 'RAA' – at the top of the new column which is introduced when the proof structure required for the rule is set up. The following example should make clear how all this works.

$$\boxed{\begin{array}{l} \qquad y \\ \text{(i)}\ \ \chi(y) \\ \text{(ii)}\ \ \boxed{\begin{array}{c} x \\ \phi(x) \end{array}} \Rightarrow \boxed{\psi(x, y)} \\ \text{(iii) Show:}\ \boxed{\boxed{\begin{array}{c} z \\ \phi(z) \end{array}} \Rightarrow \boxed{\begin{array}{c} w \\ \chi(w) \\ \psi(z, w) \end{array}}} \end{array}}$$



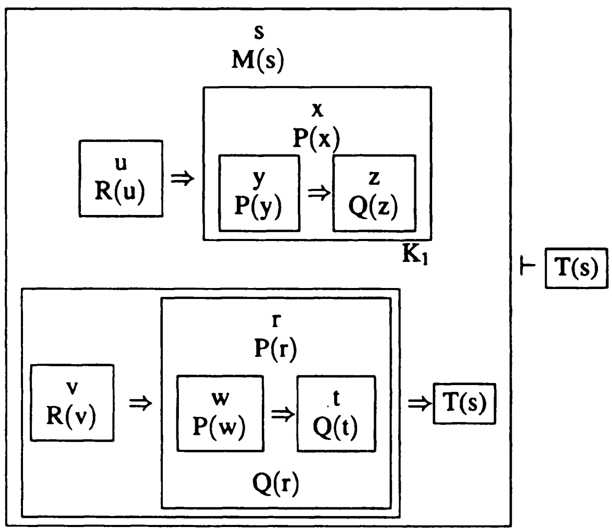




While the conventions introduced in this section have the merit of making fully explicit which rule applications a proof consists of, the extra notation has a tendency to clutter up derivations that would be perfectly transparent without it. Therefore one should in practice make a judicious rather than an exhaustive use of the new devices. In the remainder of this essay we will keep score only in this pragmatic manner.

4. Extended Inference Rules

As we have stated them, all the inference rules of our system other than 'non-empty universe' (thus: DET and DN) operate only at the level of the 'main' DRS K(S). Sometimes, however, it would be convenient to apply such a rule at some subordinate level, as for instance in the proof of the following argument



We can prove this very easily provided we are entitled to apply DET inside the sub-DRS K_1, thereby adding **Q(x)** to the conditions of K_1. Once this has been done only one further application of DET will complete the derivation. If, on the other hand, we do not apply DET within K_1, the proof is much more involved.

Intuitively application of an inference rule at a subordinate level should be all right. For suppose that a DRS K_2 is obtained from a DRS K_1 by an application of an inference rule (in the old sense, i.e. at the 'top' level). Then K_1 and K_2 are logically equivalent. So if we replace K_1 by K_2 in some larger DRS K then the resulting DRS K′ ought to be logically equivalent to K and thus the passage from K to K′ is valid.

Let us understand by an *extended application* of a given inference rule an application of that rule either at the level of the main DRS or at that of one of its sub-DRSs. And let us call *the extended proof system* that system in which extended applications of the basic inference rules are permitted (but which is like our old system in having the same inference rules and the same rules of proof). Then

THEOREM 1: Every argument that is provable in the extended system is provable in the old system.

The theorem rests on two lemmas, one to the effect that exchange of provably equivalent DRSs within some larger DRS preserves provable equivalence; and one to the effect that application of an inference rule at top level transforms a DRS into a provably equivalent one. To state these lemmas we have to extend the concept of a proof to include cases $K_1 \vdash K_2$ where $K_1 \cup K_2$ is an improper DRS. But this is straightforward. Where K_1 and K_2 are (possibly improper) DRSs, we define $K_1 \vdash K_2$ as holding just in case $K_1' \vdash K_2$, where $K_1' = \langle U_{K_1} \cup Fr(K_1) \cup Fr(K_2), Con_{K_1} \rangle$.

LEMMA 1: Suppose that K_1 and K_2 are (possibly improper) DRSs, that $K_1 \dashv\vdash K_2$, that K_1 is a sub-DRS of the DRS K, that K' results from replacing K_1 by K_2 in K and that both K and K' are pure. Then $K \dashv\vdash K'$.

Proof: We proceed by induction on the depth of embedding of K_1 within K.

Suppose that K_1, K_2, K and K' are as stated in the lemma. If the depth of embedding is 0, i.e. if $K_1 = K$, then $K' = K_2$ and there is nothing to prove. Suppose that K_1 is embedded in K at depth $n \geq 1$. Then K_1 is part of a complex condition γ, where $\gamma \in Con_{K_3}$ and K_3 is a sub-DRS of K (in the special case where $n = 1$, K_3 will coincide with K). γ will have one of the following three forms:

(i) $\neg K_1$

(ii) $K_1 \Rightarrow K_0$ (where K_0 is some other DRS)

(iii) $K_0 \Rightarrow K_1$

Let K_4 be the result of replacing K_1 by K_2 in K_3. Then we must show, for each of the cases (i)–(iii), that $K_3 \dashv\vdash K_4$. This proves the lemma. For we may argue as follows. Suppose that the lemma has been established for embeddings of depth n and that K_1 is embedded in K at depth $n + 1$. Then, since $K_3 \dashv\vdash K_4$, K_3 is embedded in K at depth n and K' is obtained from K by replacing K_3 by K_4, we may conclude $K \dashv\vdash K'$.

Case (i): K_1 occurs as part of a condition $\neg K_1$. Then K_3 has the form

$$\boxed{\begin{array}{c} x_1 \ldots x_k \\ B_1 \\ \vdots \\ B_m \\ \neg K_1 \end{array}}$$

To show $K_3 \vdash K_4$ it is enough to observe that we can construct a proof from K_3 as follows:

$x_1 \ldots x_k$
B_1
$\vdots$
B_m
$\neg K_1$
Show: $\neg K_2$
Show: $\boxed{\neg K_2}$

RAA

$\neg \boxed{\neg K_2}$
Show: $K_1'' \cup \mathring{\neg} K_1$
K_2'
~~Show~~: K_1' } P_1
$\vdots$

Here P_1 is a proof of a variant K_1' of K_1 from the variant K_2' of K_2. Such a proof exists in view of the fact that $K_2 \vdash K_1$.

The proof of K_3 from K_4 is analogous.

Case (ii): γ is $K_1 \Rightarrow K_0$. In this case K_3 has the form

$x_1 \ldots x_n$
B_1
$\vdots$
B_m
$K_1 \Rightarrow K_0$

To show that $K_3 \vdash K_4$ it suffices to complete a derivation beginning with:

$x_1 \ldots x_n$
B_1
$\vdots$
B_m
$K_1 \Rightarrow K_0$
Show: $K_2' \Rightarrow K_0'$

CP

K_2'
Show: K_0'

As in case (i) we can copy a proof of K_1' from K_2' on the right hand side. Then we can apply DET using K_1' and $K_1 \Rightarrow K_0$, so as to obtain (an alphabetic variant of) K_0'. This concludes the proof. Again the demonstration of $K_4 \vdash K_3$ is symmetrical.

Case (iii): This case is wholly analogous to case (ii).

This concludes the proof of **Lemma 1**.

LEMMA 2: Suppose that the DRS K' results from K through one application of DET, NEU or DN. Then $K \dashv\vdash K'$.

Proof: The proof is trivial. On the one hand $K' \vdash K$ since $K \subseteq K'$. On the other $K \vdash K'$, because by assumption K' results through an application of a valid proof rule to K.

This concludes the proof of **Theorem 1**.

Both the example and the theorem we have just proved concern only the case where all premises for the application of an inference rule in a subordinate DRS belong to this DRS. But this is unnecessarily restricted. When applying a rule to a sub-DRS K_1 of K it is legitimate to use not only premises that belong to Con_{K_1}, one might also use conditions and discourse referents that occur in any DRS K_2 which contains K_1 as a sub-DRS. To make this formally precise we define the *closure of* K_1 *in* K as follows: $CL(K_1, K) = \langle U, Con \rangle$, where

$$U = \bigcup_{K_1 \leq K_2 \leq K} U_{K_2}; Con = \bigcup_{K_1 \leq K_2 \leq K} Con_{K_2}$$

(where '$\leq$' stands for the relation 'sub-DRS of')

THEOREM 2:

(a) Suppose K_1 is a sub-DRS of K, that K_2 results from adding to K_1 the result of applying an inference rule to $CL(K_1, K)$, and let K′ be the result of replacing K_1 in K by K_2. Then $K \dashv\vdash K'$.

(b) Suppose that K contains a condition of the form

$$\boxed{\begin{array}{c} x_1 \ldots x_n \\ B \\ C_1 \\ \vdots \\ C_m \end{array}} \Rightarrow \boxed{\begin{array}{c} y_1 \ldots y_k \\ D_1 \\ \vdots \\ D_l \end{array}}$$

Suppose that we add an alphabetic variant B′ of B to the consequent box of this condition:

$$\boxed{\begin{array}{c} x_1 \ldots x_n \\ B \\ C_1 \\ \vdots \\ C_m \end{array}} \Rightarrow \boxed{\begin{array}{c} y_1 \ldots y_k \\ D_1 \\ \vdots \\ D_l \\ B' \end{array}}$$

Let K′ be the result of this change. Then once again $K \dashv\vdash K'$.

Theorem 2 follows from the following:

LEMMA 3: Suppose K is some DRS such that Con_K contains (i) a condition B and (ii) a complex condition γ which has one of the following forms: $\neg K_1$, $K_1 \Rightarrow K_0$, $K_0 \Rightarrow K_1$. Let K_2 result from adding some alphabetic variant B′ of B to Con_{K_1} and let K′ result from replacing K_1 in K by K_2. Then $K \dashv\vdash K'$.

Proof: Assume that γ is of the form $\neg K_1$. We prove only the interesting direction, which in this case is: $K' \vdash K$.

Suppose K is of the form

$$\begin{array}{|c|}\hline x_1 \ldots x_n \\ B \\ C_1 \\ \vdots \\ C_m \\ \neg K_1 \\ \hline \end{array}$$

and K_1 is of the form

$$\begin{array}{|c|}\hline y_1 \ldots y_k \\ D_1 \\ \vdots \\ D_l \\ \hline \end{array}$$

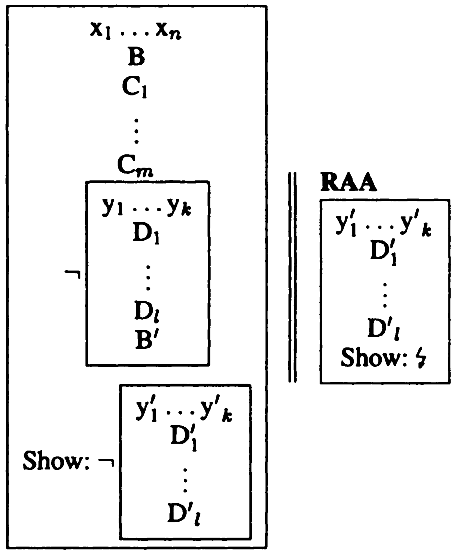
Then K_2 is of the form

$$\begin{array}{|c|}\hline y_1 \ldots y_k \\ D_1 \\ \vdots \\ D_l \\ B' \\ \hline \end{array}$$

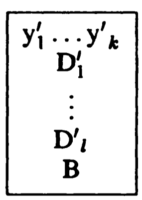
So K′ has the form

$$\begin{array}{|c|}\hline x_1 \ldots x_n \\ B \\ C_1 \\ \vdots \\ C_m \\ \neg \begin{array}{|c|}\hline y_1 \ldots y_k \\ D_1 \\ \vdots \\ D_l \\ B' \\ \hline \end{array} \\ \hline \end{array}$$

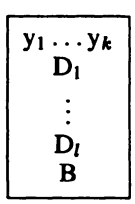
It suffices to complete the proof stage



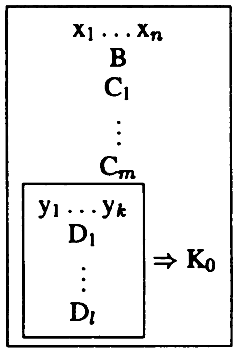
Note that this proof is already essentially complete, since



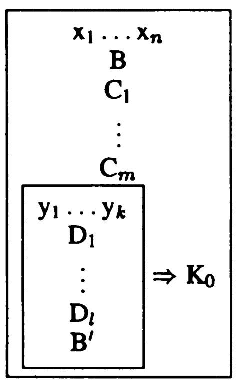
is an alphabetic variant of



Case (ii): K has the form

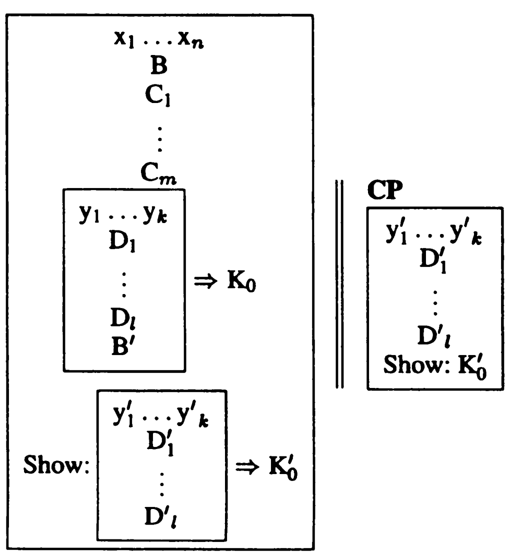


and K′ the form



Again we show $K' \vdash K$ and leave $K \vdash K'$ to the reader.

We must be able to complete the proof starting with



Note that by using DET we can get to some K_0'' from the displayed complex condition of K' the new premise on the right and the condition B of K'.

The proof of the case where γ is of the form $K_0 \Rightarrow K_1$ is left as an exercise.

Lemma 3 asserts that we may 'enrich' a given DRS K by carrying down any condition B into the constituent DRSs of complex conditions that occur in the same condition set as B. By repeating this procedure, we can "push" conditions arbitrarily far down into subordinate DRSs, so that they will be available as premises to extended applications of the inference rules in the sense of Theorem 1. This shows that any such premise may be used directly in extended applications.

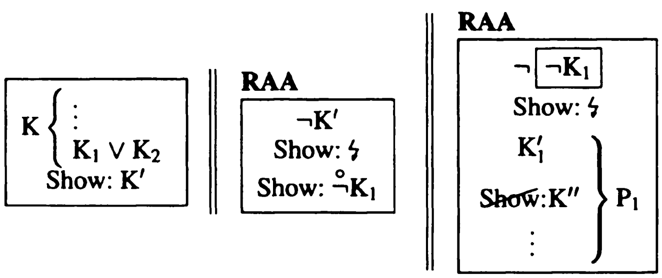
5. Disjunction and Identity

5.1. Disjunction

Suppose we have a premise DRS K containing a disjunctive condition and that we need to make use of this condition to prove some DRS K'. Intuitively one would proceed by cases: A disjunction is true iff at least one of its disjuncts is true. So in order to establish that K' follows from the disjunction (together with the remainder of K) it suffices to show that it follows from each of the disjuncts (together with the remainder of K). This method of proving things from disjunctive premises is known as the *Method of Proof by Cases*. Many proof systems for predicate logic contain a corresponding rule of proof. However, we have opted to follow (Bonevac 1986) in going a slightly different route, and to introduce instead a rule of inference, called **M**(odus) **T**(ollendo) **P**(onens). (This will have the effect of keeping the number of rules of proof down, something that turns out to be of some advantage when one wants to demonstrate soundness and completeness of the system, as we will do in Section 7.) The rule of MTP allows us to infer from a given disjunction $K_1 \vee \ldots \vee K_{i-1} \vee K_i \vee K_{i+1} \vee \ldots \vee K_n$ and the negation of one of the disjuncts, K_i, the shorter disjunction $K_1 \vee \ldots \vee K_{i-1} \vee K_{i+1} \vee \ldots \vee K_n$. In the case where the disjunction consists of only two disjuncts, the effect is that just the other disjunct remains. Then from $K_1 \vee K_2$ and $\neg K_1$ MTP enables us to infer K_2. We will concentrate on this special case to argue (informally) that MTP allows us to deduce whatever can be deduced with the Method of Proof by Cases.

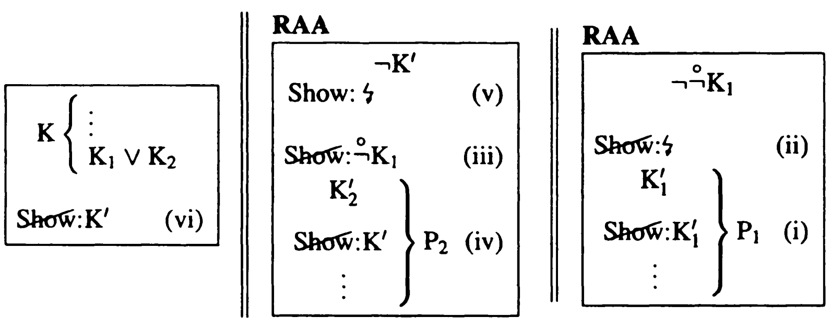
Suppose we want to deduce a DRS K' from a premise DRS K which contains the disjunctive condition $K_1 \vee K_2$. Proof by Cases allows us to carry out this deduction by deriving K' first from $K_0 \cup K_1$ and then from $K_0 \cup K_2$, where K_0 is the remainder of K (the part of K which remains when $K_1 \vee K_2$ is taken out). Suppose that we have a proof P_1 of $K_0 \cup K_1 \vdash K'$ and a proof P_2 of $K_0 \cup K_2 \vdash K'$.

To obtain a proof of K′ from K which uses MTP instead of Proof by Cases, we proceed by RAA. That is, starting from the proof stage



where K″ is an alphabetic variant of K′.

At this stage we have reached an explicit contradiction, involving the DRS K′ of the cancelled show-line on the right and the condition ¬K′ in the middle. This enables us to cancel the show-line **Show:** ↯ on the right and, with that, the show-line **Show:** $\overset{\circ}{\neg}$**K**$_1$ in the middle. MTP then permits us to infer K_2 (from the DRS of this last cancelled show-line and the disjunctive condition $K_1 \vee K_2$). Inserting the proof P_2 then gives K′, which forms another explicit contradiction with the condition ¬K′. This permits cancellation of the middle show-line **Show:** ↯, which completes the proof:



(The roman numerals indicate the order in which the show-lines get cancelled.)

The box below contains an explicit statement of MTP.

M(odus) **T**(ollendo) **P**(onens):

Suppose K contains a condition of the form $K_1 \vee \ldots \vee K_n$ together with a condition of the form $\neg K'_1$, where K'_1 is an alphabetic variant of K_1. Then we may add $K_2 \vee \ldots \vee K_n$ to K.

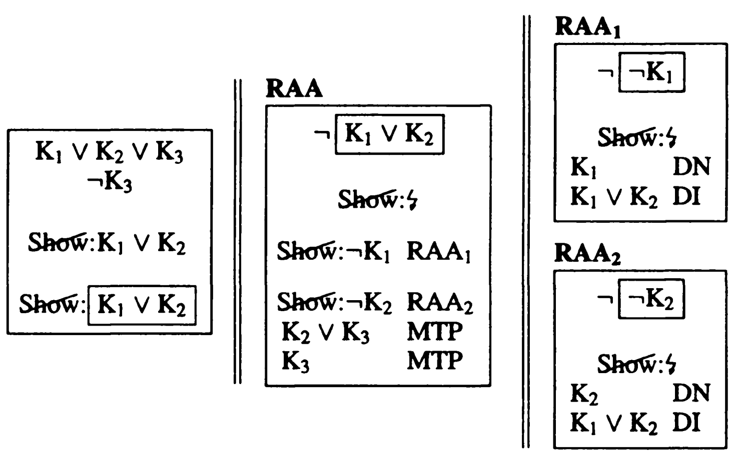
Besides MTP, which enables us to exploit disjunctions as premises, we also need a principle that allows for the deduction of disjunctive conditions. A particularly simple principle that does this is the one according to which a disjunction can be inferred from any one of its disjuncts. The corresponding inference rule, the rule of Disjunction Introduction (DI), can be stated as follows:

D(isjunction) I(ntroduction):

Let $K_1 \vee \ldots \vee K_n$ be a disjunctive condition and suppose that one of its disjunctions K_i is included in K. Then we may add $K_1 \vee \ldots \vee K_n$ to the condition set of K.

The immediate inferences which Disjunction Introduction allows us to draw are trivial to a degree which would make one suspect that additional principles for inferring disjunctions are needed. However, as we will see in Section 7, this is not so: MTP and DI are all we need in addition to the rules of Sections 3 and 4 to derive all valid arguments within the DRS language which contains disjunction as well as the constructions with which we dealt before Section 6.

Note that MTP is formulated with respect to the first disjunct only. We close this section with a sample proof, which shows that this formulation of MTP entails the more general version (which permits extraction of a disjunct whose negation is also present from other positions in a disjunction). The argument we give for disjuncts of length 3. We leave it to the reader to generalize this to disjuncts of arbitrary length.



5.2. IDENTITY

The condition $x = y$ is verified by an embedding f iff $f(x) = f(y)$. This means that if γ is any DRS-condition, neither x nor y is among the declared discourse referents of γ and γ' results from replacing zero or more occurrences of x by y in γ, then f verifies γ' iff it verifies γ. This principle constitutes one of the two rules pertaining to identity which must be added to our proof system to cover the cases of validity which depend on =. We adopt it in the form of an inference rule, the rule of *Substitution of Identicals*, which we state as follows:

Substitution of Identicals:

Suppose K contains conditions $\mathbf{x} = \mathbf{y}$ and γ such that $\mathbf{x}, \mathbf{y} \notin \underline{U}(\gamma)$. Then we may add γ' to K, where γ' results from γ by replacing one occurrence of $\mathbf{x}$ by $\mathbf{y}$.

The second principle we use rests on the fact that a condition of the form $\mathbf{x} = \mathbf{x}$ is verified by any embedding whatever (for $f(\mathbf{x})$ is always equal to $f(\mathbf{x})$!). This means that such conditions may always be added to the active part of a proof stage without running the risk that verifiability is thereby lost. We adopt this principle as a kind of degenerate inference rule, of the form

Selfidentity:

For any $\mathbf{x} \in U_K$ we may add $\mathbf{x} = \mathbf{x}$ to K.

(This rule is degenerate in that it does not involve any premises. Such degenerate rules are often called *axioms*.)

It is easily verified that the following arguments are derivable with the help of these new principles.

(i) $\boxed{\begin{array}{c} x \;\; y \\ x = y \end{array}} \vdash \boxed{y = x}$

(ii) $\boxed{\begin{array}{c} x \;\; y \;\; z \\ x = y \\ y = z \end{array}} \vdash \boxed{x = z}$

(iii) $\boxed{\begin{array}{c} x_1, \ldots, x_i, \ldots, x_n \;\; y \\ x_i = y \\ P(x_1, \ldots, x_i, \ldots, x_n) \end{array}} \vdash \boxed{P(x_1, \ldots, y, \ldots, x_n)}$

This completes the proof system for the full DRS-language defined in Definition 1. In the next section we show that this system proves all and only those arguments that are valid.

5.3. Derived Rules and Redundancies

Every argument $K \vdash K'$ that is derivable within our proof system can be interpreted as a *derived inference rule*. For suppose that the predicates occurring in K and K′ are $P_1^{n_1}, \ldots, P_k^{n_k}$ where n_i gives the number of arguments of the predicate $P_i^{n_i}$. Furthermore let for $i = 1, \ldots, k$ γ_i be a DRS-condition with $Fr(\gamma_i) = \{v_{n_1}, \ldots, v_{n_i}\}$, where $v_1, v_2, \ldots$ is a fixed enumeration of some denumerable subset of V, and let K^S and K'^S be the result of replacing every atomic condition $P_i^{n_i}(u_1, \ldots, u_{n_i})$ by

$\gamma_i[u_1,\ldots,u_{n_i}]$ (where $\gamma_i[u_1,\ldots,u_{n_i}]$ is the result of replacing in γ_i v_1 by u_1, v_2 by u_2 etc.). Then $K^S \vdash K'^S$ – in fact, the given derivation of $K \vdash K'$ can be converted into a derivation of $K^S \vdash K'^S$ by carrying out the same substitutions of γ's for P's throughout. Thus the argument $K \vdash K'$ can be understood as a schema for inferring DRSs of the *form of* K′ (i.e. DRSs which can be obtained from K′ through series of substitutions of the kind just described) from DRSs which include corresponding substitution instances of K.

In practice it will often be useful to store arguments one has already proved (in the head or in the memory of a computer) so that they can serve as additional inference rules to be used in further derivations.

From a theoretical point of view the resulting proof systems which have been expanded through the addition of derived rules are unattractive: there is theoretical virtue – at the very least the virtue of parsimony – in systems that posit no more rules than they strictly need. The term for systems lacking this virtue – they are standardly referred to as *redundant* – succeeds quite well in convoying the negative connotation such systems carry for the typical mathematician or logician.

In fact, considerable efforts have been made in the course of the modern history of logic to show that given proof systems were free of redundancy or, if not, to reduce them to systems that are.

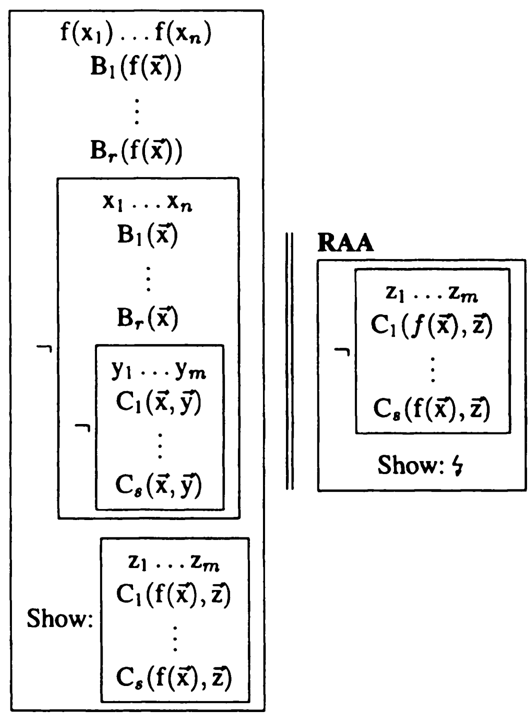
When a system is redundant it is not always possible to distinguish which rules are to be considered its derived rules and which its non-derived ones. For instance, there are systems which turn out to be redundant in that they have two rules, R_1 and R_2, such that R_1 is a derived rule of the system $S-\{R_1\}$ and R_2 is a derived rule of $S-\{R_2\}$. How S is to be reduced to a redundancy free system – whether by eliminating R_1 or by eliminating R_2 – is often not clear and must be decided by additional considerations, if these can be found.

The system we have presented in Sections 3 and 4 is redundant too; in fact, we already admitted as much in Section 3, when we announced that the rule of Double Negation would become redundant once the rule of RAA would be added. To see that this is so, we argue as follows. Suppose that Con_K contains the condition

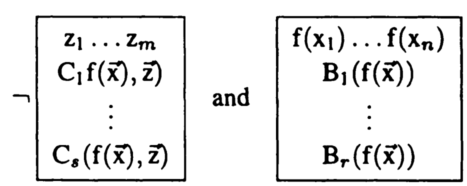
$$\neg \begin{array}{|c|} \hline x_1 \ldots x_n \\ B_1(\vec{x}) \\ \vdots \\ B_r(\vec{x}) \\ \neg \begin{array}{|c|} \hline y_1 \ldots y_m \\ C_1(\vec{x},\vec{y}) \\ \vdots \\ C_s(\vec{x},\vec{y}) \\ \hline \end{array} \\ \hline \end{array} \qquad (38)$$

and that there is a function f defined on $\{x_1, \ldots, x_n\}$ which converts the conditions B_i into conditions B'_i in Con_K. DN says that we may then add new discourse referents $\mathbf{z}_1, \ldots, \mathbf{z}_m$ to U_K and to Con_K the conditions $C_j(g(\vec{\mathbf{x}}),g(\vec{\mathbf{y}}))$, where $g(\mathbf{x}_i) = f(\mathbf{x}_i)$ and $g(\mathbf{y}_j) = \mathbf{z}_j$.

We can obtain this same inference without an appeal to DN by using RAA:



It is clear that the right hand show-line may be cancelled (and with it the one on the left), since the union of



explicitly contradicts the premise (38) of DN within Con_K.

The notion of a derived rule can also be applied to rules of proof. Formally the criterion for derivability of the rule is the same as it is in the case of inference rules: Every proof in the system S′, which we get by adding the rule to S (if it is not part of S already) can be replaced by a proof of the same argument in which the rule is not used. An example of a derived rule of proof is the rule of Proof By Cases, which we discussed in Section 7.1.

6. Soundness and Completeness

6.1. Soundness

A good proof system is one that (i) proves nothing that it should not prove, and (ii) proves everything that it should.

The first property is called soundness. A proof system $\mathcal{PS}$ is sound if it proves every valid argument – in our notation: $\mathcal{PS}$ is *sound* iff whenever $\mathcal{PS}$ proves $K \vdash K'$ it is the case for every model M and embedding f that if $M \models_f K$ then for some extension g of f $M \models_g K'$. The second property, *completeness*, is the converse of the first: $\mathcal{PS}$ is *complete* iff $K \vdash K'$ for every K, K′ such that if $M \models_f K$ then for some $g \supseteq f$ $M \models_g K'$.

We first show that our proof system is sound. We will make use of the following abbreviation: if f is an embedding and g extends f to a set of discourse referents X, i.e. dom(g) = dom(f) ∪ X, we write $f \subseteq_X g$.

The states of being explicitly and implicitly contradictory are subsumed under a single concept, that of *inconsistency*:

DEFINITION 15 A DRS K is inconsistent iff for some DRS K_1 $K \vdash K_1 \cup \overset{\circ}{\neg} K_1'$, where K_1' is an alphabetic variant of K_1.

A DRS which is not inconsistent is called *consistent*.

We proceed by induction on the length of proofs. That is, we prove by induction for each of the successive proof stages of the given proof that the premises at S semantically entail K(S).

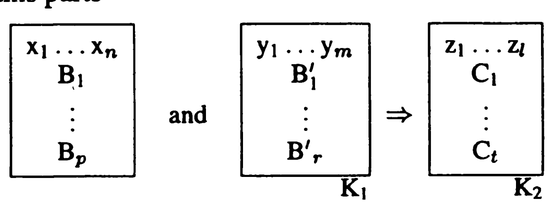
LEMMA 1: If S_n is any stage of a proof from some DRS K, M is any model and f any embedding of $U_{PR(S_{n+1})}$ into M such that $M \models_f PR(S_{n+1})$, then there is a $g \supseteq f$ such that $M \models_g K(S_{n+1})$.

Proof

1. Stage "zero". Then PR(K) = K(K) = K.
   Since $K \vdash K$ there is nothing to be proved.
2. S_{n+1} results through application of an inference rule. Here we have to look at each of the inference rules separately.

   (a) DET

   Suppose that S_{n+1} results from S_n through an application of DET. So $K(S_n)$ contains parts



while S_{n+1} results from S_n through adding a DRS K_3 of the form

$$\boxed{\begin{array}{c} z'_1 \ldots z'_l \\ C'_1 \\ \vdots \\ C'_t \end{array}}$$

More precisely, there will be a function F: $\{y_1, \ldots, y_m\} \to \{x_1, \ldots, x_n\}$, such that $F(B'_i) = B_i$ $(i = 1, \ldots, r)$, $z'_1, \ldots, z'_l$ are new discourse referents and $C'_j = G(C_j)$ where G is the function on $\{y_1, \ldots, y_m, z_1, \ldots, z_l\}$ such that $G(y_i) = F(y_i)$ and $G(z_j) = z'_j$.
Let M be a model and f an embedding from $U_{PR(S_{n+1})}$ into M such that $M \models_f PR(S_{n+1})$. So, since $PR(S_{n+1}) = PR(S_n)$, $M \models_f PR(S_n)$. By induction hypothesis there is a $g \supseteq f$ with $Dom(g) \supseteq U_{K(S_n)}$ such that $M \models_g K(S_n)$. In this case $K(S_{n+1}) = K(S_n) \cup K_3$, so it suffices to show that g can be extended to a function h with $Dom(h) = Dom(g) \cup \{z'_1, \ldots, z'_l\}$ such that $M \models_h K_3$.
The existence of such an h can be shown as follows. Since $M \models_g K(S_n)$, $M \models_g K_1 \Rightarrow K_2$. So for any $e \supseteq_{U_{K_1}} g$ such that $M \models_e K_1$ there is a $d \supseteq_{U_{K_2}} e$ such that $M \models_d K_2$. Now let e_0 be that extension of g such that $e_0(y_i) = g(F(y_i))$, and let $d_0 \supseteq_{U_{K_2}} e_0$ such that $M \models_{d_0} K_2$. Let $h \supseteq_{\{z'_1,\ldots,z'_l\}} g$ be that function such that $h(z'_i) = d_0(z_i)$ for $i = 1, \ldots, l$. Then $M \models_h C'_j$ iff $M \models_{d_0} C_j$. So, since $M \models_{d_0} C_j$, $M \models_h C'_j$. As this holds for $j = 1, \ldots, t$, $M \models_h K_3$.

(b) NEU

Obvious: Map extra discourse referent into some element of U_M (since by assumption $U_M \neq \{\}$, this will always be possible).

(c) MTP

Suppose that S_{n+1} results from S_n through an application of MTP. So $K(S_n)$ contains parts $(p \leq r, n \leq m)$

$$\neg \underbrace{\boxed{\begin{array}{c} x_1 \ldots x_n \\ B_1 \\ \vdots \\ B_p \end{array}}}_{K_0} \quad \text{and} \quad \underbrace{\boxed{\begin{array}{c} y_1 \ldots y_m \\ B'_1 \\ \vdots \\ B'_r \end{array}}}_{K_1} \quad \vee K_2 \vee \ldots \vee K_n$$

while S_{n+1} results from S_n through adding $K_2 \vee \ldots \vee K_n$.
It is easily seen that $M \models K(S_{n+1})$: Since $M \models \neg K_0$ and because K_0 is embeddable into K_1 $M \models K_1 \vee \ldots \vee K_n$ can only hold on the basis of $K_2 \vee \ldots \vee K_n$.

The cases of **DI**, **SI** and **SUI** are obvious. This concludes 2.

3. The next case to be considered concerns the introduction of show-lines. Strictly speaking we must distinguish three cases here, depending on whether the

introduction involves direct proof, conditional proof or RAA. All cases are essentially trivial, however. The reason is that in each of them the change from $K(S_n)$ to $K(S_{n+1})$ is identical to that from $PR(S_n)$ to $PR(S_{n+1})$. In the case of direct proof we have $K(S_{n+1}) = K(S_n)$ and $PR(S_{n+1}) = PR(S_n)$ whereas in the other two $K(S_{n+1}) = K(S_n) \cup K'$ and $PR(S_{n+1}) = PR(S_n) \cup K'$, where K' is the new assumption that heads the new subderivation. In each case the inductive hypothesis – if $M \models_f PR(S_n)$ then there is $g \supseteq_{U_{K_1}} f$ such that $M \models_g PR(S_n)$ carries over directly to S_{n+1}.

4. S_{n+1} results from S_n through the cancellation of a show-line by the method of direct proof. Then $K(S_{n+1}) = K(S_n) \cup K'$, where K' is the DRS in the cancelled show line. But if the line is cancelled by the method of direct proof, then $K(S_n)$ includes an alphabetic variant K'' of K'.
Suppose $M \models_f PR(S_{n+1})$. Since $PR(S_{n+1}) = PR(S_n)$, we infer by induction hypothesis that for some $g \supseteq_{K(S_n)} f$, $M \models_g K(S_n)$. Since $K'' \subseteq K(S_n)$, $M \models_g K''$. Moreover, for some function α K'' is the alphabetic variant of K' under α. Since $U_{K'} \cap U_{K(S_n)} = \{\}$, $g' = g \cup \{\langle\alpha^{-1}(y), g(y)\rangle : y \in U_{K''}\}$ is a function, and in particular it is an embedding of $K(S_n) \cup K'$ into M. Since $M \models_g K''$ and K'' is alphabetic variant of K' under α, $M \models_g K'$ by the remark on page 305 concerning equivalence of alphabetic variants. So since $K(S_{n+1}) = K(S_n) \cup K'$, $\exists g' \supseteq_{K(S_n)} f$ such that $M \models_{g'} K(S_{n+1})$.
5. Cancellation of show line on the strength of completed CP:

⋮ ~~Show~~: $K_1 \Rightarrow K_2$	**CP** K_1 ~~Show~~: K_2

$PR(S_{n+1}) = PR(S_r)$ where S_r is the last stage before the subderivation with K_1 as extra assumption started. So $PR(S_n) = PR(S_r) \cup K_1 = PR(S_{n+1}) \cup K_1$.
Suppose $M \models_f PR(S_{n+1})$. We must show $M \models_f K_1 \Rightarrow K_2$. Suppose $g \supseteq_{U_{K_1}} f$ such that $M \models_g K_1$. Then $M \models_g PR(S_n)$. So by induction hypothesis there is an h such that $M \models_h K(S_n)$. So in particular $M \models_h K_2$. Since this holds for arbitrary $g \supseteq_{U_{K_1}} f$, it follows that $M \models_f K_1 \Rightarrow K_2$.
6. Cancellation on the strength of a completed RAA:

⋮ ~~Show~~: K_1	**RAA** $\neg K_1$ ~~Show~~: $K_2 \cup \langle\{\}, \{\neg K_2'\}\rangle$

In this case $PR(S_n) = PR(S_{n+1}, K) \cup \langle\{\}, \{\neg K_1\}\rangle$.
Suppose $M \models_f PR(S_{n+1})$. We must show that there exists a $g \supseteq_{U_{K_1}} f$ such that $M \models_g K_1$. Suppose there is no such g. Then by definition of verification $M \models_f \neg K_1$. So $M \models_f PR(S_n)$. So by induction hypothesis there is $h \supseteq f$ such that $M \models_h K(S_n)$. But $K(S_n)$ contains the alphabetic variants of the parts K_2

and $\neg K_2'$. But no embedding can simultaneously verify both of these. So for no function h $M \models_h K(S_n)$. So it can not be true that $M \models_f \neg K_1$. So there is a $g \supseteq_{U_{K_1}} f$ such that $M \models_g K_1$.

6.2. COMPLETENESS

We now proceed to show that our proof system allows us to prove all valid arguments. Together with the already proved soundness result this will establish that the system permits us to derive exactly those arguments which intuitively it should.*

We have to show that (39) holds for any K_1, K_2 such that $(K_1 \cup K_2)$ is pure:

$$\text{if } K_1 \models K_2, \text{ then } K_1 \vdash K_2. \quad (39)$$

To prove this we show (following an insight originally due to Gödel) the converse, viz. that if $K_1 \nvdash K_2$, then $K_1 \not\models K_2$, or

if $K_1 \nvdash K_2$, then there is a model M and an embedding f of U_{K_1} into U_M, such that $M \models_f K_1$ and for no $g \supseteq_{U_{K_2}} f$ $M \models_g K_2$. (40)

(40) can be transformed further into

if K is consistent, then there is a model M and an embedding f from U_K into U_M such that $M \models_f K$. (41)

That (40) and (41) are equivalent can be seen as follows. First, assume (40), and suppose that K is consistent. Let $K_1 \cup \mathring{\neg} K_1'$ be some explicit contradiction (with K_1' an alphabetic variant of K_1). Then, since K is consistent, $K \nvdash K_1 \cup \mathring{\neg} K_1'$. So by (40) there are M and f such that $M \models_f K$ and for no $g \supseteq_{U_{K_2}} f$ $M \models_g K_1 \cup \mathring{\neg} K_1'$. So in particular $M \models_f K$. Conversely, assume (41) and suppose that $K_1 \nvdash K_2$. Then $K_1 \cup \mathring{\neg} K_2'$ is consistent. For suppose not. Then there is a proof beginning with a stage of the form

K_1 { ⋮ ⋮ }
$\neg K_2$
Show: $K_3 \cup \mathring{\neg} K_3'$

But then the proof beginning with

K_1 { ⋮ ⋮ }
Show: K_2

‖ **RAA**
$\neg K_2$
Show: $K_3 \cup \mathring{\neg} K_3'$

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* Throughout this section we use the term DRS to denote *non-extended* DRSs – in other words, DRSs in the sense of Definition 1.

can also be completed, establishing $K_1 \vdash K_2$. Since $K_1 \cup \overset{\circ}{\neg} K_2$ is consistent, there are by (41) M and f such that $M \models_f K_1 \cup \overset{\circ}{\neg} K_2$. So $M \models_f K_1$ and for no $g \supseteq_{U_{K_2}} f$ $M \models_g K_2$.

To show (41) we proceed as follows. Starting with a given consistent DRS K we expand K to a gigantic DRS K_ω in which, roughly speaking, every finite DRS K is "decided", i.e. for every DRS K′ either $K'' \subseteq K_\omega$, or else $\overset{\circ}{\neg} K'' \subseteq K_\omega$, where K″ is some alphabetic variant of K′. K_ω can be used to define a model M and embedding f such that $M \models_g K_\omega$. Since $K \subseteq K_\omega$, $M \models_g K$.

To construct K_ω we proceed in a denumerably infinite number of stages. At each stage we "decide" one of the infinitely many possible DRSs, in that we either add (an alphabetic variant of) the DRS in question, call it K_i, to the DRS under construction, or else add some alphabetic variant of $\overset{\circ}{\neg} K_i$. However, to make things work out just right we have to make a few special provisions. First, it will be convenient to assume that the total set of discourse referents R in which our DRS language is based and the sets $\mathbf{P}^n$ of n-place predicates are all denumerable. (Inspection of the proof will show that this restriction is inessential.) For any particular pair of finite DRSs K, K′ will contain only finitely many discourse referents and predicates. So even if R or some of the sets $\mathbf{P}^n$ were non-denumerable, the proof we will give can be used to show (21) for the given K and K′, by applying it to a collection of denumerable subsets of R and the P_i which include all the discourse referents and predicates of K and K′.

Second, given that R is denumerable we may assume that its members are given by a particular enumeration $\{\mathbf{u}_1, \mathbf{u}_2, \dots\}$. We divide this enumeration into two infinite halves (say, by putting the even-indexed discourse referents $\mathbf{u}_{2n}$ into the first half, and the odd-indexed ones, $\mathbf{u}_{2n+1}$, into the second). For mnemonic purposes we will refer to the discourse referents of the first group, C, as $\mathbf{c}_1, \dots, \mathbf{c}_n, \dots$ (i.e. $\mathbf{c}_n =_{def} \mathbf{u}_{2n}$) as they will play the role of names or *constants*, and to those in the second group, V, as $\mathbf{v}_1, \dots, \mathbf{v}_n, \dots$ (i.e. $\mathbf{v}_n =_{def} \mathbf{u}_{2n-1}$) since they will be used essentially like the usual variables of standard predicate logic.

We want to decide each (improper) DRS K with the special property that $Fr(K) \subseteq C$ and $\underline{U}(K) \subseteq V$. Since both C and V are denumerable and our DRS language also has only denumerably many predicates, it is possible to arrange all such DRSs in a sequence, $E_1, E_2, \dots, E_n, \dots$ (so that for each DRS K there is an n such that K is the DRS E_n!).

Let K be any pure, proper and consistent DRS. Let K′ be a pure alphabetic variant of K such that $U_{K'} \subseteq C$ and $\underline{U}(K') \backslash U_{K'} \subseteq V$. Clearly such a K′ exists, and it is also clear that if there are M, f such that $M \models_f K'$, then this is also the case for K. So it suffices to show that

$$\text{for some } M, f{:}U_{K'} \to U_M \text{ it is the case that } M \models_f K'. \qquad (42)$$

We do this by extending K′ to a DRS K_ω with the following properties:

(i) $U_{K_\omega} \subseteq C$

(ii) $\underline{U}(K_\omega)\backslash U_{K_\omega} \subseteq V$

(iii) For every finite, non-extended K′ such that $Fr(K') \subseteq C$ either (a) or (b):

(a) for some 1–1 function d from $U_{K'}$ into C $d(K')' \subseteq K_\omega$ where $d(K')'$ is an alphabetic variant of $d(K')$, (for $d(K')$ see Definition 9)

(b) $\overset{\circ}{\neg}K'' \subseteq K_\omega$ for some alphabetic variant K″ of K′.

(iv) For every K′ of the form $\langle\{\}, \{K_1 \Rightarrow K_2\}\rangle$ such that $\overset{\circ}{\neg}K' \subseteq K_\omega$ there is a 1–1 function d from U_{K_1} into C such that $d(K_1)' \subseteq K_\omega$ and $\overset{\circ}{\neg}d(K_2)' \subseteq K_\omega$, where $d(K_i)'$ is an alphabetic variant of $d(K_i)$ $(i = 1, 2)$.

(v) For every K′ of the form $\langle\{\}, \{K_1 \vee \ldots \vee K_n\}\rangle$ such that $K' \subseteq K_\omega$ there are conditions $\neg K'_1, \ldots, \neg K'_n$ in K_ω, where K'_i is an alphabetic variant of K_i $(i = 1, \ldots, n)$.

(vi) K_ω is consistent

(vii) K_ω is pure and proper

To obtain K_ω we proceed in the following way. We construct an infinite sequence $K_0, K_1, K_2, \ldots$ of increasing DRSs (i.e. $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$), and let K_ω be the union of these: $K_\omega = \bigcup_{i \in \omega} K_i$. The sequence $K_0, K_1, K_2, \ldots$ is defined as follows:

(i) $K_0 = K'$

(ii) K_{n+1}: Let E'_{n+1} be some alphabetic variant of E_{n+1} such that $K_n \cup E'_{n+1}$ is pure. Then

(a) Suppose $K_n \cup E'_{n+1} \cup \langle Fr(E'_{n+1}), \{\}\rangle$ is consistent. Let d be a 1–1 map from $U_{E'_{n+1}}$ into C such that the discourse referents in Ran(d) occur neither in K_n nor in E'_{n+1}.* Put

$$K_{n+1} = K_n \cup d(E'_{n+1}) \cup \langle Fr(E'_{n+1}), \{\}\rangle^{**}$$

(b) $K_n \cup E'_{n+1} \cup \langle Fr(E'_{n+1}), \{\}\rangle$ is inconsistent and E_{n+1} is neither of the form $\langle\{\}, \{K_1 \Rightarrow K_2\}\rangle$ nor of the form $\langle\{\}, \{K_1 \vee \ldots \vee K_n\}\rangle$; then

$$K_{n+1} = K_n \cup \overset{\circ}{\neg}E'_{n+1} \cup \langle Fr(E'_{n+1}), \{\}\rangle$$

(c) $K_n \cup E'_{n+1} \cup \langle Fr(E'_{n+1}), \{\}\rangle$ is inconsistent and E_{n+1} is of the form $\langle\{\}, \{K_1 \Rightarrow K_2\}\rangle$. Let d be a 1–1 map from U_{K_1} into C such that the discourse referents in Ran(d) occur neither in K_n nor in E'_{n+1}; then

$K_{n+1} = K_n \cup \overset{\circ}{\neg}E'_{n+1} \cup d(K_1)' \cup \overset{\circ}{\neg}d(K_2)' \cup \langle Fr(E'_{n+1}) \cup Ran(d), \{\}\rangle$, where $d(K_i)'$ is an alphabetic variant of $d(K_i)$ such that K_{n+1} is pure $(i = 1, 2)$.

(d) $K_n \cup E'_{n+1} \cup \langle Fr(E'_{n+1}), \{\}\rangle$ is inconsistent and E_{n+1} is of the form $\langle\{\}, \{K_1 \vee \ldots \vee K_n\}\rangle$. Let d be a 1–1 map from $\bigcup_{i \in n} U_{K_i}$ into C such that the discourse referents in Ran(d) occur neither in K_n nor in E'_{n+1}; then

* N.B. the particular choice of the alphabetic variant E'_{n+1} and of the function d do not matter; they could be made fully precise, e.g. by taking the "first" variant of E_{n+1} and the first such function d in the sense of our enumeration of V, C and the enumeration $\{E_i\}$; but we won't bother to do this.

** The last member of this union, $Fr(E'_{n+1})$, is there to insure that all discourse referents occurring free in E_{n+1} will become part of the universe of K_{n+1}, so that K_{n+1} is a proper DRS.

$K_{n+1} = K_n \cup \overset{\circ}{\neg}E'_{n+1} \cup \overset{\circ}{\neg}d(K_1)' \cup \ldots \cup \overset{\circ}{\neg}d(K_n)' \cup$
$\langle Fr(E'_{n+1}) \cup Ran(d), \{\}\rangle$,
where $d(K_i)'$ is an alphabetic variant of $d(K_i)$ such that K_{n+1} is pure $(i = 1, \ldots, n)$.

Before we proceed further we state five simple lemmas. All relevant DRSs are assumed to be pure but not necessarily proper.

LEMMA 2: $K \cup K_1$ inconsistent iff $K \vdash \overset{\circ}{\neg}K_1$.

LEMMA 3: $\langle\{\}, \{K_1 \Rightarrow K_2\}\rangle \dashv\vdash \langle\{\}, \{\neg(K_1 \cup \langle\{\}, \{\neg K_2\}\rangle)\}\rangle$.

LEMMA 4: $\langle\{\}, \{K_1 \vee \ldots \vee K_n\}\rangle \dashv\vdash \overset{\circ}{\neg}\langle\{\}, \{\neg K_1, \ldots, \neg K_n\}\rangle$

LEMMA 5: If $K \vdash K'$ and $K' \vdash K''$ then $K \vdash K''$.

LEMMA 6: Suppose K'' is an alphabetic variant of K' and both $K \cup K'$ and $K \cup \overset{\circ}{\neg}K''$ are inconsistent. Then K is inconsistent.

The proofs of these are left to the reader.

We next prove

LEMMA 7: For each n K_n is consistent.

Proof: By induction on n.

K_0: Obvious, since $K_0 = K'$ and K' consistent by assumption.

Suppose now that K_n is consistent. We want to show that K_{n+1} is consistent as well. We must distinguish four cases.

(i) $K_n \cup E'_{n+1} \cup \langle Fr(E'_{n+1}), \{\}\rangle$ is consistent. Then, since discourse referents in the range of d do not belong to either K_n or E'_{n+1}, $K_n \cup d(E'_{n+1}) \cup \langle Fr(E'_{n+1}), \{\}\rangle$ also is consistent.

(ii) $K_n \cup E'_{n+1} \cup \langle Fr(E'_{n+1}), \{\}\rangle$ is inconsistent and E_{n+1} is neither of the form $\langle\{\}, \{K_1 \Rightarrow K_2\}\rangle$ nor of the form $\langle\{\}, \{K_1 \vee \ldots \vee K_n\}\rangle$. In that case $K_{n+1} = K_n \cup \overset{\circ}{\neg}E'_{n+1} \cup \langle Fr(E'_{n+1}), \{\}\rangle$. Suppose K_{n+1} inconsistent. Then by Lemma 2 $K_n \cup \langle Fr(E'_{n+1}), \{\}\rangle \vdash E'_{n+1}$. Since $K_n \vdash K_n \cup Fr(E'_{n+1})$, by the non-empty universe rule, also $K_n \vdash E'_{n+1}$ by Lemma 5. But $K_n \cup E'_{n+1}$ is by assumption (and Lemmas 2 & 5) inconsistent. Hence K_n is inconsistent, contrary to the induction hypothesis. So K_{n+1} is consistent.

(iii) $K_n \cup E'_{n+1} \cup \langle Fr(E'_{n+1}), \{\}\rangle$ is inconsistent and E_{n+1} is of the form $\langle\{\}, \{K_1 \Rightarrow K_2\}\rangle$. Then $K_{n+1} = K_n \cup \overset{\circ}{\neg}E'_{n+1} \cup d(K_1)' \cup \overset{\circ}{\neg}d(K_2)' \cup \langle Fr(E'_{n+1}), \{\}\rangle$, where $d(K_1)'$ and $d(K_2)'$ are as in (c) of the definition of K_{n+1}.

Suppose K_{n+1} is inconsistent. Then by Lemma 2

$K_n \cup \overset{\circ}{\neg}E'_{n+1} \vdash \overset{\circ}{\neg}\langle d(K_1)' \cup \overset{\circ}{\neg}d(K_2)'\rangle$

and by Lemma 3
$K_n \cup \mathring{\neg} E'_{n+1} \vdash \langle \{\}, \{d(K_1)' \Rightarrow d(K_2)'\} \rangle$
So $K_n \cup \mathring{\neg} E'_{n+1} \vdash E''_{n+1} \cup \mathring{\neg} E'_{n+1}$, where E'_{n+1} is an alphabetic variant of E''_{n+1}. So $K_n \cup \mathring{\neg} E'_{n+1}$ is inconsistent. By Lemma 6 K_n is inconsistent.

(iv) $K_n \cup E'_{n+1} \cup \langle Fr(E'_{n+1}), \{\} \rangle$ is inconsistent and E_{n+1} is of the form $\langle \{\}, \{K_1 \vee \ldots \vee K_n\} \rangle$. Then $K_{n+1} = K_n \cup \mathring{\neg} E'_{n+1} \cup \mathring{\neg} d(K_1)' \cup \ldots \cup \mathring{\neg} d(K_n)' \cup \langle Fr(E'_{n+1}), \{\} \rangle$, where $d(K_1)' \ldots d(K_n)'$ are as in (d) of the definition of K_{n+1}.
Suppose K_{n+1} is inconsistent. Then by Lemma 2
$K_n \cup \mathring{\neg} E'_{n+1} \vdash \mathring{\neg} \langle \{\}, \{ \neg K_1, \ldots, \neg K_n \} \rangle$
and by Lemma 4
$K_n \cup \mathring{\neg} E'_{n+1} \vdash d(K_1)' \vee \ldots \vee d(K_n)'$
Therefore K_n must be inconsistent.

Corrollary 8: K_ω is consistent.

Proof: Suppose not. Then there will be a proof of some contradiction from K_ω. This proof will use only a finite portion of K_ω. This portion will be wholly included in K_n for some n. So K_n inconsistent, contrary to what has just been shown.

This establishes the sixth of the seven properties of K_ω listed on page 343. All the other properties can be verified directly from the construction.

K_ω determines a model M which verifies all and only those DRSs E such that some variant of E is included in K_ω. The idea behind the definition of M is a very simple one. Its universe U_M is to consist of the discourse referents of K_ω which we referred to as constants. Moreover, for any predicate P^n the extension of P^n in M is to be the set of all n-tuples $\langle \mathbf{c}_1, \ldots, \mathbf{c}_n \rangle$ such that $P(\mathbf{c}_1, \ldots, \mathbf{c}_n) \in Con_{K_\omega}$. A fairly straightforward induction on the complexity of DRSs and conditions thus shows that the DRSs and conditions contained in K_ω are all verified in M by embeddings that map the constants $\mathbf{c}_i$ onto themselves. This is almost right but not quite. The reason why it is not quite right has to do with identity. Suppose Con_{K_ω} contains the condition $\mathbf{c}_i = \mathbf{c}_j$ with $i \neq j$. Then the embedding f which maps $\mathbf{c}_i$ onto $\mathbf{c}_i$ and $\mathbf{c}_j$ onto $\mathbf{c}_j$ will *not* verify $\mathbf{c}_i = \mathbf{c}_j$. For the values associated by f with $\mathbf{c}_i$ and $\mathbf{c}_j$, i.e. $\mathbf{c}_i$ and $\mathbf{c}_j$ themselves, are not identical. In order that the conditions will be verified by f, f must map $\mathbf{c}_i$ and $\mathbf{c}_j$ onto the *same* element of U_M.

To achieve this we group constants $\mathbf{c}_i$, $\mathbf{c}_j$ such that $\mathbf{c}_i = \mathbf{c}_j$ is a condition of K_ω into groups and then make these groups the members of U_{K_ω} and the f-values of the constants they contain. The deduction properties of '=' guarantee that this construction works the way it is meant to.

We define the relation between the constants in C as follows:

$\mathbf{c}_i \sim \mathbf{c}_j$ iff the condition $\mathbf{c}_i = \mathbf{c}_j$ belongs to Con_{K_ω}.

It is easy to show, using the results mentioned on page 335,

1. '$\sim$' is an equivalence relation on U_{K_ω}: i.e. for all $\mathbf{c} \in U_{K_\omega}$ $\mathbf{c} \sim \mathbf{c}$, for all $\mathbf{c}, \mathbf{c}' \in K_\omega$ if $\mathbf{c} \sim \mathbf{c}'$, then $\mathbf{c}' \sim \mathbf{c}$ and for all $\mathbf{c}, \mathbf{c}', \mathbf{c}'' \in U_{K_\omega}$ if $\mathbf{c} \sim \mathbf{c}'$ and $\mathbf{c}' \sim \mathbf{c}''$ then $\mathbf{c} \sim \mathbf{c}''$.
2. if $\mathbf{c}_1 \sim \mathbf{c}'_1, \ldots, \mathbf{c}_n \sim \mathbf{c}'_n$ then the condition $P^n(\mathbf{c}_1, \ldots, \mathbf{c}_n)$ belongs to Con_{K_ω}.

For any $\mathbf{c} \in U_{K_\omega}$ let $[\mathbf{c}]$ be the equivalence class, relative to '$\sim$', generated by $\mathbf{c}$: $[\mathbf{c}] = \{\mathbf{c}' : \mathbf{c} = \mathbf{c}' \in Con_{K_\omega}\}$. M and the relevant embedding $f{:}U_K \rightarrow U_M$ are defined by:

(i) $U_M = \{[\mathbf{c}] : \mathbf{c} \in U_{K_\omega}\}$

(ii) For each n-place predicate P
$Pred_M(P) = \{\langle[\mathbf{c}_1], \ldots, [\mathbf{c}_n]\rangle : \mathbf{c}_1, \ldots, \mathbf{c}_n \in C \text{ and } P(\mathbf{c}_1, \ldots, \mathbf{c}_n) \in Con_{K_\omega}\}$

(iii) $f(\mathbf{c}) = [\mathbf{c}]$ for each $\mathbf{c} \in C$

Note that (ii) does not depend on the choice of the representative of $[\mathbf{c}]$. To show this we have to verify that if $\mathbf{c}_i = \mathbf{c}'_i \in Con_{K_\omega}$ for some $i \in n$, then $P(\mathbf{c}_1, \ldots, \mathbf{c}_i, \ldots, \mathbf{c}_n) \in Con_{K_\omega}$ iff $P(\mathbf{c}_1, \ldots, \mathbf{c}'_i, \ldots, \mathbf{c}_n) \in Con_{K_\omega}$. This follows directly by SUI.

We finally prove for arbitrary DRS-conditions γ such that $Fr(\gamma) \subseteq C$ and $\underline{U}(\gamma) \subseteq V$:

$$M \models_f \gamma \text{ iff } \gamma' \in Con_{K_\omega} \text{ for some alphabetic variant } \gamma' \text{ of } \gamma. \tag{43}$$

The proof is by induction on the complexity of γ.

(i) When γ is an atomic condition, then (43) follows directly from the definitions of $Pred_M(P)$ and f.

(ii) $\gamma = \neg K_1$
Suppose $M \models_f \gamma$. Then for no $g \supseteq_{U_{K_1}} f$ $M \models_g K_1$. By the induction hypothesis for no d: $U_{K_1} \rightarrow C$ $d(K_1)' \subseteq K_\omega$ (For if there were such a d, then $M \models_d \delta$ for each of the conditions δ of K_1, and so $M \models_{f \cup d} K_1$, contrary to assumption.). The DRS K_1 will occur somewhere in our enumeration $E_1, \ldots, E_n, \ldots$ say as E_{n+1}. Suppose that in the formation of K_{n+1} some alphabetic variant of E_{n+1}, E'_{n+1}, had been added (rather than its negation). Then we would have had
$K_{n+1} = K_n \cup d(E'_{n+1}) \cup \langle Fr(E'_{n+1}), \{\}\rangle$.
But then $M \models_f d(E'_{n+1})$, since $d(E'_{n+1}) \subseteq K_\omega$. So $M \models_{f \cup d} d(E'_{n+1})$. And since E'_{n+1} is an alphabetic variant of K_1, $M \models_{f \cup d} K_1$; but we saw that this contradicts the assumption.
Now suppose $\gamma' \in Con_{K_\omega}$ for some alphabetic variant γ' of γ. Thus $\neg K'_1 \in Con_{K_\omega}$ where K' is an alphabetic variant of K_1. Suppose for some d: $U_{K_1} \rightarrow C$ $M \models_{f \cup d} K_1$. Then $M \models_f d(K_1)$. So for each $\delta \in Con_{d(K_1)}$ $M \models_f \delta$. So by induction hypothesis $\delta \in Con_{K_\omega}$. But then $d(K_1) \subseteq K_\omega$. Since also $\neg K'_1 \in Con_{K_\omega}$, K_ω would be inconsistent, contrary to what we have shown. So for no d: $U_{K_1} \rightarrow C$ $M \models_{f \cup d} K_1$. So $M \models_f \neg K_1$.

(iii) $\gamma = K_1 \Rightarrow K_2$

Assume $M \models_f \gamma$. Let $\langle\{\}, \{K_1 \Rightarrow K_2\}\rangle$ be the DRS E_{n+1}. If in the formation of K_{n+1} E'_{n+1} has been added, then we are done. Suppose not. Then

$K_{n+1} = K_n \cup \mathring{\neg}E'_{n+1} \cup d(K_1)' \cup \mathring{\neg}d(K_2)' \cup \langle Fr(E'_{n+1}) \cup Ran(d), \{\}\rangle$.

Since $d(K_1)' \subseteq K_\omega$ for some alphabetic variant of $d(K_1)$ we get by the induction hypothesis, that $M \models_f d(K_1)$. In the same way we infer $M \models_f \mathring{\neg}d(K_2)$. So $M \models_{f \cup d} K_1$ and for no $h \supseteq_{U_{K_2}} f \cup d$ $M \models_h K_2$. But this means that $M \not\models_f K_1 \Rightarrow K_2$, contrary to assumption. So it cannot have been the case that $\mathring{\neg}E'_{n+1}$ was added to K_n. So E'_{n+1} has been added. Consequently $\gamma' \in Con_{K_\omega}$ for some alphabetic variant γ' of γ.

Conversely, suppose $\gamma' \in Con_{K_\omega}$ for some variant γ' of γ. Evidently γ' is of the form $K'_1 \Rightarrow K'_2$. Let g be any function such that $g \supseteq_{U_{K_1}} f$ and $M \models_g K_1$. By induction hypothesis $Con_{K''_1} \subseteq Con_{K_\omega}$ for some alphabetic variant K''_1 of K_1. Suppose for no $h \supseteq_{U_{K_2}} g$ $M \models_h K_2$. Then $M \models_g \mathring{\neg}K_2$. Then by induction hypothesis and the same reasoning as under (ii) $\neg K''_2 \in Con_{K_\omega}$ for some variant K''_2 of K_2. But then K_ω inconsistent, contrary to Corollary 8. So there must be an $h \supseteq_{U_{K_2}} g$ such that $M \models_h K_2$. Since this holds for arbitrary g, $M \models_f K_1 \Rightarrow K_2$.

(iv) $\gamma = K_1 \vee \ldots \vee K_n$

Assume $M \models_f \gamma$. Let $\langle\{\}, \{K_1 \vee \ldots \vee K_n\}\rangle$ be the DRS E_{n+1}. If in the formation of K_{n+1} E'_{n+1} has been added, then we are done. Suppose not. Then

$K_{n+1} = K_n \cup \mathring{\neg}E'_{n+1} \cup \mathring{\neg}d(K_1)' \cup \ldots \cup \mathring{\neg}d(K_n)' \cup \langle Fr(E'_{n+1}) \cup Ran(d), \{\}\rangle$.

Since $\mathring{\neg}d(K_i)' \in Con_{K_\omega}$ for some alphabetic variant of $d(K_i)$ we get by the induction hypothesis, that $M \models_f \mathring{\neg}d(K_i)$ (for all $i = 1, \ldots, n$). But this means that $M \not\models_f K_1 \vee \ldots \vee K_n$, contrary to assumption.

Conversely, suppose $\gamma' \in Con_{K_\omega}$ for some variant γ' of γ. Evidently γ' is of the form $K'_1 \vee \ldots \vee K'_n$. Suppose for no $i = 1, \ldots, n$ there is a $g \supseteq_{U_{K_i}} f$ such that $M \models_g K_i$ then for each i there is no $g \supseteq_{U_{K_i}} f$ such that $M \models_g K_i$ and so $M \models_f \mathring{\neg}K_i$. By induction hypothesis and the same reasoning as under (ii) we get for all $i = 1, \ldots, n$ $\neg K''_i \in Con_{K_\omega}$ for some variant K''_i of K_i. But then K_ω inconsistent, contrary to Corollary 8.

This completes the proof of (24).

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