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Quantifiers Defined by Parametric Extensions

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Abstract This paper develops a metaphysically flexible theory of quantification broad enough to incorporate many distinct theories of objects. Quite different, mutually incompatible conceptions of the nature of objects and of reference find representation within it. Some conceptions yield classical first-order logic; some yield weaker logics. Yet others yield notions of validity that are proper extensions of classical logic.

Keywords Quantification · Semantics · Metaphysics · Objects · Ontology

Since Quine declared that 'to be is to be a value of a variable' [9], philosophers and logicians have recognized an intimate connection between quantification and ontology. Most directly, they have seen the quantificational structure of a theory as revealing its ontological commitments. This presupposes an objectual interpretation of the quantifiers, according to which $\forall x A$ is true iff every object in the domain satisfies A.

But many philosophers have also seen the logic of quantification as cloaking a theory of objects. Intuitionists, for example, have felt compelled to rewrite the semantics of quantification (and other logical operators) to accord with their nonclassical

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conception of the objects of mathematics. Free logic and substitutional quantification have similarly arisen from, *inter alia*, ontological concerns.

In this paper we develop a theory of quantification broad enough to incorporate many distinct theories of objects. We do not want to claim that the theory is ontologically neutral in any absolute sense. But quite different, mutually incompatible conceptions of the nature of objects and of reference find representation within it. Some conceptions yield classical first-order logic; some yield weaker logics. Yet others yield notions of validity that are proper extensions of classical logic.

The conceptions we investigate share a broadly constructivist outlook: they see the domain of objects under discussion as constructed in stages. They interpret the quantifiers as ranging over a domain of constructible objects. Being a value of a variable, in these conceptions, has a determinate and unified meaning: being constructible. They construe the existential and universal quantifiers as duals. In this respect they differ from intuitionistic logic, which from this perspective treats the universal quantifier as ranging over constructible objects but the existential quantifier as ranging over those already constructed.[1]

Thinking of a domain as constructed stage-by-stage has appeal in a variety of contexts, including:

– *Mathematics*. We might think of mathematical objects—e.g., natural numbers, real numbers, sets—as constructed stage-by-stage by way of mental activity, proofs, the specification of algorithms, or other methods. We might also view abstraction as proceeding stage-by-stage to construct abstract objects by a process akin to Fine's procedural postulation [5].
– *Fiction*. We might think of works of fiction as constructing fictional objects, both by introducing characters as the fiction proceeds and then also by specifying more and more information about them.
– *Discourse interpretation*. The same is true of discourse in general. We can think of a discourse as introducing entities—discourse referents [6, 7]—and specifying information about them. The information is initially partial, as is the domain of discourse referents, but it typically grows as the discourse proceeds.

There are two kinds of motivation for a characterization of quantifiers in terms of parametric substitution:

(i) the intuition (already spoken of above) that the quantifiers range over a growing, gradually unfolding universe; and
(ii) the 'nominalist' conception of quantification according to which a quantified statement is true just in case all its substitution results are true (i.e. those state-

[1]Constructivists generally count an existential sentence true if an object satisfying the corresponding open sentence has been constructed—or if an algorithm for constructing it has been specified. That suggests distinct interpretations of the stage-by-stage constructions we model in this paper; we might classify objects for the construction of which an algorithm has been specified at a given stage as available at that stage or as available only at some future stage.

ments that result from substituting constants in the statement for the variable bound by the quantifier).

A genuinely nominalist conception of a substitutional definition of the quantifiers should be neutral on the question whether the substituted constants actually denote objects or play a merely nominal part, as 'potential designators,' whose logical and semantic role does not depend on whether they actually designate something or not; or, a third possibility, constants might be seen as designators of objects *in statu nascendi*—a bit like pouches of marsupials, the contents of which develop stage by stage into mature objects. There is much more to say about this second aspect of parametric substitution than we do in this paper.

It has become common practice within model-theoretic semantics to state the truth-conditions of quantified formulae in substitutional terms, assuming that every object in the model has a name. In the formal developments which take up the bulk of this paper we have found this technical ploy convenient and have adopted it partly for this reason. But *the ploy* is compatible with many different conceptions of reference and existence—conceptions that are to be made formally explicit through assumptions about the nature and structure of the parametric models in terms of which our account of quantification is formulated. As we define them, our parametric models are partially ordered sets of diagrams, where diagrams are (as standardly in model theory) functions from atomic sentences to truth values. Different conceptions of reference and ontology take the form of different assumptions about the nature of these diagrams and of structural conditions that are implied by these assumptions. For instance, on one assumption, diagrams are determined by sets of objects together with assignments of extensions to the predicates of the language on these sets and denotations to its constants; on another the role of the constants occurring in the atomic sentences for which a diagram is defined would be exhausted by what can be said about the truth values of the sentences involving them. But these are just two from an open-ended set of possibilities.

In Section 1 we distinguish minervan from marsupial conceptions of the nature of objects and proper-name conceptions from demonstrative conceptions of the nature of reference. In Section 2 we define parametric models and a variety of constraints on them that result from combinations of those conceptions that affect the logic of constructible domains. Models reflecting a minervan conception of objects together with a proper-name conception of reference are, under natural and specifiable conditions, strong nets; those reflecting a marsupial conception of objects together with a demonstrative conception of reference are, again under natural and specifiable conditions, weak nets. Open models are a special case of the latter in which the only constraint on constructibility is logical consistency. In Section 3 we investigate the resulting logics, showing that strong and weak nets generate classical logic while open models have decidable $\forall\exists$ theories. Section 4 deals with the logic of the class of all parametric models. It formulates an axiomatization of this non-classical logic, for which completeness is proved by an adaptation of the method of semantic tableaux to the structure of parametric models.

1 Conceptions of Objects and Reference

Constructivist conceptions of arithmetic, such as Brouwer's, have it that the natural numbers are never all present at a single stage of construction. Nevertheless, each new number is, as soon as it is constructed, 'fully there': Its position in the number sequence and its arithmetical properties are completely fixed.

But thinking of a domain as constructed in stages does not require that, even within arithmetic. Consider, for instance, the number π as defined in terms of a constructively specified sequence of rational numbers, e.g., by an effective rule for calculating the successive digits of its decimal expansion. We might see the successive results of this calculation as providing partial information about π, and regard π itself as emerging only gradually, with properties that grow more definite as the calculation progresses. Thus, the construction of a domain of reals containing π would involve, first, an act of introducing, at some given stage s, π as the number determined by the given rule of calculation, and then, at the stages following s, the steps involved in carrying out its calculation according to the rule. So treated, π is not 'fully there' when it appears on the scene. Rather its characteristics unfold gradually as the calculation determines its decimal expansion with ever greater precision.

For a different type of example, consider domains of fictional objects. Most fictional individuals develop as the work of fiction to which they belong unfolds. The work of fiction determines what is true of each fictional entity it creates. Consequently, fictional entities are seldom if ever fully determinate, in the sense that all predicates in principle applicable to the general kinds they instantiate are either definitely true or definitely false of them. Many properties of a fictional object are determined only as the work progresses, and well after the entity is first introduced. This phenomenon of gradual determination is especially noteworthy in connection with literary works that comprise several volumes, the earlier of which have appeared when later volumes are still to be written.

Constructions that produce new elements that are complete as soon as they emerge are *minervan* constructions. Constructions that produce entities that are fledgling at first and have to be nurtured to maturity as the structure unfolds we call *marsupial*.[2]

Each conception of objects sees the quantifiers as ranging over a domain existing only as the virtual limit of a hierarchy of approximating stages. At any stage of the

[2]Some approaches to real number theory yield a hierarchy of stages that fits neither the minervan nor the marsupial conception. The entities that make up the finite stages of this hierarchy are not the real numbers themselves. Each such segment has a transitional status; at the next stage it is replaced by a couple of smaller segments, which it yields by division and which classify the potential reals more finely. The real numbers themselves are not entities at any finite stage, but have the status of limits that can be approximated by sequences of segments that delimit them ever more closely. Constructions of this kind, which produce at each stage classifications of the targeted entities that are subsequently replaced by finer classifications, we call *amoebic*. The reals are not the only ontological category for which an 'amoebic' conception makes sense. Another candidate is a constructivist version of the bundle theory of objects, for example, which might construe objects as infinite limits of finite bundles of properties. Vague boundaries might be understood as infinite limits of precisifications. Quantities might be understood as infinite limits of measurements. In all such cases, we would have to extend the theory we present here to incorporate transfinite limits among its methods of construction.

progression, only part of the domain has been created. When we try to evaluate at such a stage a statement containing quantifiers, we should take into account not only what has been constructed but also what may be constructed at later stages.

Thus, suppose we have reached some stage of the construction, and ask, with respect to it, whether an existential sentence $\exists x\varphi$ is true. The answer is 'yes' if there is some later stage of the construction at which there is an object satisfying φ, or, on a substitutional interpretation, if there is a name c available at that stage such that $(\varphi)c/x$ is true there. Similarly, $\forall x\varphi$ should be true at a stage s iff, for any later stage s' and object a to be found there, a satisfies $\varphi(x)$ at s', or, on a substitutional approach, iff $\varphi(c)$ is true at all later stages for all names c available at those stages.

These considerations lead us to the following preliminary definitions. A parametric structure for a first-order language L is a partially ordered set of 'stages,' where each stage involves a certain collection of objects and determines whether they satisfy the predicates of L. For a sentence of the form $\exists x\varphi$ to be true at a stage s of a parametric structure, it is necessary and sufficient that there be some 'later' stage s' such that φ has a true substitution instance at s'.

When a parametric structure represents a purely minervan construction, it is plausible to assume that each stage settles all questions of predication involving predicates of L and objects available at that stage. Thus, each stage determines what in the model theory of first-order predicate logic is called a (complete) *diagram*, i.e., a valuation of the atomic sentences of some given language. A parametric structure for L is *bivalent* iff all its stages are complete diagrams for L. Parametric structures adequate to minervan constructions, then, are bivalent.

If the domain is generated by a nonminervan construction, the assumption that the stages of the parametric structure determine complete diagrams may no longer be tenable. The objects such constructions yield may be more or less indeterminate at birth. So, in particular, questions of predication involving the predicates of L may be settled not when the objects enter the structure but only at some later stage. Thus, the stages of such a structure cannot be expected to determine bivalent first-order models for L. In general, the models will be partial.

Nonbivalent structures have great formal and philosophical interest, but present a host of technical complications which we want to set aside here. (Because we shall limit ourselves to discussing bivalent structures, we shall drop the qualification 'bivalent' from now on.) We limit attention to bivalent models, not merely to simplify our results, but to isolate a particular kind of partiality. Parametric structures are partial in at least two respects. They may, at a given stage, convey only partial information about the objects present at that stage. But they may also convey partial information about the domain as a whole, by including only a proper subset of it. It is the nature of this second respect in which the structures are partial—that in which a stage of construction reveals the domain only partially because it covers only some part of the domain—that we will be investigating in this paper.

Parametric structures with a *supreme* stage—i.e., a stage which includes all others—behave essentially like classical models. The parametric truth definition produces exactly the same set of true sentences as the classical definition when it is applied directly to the supreme stage. But parametric structures need not have supreme stages. They need not even have maximal stages, i.e., stages not properly included in any others. For example, a parametric structure may reflect the construction of the natural numbers by having an increasing sequence of finite stages, but no infinite stage.

Many applications give rise to parametric structures that have a minimal stage. In particular, there are many for which the minimal stage is empty: The construction that the parametric structure reflects starts out with nothing, introducing elements only as it unfolds. Here the initial stage is not just minimal in the weak sense that no other stage in the structure is properly included in it, but also in the strong sense that it is included in every other stage. Clearly a parametric structure can have no more than one stage that is minimal in the strong sense. If a parametric structure has a unique strongly minimal stage, we refer to that stage as its *core*. From a formal perspective, parametric models without a core do not add much of interest to what can be learned from models with a core. We therefore assume all our parametric models to have a core.

The second distinction we wish to stress is that between *proper-name* conceptions and *demonstrative* conceptions of reference. In natural language, proper names act as persistent labels of the objects they denote. Demonstratives, in contrast, are not tied to particular objects once and for all, but can designate different objects on different occasions of use. We can think of the relation holding between the constants of L and the parametrically represented objects they denote in terms of either paradigm. On the proper-name conception, names are anchored to particular objects; each name serves as a tag designating a single object throughout the stages of a parametric structure. Each constant designates the same object at each stage at which it exists. Replacing constants with constants in a discourse, by means of a bijective map, would in general alter the content of that discourse: after the replacement, the discourse would speak about different objects.

On the demonstrative conception, in contrast, constants function as 'pegs' on which assertions hang predicates [8]. There is no reason to expect that a constant should name the same entity in different stages of a structure, or, conversely, that an object should retain the same 'designator' throughout the different stages of the structure that contain the object. The objects designated by c_1, c_2, etc., at a stage might just as well have been named by c'_1, c'_2, etc. Substituting constants for constants, again bijectively, should on this conception leave the content of a discourse unchanged. Parametric structures adequate to the demonstrative conception are purely qualitative. They can be expected to be invariant under permutations of constants, in the sense that any possible stage obtainable by bijectively replacing the constants in a stage of the structure also belongs to the structure.

Suppose that M is a parametric structure for a growing domain D of individuals and that the constant c acts as a demonstrative in M in that it refers to the object a of D at stage s_1 and to a different object b at stage s_2. Then, analyzing $\exists x Rcx$ as true at stage s_1 in M iff there is some later stage s_2 at which Rcc' is true for some constant c', we may find ourselves compelled to pronounce $\exists x Rcx$ true at s_1 because, at s_2, where c denotes b, R holds between b and some other object, even though a is nowhere R-related to any object whatever. Clearly this would be wrong. So, demonstratively referring constants should as a rule be shunned in parametric analyses of quantification.

In one type of situation, however, the demonstrative conception seems adequate. Sometimes the identity of the objects available at any stage of domain formation depends exclusively on their properties and on their relations to one another. Any two objects a and b that occur at a given stage and are indistinguishable in these terms cannot really be distinguished at all (much as two elementary particles that follow the Bose-Einstein statistics and have identical quantum numbers are indistinguishable). If a later stage has two objects a' and b', both of which share all the characteristics of a and b, but which nevertheless can be distinguished—say, by bearing incompatible relations to some third object—there is no way of telling whether a' is a and b' b or vice versa.

When identity is thus strictly qualitative, proper names are inappropriate; they would risk endowing their referents with a spurious cross-stage identity. The only proper role for constants in the stages of parametric structures which represent such domains is to provide stage-bound 'pegs' on which the predicates expressing the relevant properties and relations can be hung.

We have discussed two 'dimensions' along which the intuitions underlying a parametric approach to quantification may vary: (1) the nature of the entities that are spoken about in the components of a parametric structure; and (2) the reference relation between constants and the entities they designate. Since variations along these dimensions seem to some extent independent, a parametric approach may be based on many distinct combinations of motivations and conceptions. Hence, most questions arising from the parametric treatment of quantifiers require unraveling into a number of strands before their philosophical significance becomes fully transparent. In what follows we address only a fraction of these questions. We hope that the issues we have chosen to address may stimulate others to deepen, refine and extend the results reported here.

2 Definitions and Constraints

The languages we study in this paper are all languages of first-order predicate logic. We will restrict ourselves to languages containing only predicates and individual constants, but lacking identity and function constants. Various notions of identity can be defined in the different parametric logics we will develop. But this is a topic for exploration beyond the present paper.

2.1 Definitions

Each language L we consider will have: (i) an infinite set of variables $x_1, x_2, x_3, \ldots$. (ii) the logical constants $\neg, \rightarrow, \forall$, (iii) a set of predicates, $PredL$, each with its own -arity $i > 0$ (so, to be exact, $PredL$ is a function from symbols to positive integers); (iv) a set $ConL$ of individual constants $c_1, c_2, c_3, \ldots$.[3]

[3]Our languages are without predicates of -arity 0; that is, they are without propositional constants. This restriction has been imposed merely for convenience: excluding propositional constants simplifies both the

Formulae, sentences and atomic formulae of L are defined as usual. By a *basic* formula of L we understand a formula that is either atomic or the negation of an atomic formula. The logical operators $\wedge$, $\vee$, $\leftrightarrow$ and $\exists$ are defined as usual in terms of $\neg, \rightarrow, \forall$. This is legitimate in view of our bivalence assumption—the restriction to parametric models whose stages are *diagrams*, in which the truth values of all atomic sentences about objects that belong to the stage are decided. As will be made explicit in the definition below of truth for parametric models, this restriction entails that the propositional part of all the parametric logics we will consider is classical. Sometimes we will proceed as if $\wedge$, $\vee$, $\leftrightarrow$ and $\exists$ are primitives of L. In that case their definitions will play the role of axioms, rather than prescriptions for converting formulas with occurrences of $\wedge$, $\vee$, $\leftrightarrow$ or $\exists$ into the canonical notation that does not include these symbols. We allow for variation in the set of non-logical constants and identify each language that results from the choice of a set of nonlogical constants with that set; this enables us to talk about languages as set-theoretic objects. For instance, $L_1 \cap L_2$ is that language L such that $PredL = PredL_1 \cap PredL_2$ and $ConL = ConL_1 \cap ConL_2$. If L is a language and C a set of individual constants, then $L(C)$ is the language L' that is like L except that $ConL' = ConL \cup C$.

We assume a proper class V of entities to be used as individual constants that is disjoint from the languages L for which we formulate our theorems. The proofs of these theorems, however, will involve 'mixed' languages, which are the result of extending the language L referred to in the theorem with constants from V.

A *valuation* for L is a function from atomic sentences of L into {0, 1}, where 1 stands for truth and 0 for falsehood. So, when $ConL$ is empty, the only valuation for L is the empty set. (Recall that L has no propositional constants.) Each valuation W uniquely determines the smallest language for which it is a valuation. We refer to this language, $L(W)$, as the language *of* W. A valuation W is called a *diagram* iff its domain consists of all atomic sentences of the language $L(W)$. (Thus a diagram is a valuation that is 'complete' for its own language.) From now on D, D', D_1, etc., always stand for diagrams. Whenever $D \subseteq D'$, D' *extends* D. Note that if a diagram D' properly extends a diagram D, then $L(D) \subset L(D')$. If $ConL(D') = ConL(D) \cup C$, and D' extends D, then we write $D \subseteq_C D'$. (We write $D \subseteq_c D'$ instead of $D \subseteq_{\{c\}} D'$, and $ConD$ rather than $ConL(D)$.) For any diagram D and language L, there is a unique diagram $D' \subseteq D$ whose language is $L(D) \cap L$. We denote this diagram as $D \restriction L$. We sometimes write $D(c_1, ..., c_n)$ to indicate that $\{c_1, ..., c_n\}$ is the set $ConD$. For any terms $t_1, ..., t_n D(t_1, ..., t_n)$ is the 'diagram' which assigns to any atomic formula $\varphi(t_1/c_1, ..., t_n/c_n)$ the same truth value that $D(c_1, ..., c_n)$ assigns to φ. Let f be a function from constants to constants such that the domain of f includes $ConD$. By $f(\varphi)$ we understand the formula resulting from replacing each constant c_i in φ by $f(c_i)$, and by $f(D)$ we understand the diagram such that, for every atomic sentence φ of L, $f(D)(f(\varphi)) = D(\varphi)$. If f is a one-one function from $ConD_1$ onto

formulation of a number of our theorems and often also their proofs. Not surprisingly, our results can be generalized to languages with propositional constants, albeit at the cost of a certain amount of extra work.

$ConD_2$, we say that D_1 and D_2 are *equivalent modulo* f (in symbols, $D_1 \approx_f D_2$) iff $D_2 = f(D_1)$.

Let L be a language and U a subset of V. A *parametric model* M *for* L *and* U is a set of diagrams such that (i) for each diagram $D \in M$, $ConD \subseteq ConL \cup U$; (ii) there is exactly one diagram $D_0(M) \in M$, M's *core*, whose language is L; (iii) the core of M is included in every member of M. By a *parametric model for* L we understand a parametric model for L and some subset U of V. Any parametric model M for L uniquely determines a particular strongly minimal set U_M such that M is a parametric model for L and U, namely, the set $\bigcup\{ConD : D \in M\}$. Throughout this paper we shall exclude the trivial parametric model $\emptyset$; doing so simplifies the logic of the quantifiers without sacrificing anything significant.

Let M be a parametric model for some language L and let D be an element of M. The *truth value of a sentence* φ of $L(D)$ *at* D *in* M, written $[\![\varphi]\!]_{D,M}$, is defined as follows:

(2.1) If φ is atomic, then $[\![\varphi]\!]_{D,M} = D(\varphi)$.

(2.2) $[\![\neg\varphi]\!]_{D,M} = 1$ iff $[\![\varphi]\!]_{D,M} = 0$, and $[\![\neg\varphi]\!]_{D,M} = 0$ otherwise.

(2.3) $[\![\varphi \rightarrow \psi]\!]_{D,M} = 1$ iff $[\![\varphi]\!]_{D,M} = 0$ or $[\![\psi]\!]_{D,M} = 1$; otherwise, $[\![\varphi \rightarrow \psi]\!]_{D,M} = 0$.

(2.4) $[\![\forall x_i\varphi]\!]_{D,M} = 1$ iff $\forall D' \in M \forall c \in ConD'(D \subseteq D' \rightarrow [\![(\varphi)c/x_i]\!]_{D',M} = 1)$; otherwise, $[\![\forall x_i\varphi]\!]_{D,M} = 0$.[4]

This is entirely standard, except for the quantificational clause. (2.4) specifies that a universal sentence is true at a diagram in a model just in case all its instances are true at all extensions of that diagram in the model.

By the standard definition of the existential quantifier, we can convert (2.4) into a clause for existentially quantified sentences:

(2.5) $[\![\exists x_i\varphi]\!]_{D,M} = 1$ iff $\exists D' \in M \exists c \in ConD'(D \subseteq D' \wedge [\![(\varphi)c/x_i]\!]_{D',M} = 1)$; otherwise, $[\![\exists x_i\varphi]\!]_{D,M} = 0$.

An existential sentence is true at a diagram in a model, then, just in case an instance of it is true in an extension of that diagram in the model. The existential and universal quantifiers are duals; this differentiates our approach from an intuitionistic one, in which the existential clause would contain not an extension D' of D but D itself.

The truth value of a sentence φ of L in M, $[\![\varphi]\!]_M$, is by definition $[\![\varphi]\!]_{D_0(M),M}$. φ is *valid* iff φ is true in every parametric model for L. The set of all sentences true in M is $Th(M)$, the *theory* of M. We shall often write $D \models_M \varphi$ in lieu of $[\![\varphi]\!]_{D,M} = 1$, and $D \not\models_M \varphi$ in lieu of $[\![\varphi]\!]_{D,M} = 0$. Similarly $M \models \varphi$ ($M \not\models \varphi$) are alternative notations for $[\![\varphi]\!]_M = 1(0)$. Note that, if $D, D' \in M$ and $D \subseteq D'$, $D \models_M \forall x\varphi$ only if $D' \models_M \forall x\varphi$, and $D' \models_M \exists x\varphi$ only if $D \models_M \exists x\varphi$. Let φ be a sentence and Γ be a set of sentences of L. Then φ is a *parametric consequence* of Γ, in symbols $\Gamma \models \varphi$, iff for all parametric models M, if for all $\psi \in \Gamma$ $[\![\psi]\!]_M = 1$, then $[\![\varphi]\!]_M = 1$. φ is *(parametrically) valid* iff $\emptyset \models \varphi$.

[4]The substitution of a term t for all free occurrences of a variable x in a formula φ will be denoted either as $(\varphi)t/x$ or as $[\varphi]t/x$, depending on which notation seems more perspicuous in any given context.

2.2 Constraints on Parametric Models

The constants in parametric models may refer in the manner of demonstratives or proper names. The following two ways of deriving a parametric model from a standard classical model reflect these two conceptions of reference. Let $\mathfrak{M} = < A, F >$ be a classical model for a language L without individual constants. (A is the universe of $\mathfrak{M}$, and F its assignment function.) First, let U be a subset of V of the same cardinality as A, and let f be a one-one map between A and U. For any $a \in A$, think of $f(a)$ as the name of a. Given this naming function, we can associate with $\mathfrak{M}$ a parametric model $M^*(\mathfrak{M})$ as follows: Let D be the f-diagram of $\mathfrak{M}$, that is, the function assigning truth to all and only atomic sentences $P(f(a_1), ..., f(a_n))$ for $a_1, ..., a_n \in A$ such that $\mathfrak{M} \models P(x_1, ..., x_n)(a_1, ..., a_n)$. (That is, $\mathfrak{M}$ satisfies $P(x_1, ..., x_n)$ under an assignment that assigns the a_i to the x_i). Let $M^*(\mathfrak{M})$ be the set of all diagrams included in D whose language includes L. The constants in $M^*(\mathfrak{M})$ are attached to the objects they name in a manner that exemplifies the proper-name conception.

There is also a way to obtain a parametric model from $\mathfrak{M}$ that reflects the demonstrative conception. Let U' be any subset of V of cardinality greater than or equal to that of A. Intuitively, U' is to provide a set of constants each of which can name any element in A. The parametric model $M^{**}(\mathfrak{M})$ derived from $\mathfrak{M}$ that reflects this intuition can be defined as follows. Let B be any subset of A and f be any injection of B into U'. Let the diagram $D(\mathfrak{M}, B, f)$ determined by $\mathfrak{M}$, B and f be the function assigning truth to all and only atomic sentences $P(f(b_1), ..., f(b_n))$ for $b_1, ..., b_n \in B$ such that $\mathfrak{M} \models P(x_1, ..., x_n)(b_1, ..., b_n)$. $M^{**}(\mathfrak{M})$ is to be the set of all diagrams $D(\mathfrak{M}, B, f)$ for $B \subseteq A$ and f an injection of B into U'.

In general, $M^*(\mathfrak{M})$ and $M^{**}(\mathfrak{M})$ are not equivalent. In fact, $Th(M^*(\mathfrak{M}))$ is exactly the classical theory of $\mathfrak{M}$, while $Th(M^{**}(\mathfrak{M}))$ can be very different.[5]

The first of these two parametric models has properties distinctive of models representing a minervan domain and reflecting the proper-name conception of designation: any two diagrams of the structure must be compatible. If, however, the naming procedures associated with different stages of a structure allow using the same constant (i.e., the same symbol) to designate *distinct* elements of the domain, then the derived parametric model may contain diagrams that are formally inconsistent with each

[5]For an example let L be the language $\{R, S\}$, where R and S are 2-place predicates, and let $\mathfrak{M}$ be a model for L with universe $A = \{a_1, a_2, b_1, b_2\}$ in which $[[R]]_{\mathfrak{M}} = \{< a_1, b_1 >\}$ and $[[S]]_{\mathfrak{M}} = \{< a_2, b_2 >\}$. In $\mathfrak{M}$ the sentence $\exists x(\exists y Rxy \wedge \exists z Sxz)$ is clearly false. But in $M^{**}(\mathfrak{M})$ this sentence is true. For let B_0, B_1, B_2 be the sets $\{a_1\}$, $\{a_1, b_1\}$, $\{a_2, b_2\}$, and let f_0, f_1, f_2 be the functions with domains B_0, B_1, B_2, respectively, and defined by: $f_0(a_1) = c_1$; $f_1(a_1) = c_1$, $f_1(b_1) = c_2$; $f_2(a_2) = c_1$, $f_2(b_2) = c_2$. Then the pairs (B_0, f_0), (B_1, f_1) and (B_2, f_2) determine the diagrams D_0, D_1, D_2 of $M^{**}(\mathfrak{M})$ given by: (i) $D_i(Rc_1c_1) = D_i(Sc_1c_1) = 0$ for $i = 0, 1, 2$; (ii) $D_1(Rc_1c_2) = 1$; $D_1(Rc_2c_1) = D_1(Rc_2c_2) = D_1(Sc_1c_2) = D_1(Sc_2c_2) = D_1(Sc_2c_1) = 0$; (iii) $D_2(Rc_1c_2) = 1$; $D_2(Rc_2c_1) = D_2(Rc_2c_2) = D_2(Sc_1c_2) = D_2(Sc_2c_2) = D_2(Sc_2c_1) = 0$. Since $D_0 \subseteq D_1$ and $D_0 \subseteq D_2$, the sentence $\exists y Rc_1 y \wedge \exists z Rc_1 z$ is true in $M^{**}(\mathfrak{M})$ at diagram D_0. So, the sentence $\exists x(\exists y Rxy \wedge \exists z Rxz)$ is true in $M^{**}(\mathfrak{M})$ at its empty core. This example also confirms the informal observation we made earlier that the demonstrative conception of reference poorly fits the minervan conception of objects if demonstrative designators for these objects are used in a substitution-based definition of truth.

other. This situation can also arise on the marsupial conception, irrespective of the naming procedures used. A marsupial object introduced at stage s might develop in incompatible ways and thus give rise to incompatible extensions of the diagram associated with s. But if the properties of each object are fully determined as soon as it enters the stage, and if the name it receives there remains its name at all later stages, this cannot happen; all diagrams must be compatible.

The model $M^*(\mathfrak{M})$ derived from $\mathfrak{M}$ by the first method has this property. In fact, for any two diagrams D_1 and D_2 in $M^*(\mathfrak{M})$ there is a diagram D_3 in $M^*(\mathfrak{M})$ such that $D_1 \subseteq D_3$ and $D_2 \subseteq D_3$. We call any parametric model that satisfies this condition a *strong net*.[6]

It might be thought harmless to assume that every parametric model has the property of *reductive completeness*: a model M for L is *reductively complete* iff, for every $D \in M$ and every language L' such that $L \subseteq L' \subseteq L(D)$, $D \restriction L' \in M$. Indeed, this seems natural if we adopt the proper-name conception of constants and view objects as fully defined in advance. A problem arises, however, if we try to describe the natural numbers with 0 and successor function S by a parametric model based on the demonstrative conception of constants (as might be suggested by a reading of [1], for example), even if each diagram in the model represents an initial segment of the natural numbers. For suppose that M has a diagram representing the segment $<0, 1, 2>$ in which 0, 1, and 2 are named by the constants c_0, c_1, c_2, respectively; and that there is also a diagram $D' \in M$ representing $<0, 1, 2, 3>$ in which c_0 names 0, c_1 names 1, and c_2 names the number 3. Then $D \subseteq D'$, and so both $\exists x(Sc_0x \wedge Sxc_2)$ and $\exists x \exists y(Sc_0x \wedge Sxy \wedge Syc_2)$ will be true in M at D. The set-theoretic inclusion relation between diagrams no longer establishes that the including diagram extends the information in the included diagram.

2.3 The Demonstrative Conception

We now consider some properties of parametric models that are plausible on the demonstrative conception, which implies that each constant can be used to name any object whatever. This suggests that a parametric model that realizes all possibilities should, for each collection of objects about which it speaks, and each sufficiently large collection of constants, contain a diagram in which these constants name the objects in any permutation. On the demonstrative conception, the relation between a term c and its designatum is nonpersistent; its significance is limited to a particular stage. In fact, that *this* term is used to designate the object at this stage rather than some other should be seen as accidental and semantically unimportant. The choice of designators occurring in a diagram is irrelevant to the information it conveys; any diagram obtained by replacing the designators on a one-to-one basis by others would

[6] Not all parametric models reflecting both the proper-name and minervan conceptions are strong nets. The process of stage development may be nondeterministic; the emergence of new objects of one kind—that is, satisfying one set of predicates—may prevent the emergence of individuals of some other kind. Thus, among the diagrams of a parametric model there may be some including objects of the first kind and some including objects of the second kind, but none including objects of both kinds.

carry the same information and, by considerations of symmetry, should have an equal right to be part of a parametric model reflecting the demonstrative conception. We can make this principle structurally explicit by stipulating that a parametric structure based on the demonstrative conception be closed under such replacements. Thus, for instance, such a parametric model M should satisfy the following property, which we refer to as *closure under permutation*: if $D \in M$ and f is a permutation of $Con D$, then M also contains the diagram D' such that, for any ψ in the domain of D, $D'(f(\psi)) = D(\psi)$.

Another formal property of parametric models apparently entailed by the informal considerations above is the *universal understudy property*, so-called because, in parametric models with this property, any group of constants not already in a diagram may be made to play the roles of a group of that diagram's constants. In this sense, any constant in the universe of the model but playing no role in a diagram may serve as 'understudy' for any constant that is playing a role in that diagram. (The 'roles' here are the satisfactional roles of [4]: maps from atomic formulae with one free variable x into $\{0, 1\}$.)

Let M be a parametric model. M has the *universal understudy property* iff the following holds for arbitrary diagrams D_1, D_2, $D_3 \in M$ such that $D_1 \subseteq D_3$, $D_1 \approx_f D_2$ for some f, and $|U_M - Con D_2| \geq |Con D_3 - Con D_1|$: If C is any set of constants included in U_M disjoint from $Con D_2$ and g any one-one function from $Con D_3 - Con D_1$ onto C, there is a diagram $D_4 \in M$ such that $D_3 \approx_{f \cup g} D_4$ and $D_2 \subseteq D_4$.

Of some technical importance is the question whether a parametric model may contain diagrams that exhaust the stock of its constants. In some special cases, it seems intuitively clear that no such diagram should exist: for instance, when we think of the natural numbers as given by a parametric structure, each component of which contains only finitely many. There are also parametric models in which a diagram exhausting the set of constants does exist. But the clear examples of this, such as the parametric model derived from a classical model in the manner described in the first paragraph of 2.1, are of little interest. Moreover, these models easily convert into elementarily equivalent structures without a diagram exhausting the set of constants. Since working with parametric models from which such exhaustive diagrams are absent turns out to have considerable practical advantage, we shall concentrate on parametric models that possess this feature in a strong form. We say that a parametric model M *has an inexhaustible set of constants* iff for each D_1, $D_2 \in M$ there is a subset C of U_M such that $C \cap Con D_1 = \emptyset$ and $|C| = |Con D_2|$. Given our assumption that $M \neq \emptyset$, this condition implies the 'weaker' property that, for all $D \in M$, there is a $c \in U_M$ such that $c \notin Con D$. Note that the same argument also leads to the slightly stronger conclusion that if $D \in M$ and n is a natural number, then there is a set $C \subseteq U_M$ disjoint from $Con D$ which has cardinality n.

If M has an inexhaustible set of constants and possesses the universal understudy property, then it also has another property, the *existential understudy property*. If a parametric model has this property, each collection of constants playing roles in a diagram has a team of 'understudies,' not contained in the diagram, who can take over the roles played by members of the collection. More formally, M has the *existential understudy property* iff the following holds for arbitrary diagrams D_1, D_2, $D_3 \in M$ such that $D_1 \subseteq D_2$, $D_1 \subseteq D_3$: there exists a set $C' \subseteq U_M$ such that $C' \cap (Con D_3 -$

$Con D_1) = \emptyset$, a one-one function f from $Con D_2 - Con D_1$ onto C' and a diagram $D_4 \in M$ such that $D_2 \approx_g D_4$ where g is the union of f and the identity function on $Con D_1$.

To verify that, if M has an inexhaustible set of constants and possesses the universal understudy property, it also has the existential understudy property, assume the antecedent. Let $D_1, D_2, D_3 \in M$, where $D_1 \subseteq D_2$ and $D_1 \subseteq D_3$. By inexhaustibility, there is a set $C \subseteq U_M$ such that $C \cap Con D_3 = \emptyset$ and $|C| = |Con D_2|$. There is thus a subset $C' \subseteq C$ such that $|C'| = |Con D_2 - Con D_1|$ and $C' \cap Con D_1 = \emptyset$. Since $C' \subseteq C$, $C' \cap (Con D_3 - Con D_1) = \emptyset$. And, since $|C'| = |Con D_2 - Con D_1|$, there is a one-one function f from $Con D_2 - Con D_1$ onto C'. Note that $D_1 \approx_i D_1$, where i is the identity map on $Con D_1$. By the universal understudy property (reading 'D_1' and 'D_2' for 'D_2' and 'D_3', respectively), there is a $D_4 \in M$ such that $D_2 \approx_g D_4$ and $D_1 \subseteq D_4$, where $g = f \cup i$. Thus, M has the existential understudy property.

Also, if M has the universal and existential understudy properties and has an empty core, then M is closed under permutation. For suppose that M has both understudy properties and that $\emptyset \in M$. Let $D \in M$ and let h permute $Con D$. By the existential understudy property, there is a set $C' \subseteq U_M$ such that $C' \cap Con D = \emptyset$, a one-one function g (viz., $h \circ f^{-1}$) from $Con D$ onto C', and a diagram $D'' \in M$ such that $D \approx_f D''$. By the universal understudy property (reading '$\emptyset$' for 'D_1' and 'D_2', and 'D''' for 'D_3'), for any one-one function g' from $Con D''$ onto $Con D$ there is a $D' \in M$ such that $D'' \approx_{g'} D'$. This is true in particular for the function $g = f^{-1} \circ h$. So let D' be such that $D'' \approx_g D'$. Then $Con D' = Con D$. If ψ is any atomic sentence of $L(D)$, then $D(\psi) = D''(f(\psi)) = D'(h(f^{-1}(f(\psi)))) = D'(h(\psi))$. So, $D' \in M$. It follows that M is closed under permutation.

2.4 The Marsupial Conception

Our discussion of the example of successor arithmetic shows that it would be wrong to postulate in general that parametric models are reductively complete. Nevertheless, reductively complete models constitute a conceptually and mathematically important class. Reductive completeness is plausible, for instance, if we think of the domain of discourse of any given diagram D as consisting of entities that are determined just to the extent that their properties and mutual relations are defined in D and that can be specified more precisely in any way compatible with D. Against this background reductive completeness can be understood as saying that a set of objects which has reached a given degree of specificity (given by diagram D) could have reached this stage via an intermediate stage characterized by the properties and relations of some proper sublanguage L' of the language of D. Another principle that is compatible with the general idea of entities developing gradually into more definite 'contours' is that a set of entities that has reached a certain degree of specificity, given by a diagram D of a model M, may develop further in any way compatible with what is determined about it in D and that that can be made explicit with the linguistic means available in M. Formally this principle is captured in (2.6).

(2.6) Whenever $D_1, D_2 \in M$ and D_3 is a diagram such that $D_1 \subseteq D_3$ and $L(D_3) \subseteq L(D_2)$, then $D_3 \in M$.

Among the parametric models that satisfy (2.6) there are the so-called *open* models [3]. Open models are the topic of Section 3.3.

As we have seen, strong nets seem to reflect the proper-name conception of constants and the minervan conception of objects. There is a slightly weaker property the motivation for which is not tied to these two conceptions, and which will also be important in the formal developments below. A parametric model M is a *weak net* iff, whenever $D_1, D_2 \in M$ and $D_1 \upharpoonright (L(D_1) \cap L(D_2)) = D_2 \upharpoonright (L(D_1) \cap L(D_2))$, there is a $D_3 \in M$ such that $D_1 \cup D_2 \subseteq D_3$. Weak nets capture the idea that the presence of some objects can never be a reason for preventing others from being added: if we start out with a given collection of objects A, and extend this collection on the one hand to $A \cup B$ and on the other to the collection $A \cup C$, then, since the presence of the objects in B cannot impede the addition of the objects in C, it must be possible to add the elements of both B and C to A all at once.

So far we have analyzed the quantifiers by considering, when evaluating a quantified sentence at a diagram D of a parametric model M, all extensions of D in M. A variant of this analysis is to consider extensions of D by just one constant. For the universal quantifier, this leads to the alternative truth clause:

(2.7) $[\![\forall x_i \varphi]\!]_{D,M} = 1$ iff $\forall D' \in M \forall c \in Con D' (D \subseteq_c D' \rightarrow [\![(\varphi)c/x_i]\!]_{D',M} = 1)$; otherwise, $[\![\forall x_i \varphi]\!]_{D,M} = 0$.

The two clauses (2.4) and (2.7) do not, in general, produce the same results. By restricting the quantifier on the right to one-element extensions of D we make it easier for the universal sentence on the left to come out true at D than it is according to (2.4). In the next section, however, we find special conditions under which the two clauses become interchangeable.

3 The Logic of Parametric Substitution

Which sentences of a first-order language L are parametrically valid? That is, which sentences are true in all parametric models for L? As we will show in Section 4, the set of parametrically valid sentences is a proper subset of the classically valid sentences of L. However, in a great many special cases, where the parametric models are assumed to satisfy some further conditions, the corresponding set of parametrically valid sentences will include all classically valid sentences. In fact, in some cases, it will properly include them.

3.1 Strong Nets

We begin by considering parametric models based on the minervan view of objects and the proper-name conception of constants. Among the parametric models reflecting these two conceptions there are in particular the strong nets—those models in which different extensions of a given diagram are always compatible with each other, to the point of being both included in a diagram that also belongs to the model. In what comes next we focus on strong nets.

Our first theorem asserts that, if a parametric model M is a strong net, then it verifies all theorems of classical logic.

Theorem 1 *Suppose M, a parametric model for L, is a strong net. Then $Th(M)$ includes every sentence of L that is classically valid.*

To establish Theorem 1, we first prove the following lemma:

Lemma 2 *Suppose M, a parametric model for L, is a strong net. Then, for any sentence φ of L and diagrams $D_1, D_2 \in M$, if $L(\varphi) \subseteq L(D_1)$ and $D_1 \subseteq D_2$, then $D_1 \models_M \varphi$ iff $D_2 \models_M \varphi$.*

Proof We proceed by induction on the complexity of φ to show that whenever D_1 and D_2 of M are such that $D_1 \subseteq D_2$ and $L(\varphi) \subseteq L(D_1)$, then $D_1 \models_M \varphi$ iff $D_2 \models_M \varphi$. The only interesting cases are those where φ begins with a quantifier. We only consider the one where the quantifier is existential. Let φ be of the form $\exists x \psi(x, \mathbf{c})$. Suppose first that $D_1 \models_M \varphi$. Then for some $D_3 \supseteq D_1$, and some constant $c_0 \in L(D_3)$, $D_3 \models_M \psi(c_0, \mathbf{c})$. Since M is a strong net, there is a D_4 in M such that $D_2 \subseteq D_4$ and $D_3 \subseteq D_4$. So, by the induction hypothesis, $D_3 \models_M \psi(c_0, \mathbf{c})$ implies that $D_4 \models_M \psi(c_0, \mathbf{c})$. Since $D_2 \subseteq D_4$, it follows that $D_2 \models_M \varphi$. Conversely, suppose that $D_2 \models_M \varphi$. Then, for some D_3 such that $D_2 \subseteq D_3$ and $c_0 \in L(D_3)$, $D_3 \models_M \psi(c_0, \mathbf{c})$. Since $D_1 \subseteq D_3$, $D_1 \models_M \varphi$.

To prove the theorem, let M be a strong net. We shall show that all axioms of an axiomatization of the predicate calculus are true in M, and that the inference rules of this axiomatization preserve truth in M. We use Quine's axiomatization in [10], eliminating the rule of substitution in favor of a schematic presentation, so that modus ponens is its only rule of inference. That this rule preserves truth in M is obvious from the truth definition.

So, it suffices to check that each axiom is true in M. (We already noted that the propositional logic of all parametric models we consider in this paper is classical.) We need consider only the axioms that concern quantification. The system contains two types of these (its primitives include only the universal quantifier):

(3.1) All closures of formulae of the form $\forall x \varphi \to \varphi y/x$ (y free for x in φ);
(3.2) All closures of formulae of the form $\forall x(\varphi \to \psi) \to (\varphi \to \forall x \psi)$ (where x is not free in φ).

For (3.1), let D_1 be a diagram in M. $D_1 \models_M \forall \mathbf{z} \forall y(\forall x \varphi(x, \mathbf{z}) \to \varphi(y, \mathbf{z}))$ iff, for every $D_2 \in M$ such that $D_1 \subseteq D_2$ and every $\mathbf{c}_1, c_2 \in L(D_2)$,

(3.3) $D_2 \models_M \forall x \varphi(x, \mathbf{c}_1) \to \varphi(c_2, \mathbf{c}_1)$.

(3.3) is true iff either $D_2 \not\models_M \forall x \varphi(x, \mathbf{c}_1)$ or $D_2 \models_M \varphi(c_2, \mathbf{c}_1)$. Suppose $D_2 \models_M \forall x \varphi(x, \mathbf{c}_1)$. Then, for all $D_3 \in M$ such that $D_2 \subseteq D_3$ and $c_3 \in L(D_3)$, $D_3 \models_M \varphi(c_3, \mathbf{c}_1)$. Since $D_2 \subseteq D_2$ and $c_2 \in L(D_2)$, it follows that $D_2 \models_M \varphi(c_2, \mathbf{c}_1)$. So

(3.3) holds for D_2. Since this is so for all $D_2 \supset D_1$, (3.1) holds at D_1. (Note that this result holds for all parametric models, not just for strong nets.)

For (3.2), $D_1 \models_M \forall \mathbf{z}(\forall x(\varphi(\mathbf{z}) \rightarrow \psi(x, \mathbf{z})) \rightarrow (\varphi(\mathbf{z}) \rightarrow \forall x \psi(x, \mathbf{z})))$ iff, for all $D_2 \in M$ such that $D_1 \subseteq D_2$ and all $\mathbf{c}_2 \in L(D_2)$

(3.3.1) if $D_2 \models_M \forall x(\varphi(\mathbf{c}_2) \rightarrow \psi(x, \mathbf{c}_2))$ then $D_2 \models_M \varphi(\mathbf{c}_2) \rightarrow \forall x \psi(x, \mathbf{c}_2)$.

This is true iff whenever $D_2 \models_M \varphi(\mathbf{c}_2)$ and $D_2 \models_M \forall x(\varphi(\mathbf{c}_2) \rightarrow \psi(x, \mathbf{c}_2))$ then $D_2 \models_M \forall x \psi(x, \mathbf{c}_2)$. Suppose $D_2 \models_M \varphi(\mathbf{c}_2)$ and $D_2 \models_M \forall x(\varphi(\mathbf{c}_2) \rightarrow \psi(x, \mathbf{c}_2))$; then for all $D_3 \in M$ such that $D_2 \subseteq D_3$ and all $c_3 \in L(D_3)$ $D_3 \models_M \varphi(\mathbf{c}_2) \rightarrow \psi(c_3, \mathbf{c}_2)$. Since $D_2 \models_M \varphi(\mathbf{c}_2)$, and M is a strong net, it follows from Lemma 2 that $D_3 \models_M \varphi(\mathbf{c}_2)$. So $D_3 \models_M \psi(c_3, \mathbf{c}_2)$. Since this holds for arbitrary choice of $D_3 \supseteq D_2$ and $c_3 \in L(D_3)$, $D_2 \models_M \forall x \psi(x, \mathbf{c}_2)$. This establishes that (3.3.1) holds for D_2. Since this applies to all $D_2 \supseteq D_1$, (3.1) holds at D_1. □

In Section 2.2, we defined a construction associating a parametric model $M^*(\mathfrak{M})$ with any classical model $\mathfrak{M}$. Two observations that we made in connection with that construction are repeated here as Theorem 3 and Theorem 4:

Theorem 3 $Th(M^*(\mathfrak{M})) = Th(\mathfrak{M})$.

Theorem 4 $M^*(\mathfrak{M})$ *is a strong net.*

Combining Theorems 3 and 4 with Theorem 1, we obtain:

Theorem 5 *Let K be the class of all parametric models that are strong nets. Then $Th(K)$ is the set of theorems of classical first-order logic.*

3.2 Weak Nets

Weak nets do not always include all theorems of classical logic within their theories. However, if a weak net also has the understudy properties, then its theory is again always classical. To prove this, we first show the following lemmas.

Lemma 6 *Suppose that M is a parametric model with an inexhaustible set of constants and the universal understudy property, $D \in M$, f is a one-one function from $Con D$ into U_M and $f(D) \in M$. Then, for any sentence φ of $L(D)$, $D \models_M \varphi$ iff $f(D) \models_M f(\varphi)$.*

Proof By induction on the complexity of φ. Assume that D, f are as in the statement of the lemma. The only interesting cases are those where φ begins with a quantifier; we consider only the existential quantifier. Suppose φ has the form $\exists x \psi(x)$. Assume first that $D \models_M \varphi$. Then there is a subset C of U_M, a $c \in C \cup Con D$ and a $D' \in M$ such that $D \subseteq_C D'$ and $D' \models_M \psi(c)$. Since M has an inexhaustible set of constants, $|U_M - Conf(D)| \geq |Con D' - Con D|$. So, there is a set $C' \subseteq U_M$ such that $Conf(D) \cap C' = \emptyset$ and a one-one function g from $Con D' - Con D$ onto

C'. By the universal understudy property, there is a diagram $D'' \in M$ such that $D'' = (f \cup g)(D')$. Since $f \cup g$ is a one-to-one correspondence between D' and D'', we have, by the induction hypothesis, that $D' \models_M \psi(c)$ iff $D'' \models_M (f \cup g)(\psi(c))$. But $(f \cup g)(\psi(c)) = (f(\psi))(c')$, where $c' = (f \cup g)(c)$. So, since $D' \models_M \psi(c)$, $D'' \models_M f(\psi)(c')$. Since $f(D) \subseteq (f \cup g)(D') = D''$, $f(D) \models_M \exists x f(\psi(x))$, i.e., $f(D) \models_M f(\varphi)$. In the same way we can show that if $f(D) \models_M f(\varphi)$, then $D \models_M \varphi$. The other cases of the induction are straightforward. □

Lemma 7 *Suppose M is a weak net and has the understudy properties. Then, for any $D_1, D_2 \in M$ such that $D_1 \subseteq D_2$, and every sentence φ of $L(D_1)$, $D_1 \models_M \varphi$ iff $D_2 \models_M \varphi$.*

Proof By induction on the complexity of φ. We consider only the case where φ is $\exists x \psi(x)$. Suppose $D_1 \models_M \varphi$. Then, for some $D_3 \in M$ such that $D_1 \subseteq D_3$ and some constant $c_3 \in L(D_3)$, $D_3 \models_M \psi(c_3)$. Let C be the set $(Con D_3 - Con D_1) \cap (Con D_2 - Con D_1)$. Since M has the existential understudy property, there is a set $C' \subseteq U_M$ such that C' is disjoint from $Con D_2$ and there is a one-one map f from C onto C' such that $f(D_3) \in M$. By lemma 6, $f(D_3) \models_M \psi(f(c_3))$. Since, moreover, $D_2 \restriction L(D_1) = f(D_3) \restriction L(D_1) = D_1$, there exists, because M is a weak net, a diagram $D_4 \in M$ such that $D_2 \subseteq D_4$ and $f(D_3) \subseteq D_4$. By the induction hypothesis, $D_4 \models_M \psi(f(c_3))$ and so $D_2 \models_M \exists x \psi(x)$. The converse follows quickly from (2.5), the recursion clause for $\exists$ in the truth definition. □

Theorem 8 *Suppose that M is a weak net and has the understudy properties. Then, for each sentence φ of $L(M)$ that is a theorem of classical first-order logic, $M \models \varphi$.*

Proof Like that for Theorem 1, with Lemma 7 replacing Lemma 2. □

3.3 Open Models

An open model is a parametric model satisfying (2.6). We can approximate the idea behind (2.6)—of a model for a language L in which all possibilities are realized—without running afoul of the set-theoretic paradoxes by considering parametric models containing all diagrams up to some fixed infinite cardinality κ. Once κ and a set W of cardinality κ, which is to serve as inexhaustible supply of individual constants and therefore must be disjoint from the set of expressions of L, have been chosen, that determines the open model modulo a choice for the model's core. Formally, let κ and W be as indicated, let C be a subset of W with $| C | < \kappa$ and let D be a diagram for the language $L(C)$. Then *the open model of cardinality κ for W and D*, $O_{\kappa,W,D}$, is the set $\{D' \supseteq D : \exists U(Con D \subseteq U \subseteq W \wedge | U | < \kappa \wedge Con D = U)\}$.

Since every constant in W occurs in some diagrams of $O_{\kappa,W,D}$, W can be recovered from $O_{\kappa,W,D}$. Likewise the core D is recoverable from $O_{\kappa,W,D}$. So if O is any open model of cardinality κ, we can refer to the set W and the core D such that O is the open model of cardinality κ for W and D as 'W_O' and 'D_O', respectively. It should be clear that when W and W' are two sets of cardinality κ, such that $ConD \subseteq W \cap W'$, then $O_{\kappa,W,D}$ and $O_{\kappa,W',D}$ are isomorphic (in the intuitively obvious sense) and thus assign the same truth values to all sentences of L. For our purposes any two such models can be identified. So, we will treat this isomorphism class as if it were a single model. We assume that a representative of each isomorphism class has been selected, without bothering to make the selection procedure explicit, and refer to these models as *the open model for L of cardinality κ with core D*, for any possible choice of L, κ, and D.

Among the open models there are in particular those where the language L is without individual constants and the model's core is the empty diagram $\emptyset$. We refer to the (isomorphism class of the) model $O_{\kappa,W,\emptyset}$ (where W is some set of cardinality κ disjoint from L) as *the open model of cardinality κ*. The results we will prove in this section are formulated for such models, but generalizations to models with non-empty cores are possible.

The main (and probably not surprising) upshot of our formal results is that all open models O for a given language L determine the same theory $Th(O)$. Moreover, this theory is decidable. It can be characterized axiomatically in quite simple terms. That the size of κ does not make a difference to $Th(O)$ has to do with the fact that for open models the quantifier clauses (2.4) and (2.7) are equivalent. (See lemma 11 below.) This means that, once again, we can use (2.7) when applying the method of proof by induction on the syntactic complexity of formulas of L. And that in its turn means that the truth values of any sentence ϕ in an open model O (at its empty core) will be fully determined by those diagrams of O that each involve only a finite number of individual constants. It is this feature—that only diagrams with finitely many constants are required to determine the truth value of a sentence at $\emptyset$—that is at the heart of the proof of Theorem 13, according to which the theory of an open model is decidable; and it is equally crucial to the proof of Theorem 14, according to which $Th(O)$ has a comparatively simple axiomatization, consisting entirely of ∀∃-sentences.[7]

Theorem 9 is a preparation for the more telling results that follow.

Theorem 9 *Let O be an open model of infinite cardinality κ. Then (i) O is a weak net; (ii) O has the universal understudy property; (iii) O has the existential understudy property; and (iv) O is reductively complete.*

Proof Let O be an open model of infinite cardinality κ. (i) Let $D_1, D_2 \in O$ and suppose that $D_1 \upharpoonright (L(D_1) \cap L(D_2)) = D_2 \upharpoonright (L(D_1) \cap L(D_2))$. Then $D_1 \cup D_2$ is a well-defined function. Moreover, because $\kappa \geq \aleph_0$, $|ConD_1 \cup ConD_2| < \kappa$. Let

[7] Note that the simplicity of the axioms that can be used to axiomatize $Th(O)$ does not by itself entail the decidability of $Th(O)$, since the only sets of such simple axioms for $Th(O)$ might not themselves be decidable. We return to this point below when proving $Th(O)$'s decidability. We thank an anonymous reviewer for suggesting that we bring forward to its present location the point that the class of ∀∃-formulas of predicate logic should not be confused with the decidability of ∀∃ theories.

$L' = L \cup ConD_1 \cup ConD_2$. We have to show that O contains a diagram D_3 for the language L' such that $D_1 \cup D_2 \subseteq D_3$. We define D_3 as follows:

(a) if φ is an atomic sentence of $L(D_1) \cup L(D_2)$, then $D_3(\varphi) = (D_1 \cup D_2)(\varphi)$;
(b) if φ is an atomic sentence of $L' \setminus (L(D_1) \cup L(D_2))$, then $D_3(\varphi) = 0$.

It is easy to check that D_3 is a diagram for the language L' and that $D_1 \cup D_2 \subseteq D_3$. By the definition of 'open model of cardinality κ,' $D_3 \in O$.

(ii) Let $D_1, D_2, D_3 \in O$ be such that $D_1 \subseteq D_3$, $D_1 \approx_f D_2$ for some one-one function f, D_3 is f-invariant, $|U_O - ConD_2| \geq |ConD_3 - ConD_1|$, and g a one-one function from $ConD_3 - ConD_1$ onto a set of constants $C \subseteq U_O$ disjoint from $ConD_2$. Since $|ConD_3 - ConD_1| < \kappa$, $|C| < \kappa$. Also, $|ConD_2| < \kappa$; so |the range of $(f \cup g)| = |ConD_2 \cup C| < \kappa$. Let D_4 be the diagram defined by $D_4(P(c'_1, ..., c'_n)) = D_3(P(c_1, ..., c_n))$, where $c_1, ..., c_n$ are any constants in $ConD_3$ such that for $i \in \{1, ..., n\}$, $c'_i = (f \cup g)(c_i)$. Note that the definition of c'_1 is proper. It is also easy to see that, since $D_1 \approx_f D_2$, $D_2 \subseteq D_4$ and, finally, that $D_3 \approx_{f \cup g} D_4$, and so, by the definition of 'open model,' $D_4 \in O$. This shows that O has the universal understudy property. (iii) That O also has the existential understudy property follows from (ii) together with the fact that it has an inexhaustible set of constants. (iv) Obvious. □

In view of Theorem 8, Theorem 9 yields the following corollary:

Theorem 10 *Let O be an open model for some language L of infinite cardinality κ. Then $Th(O)$ is a first-order theory of L.*

The following lemma is needed for a general characterization of the logic of open models of infinite cardinality κ.

Lemma 11 *If a parametric model M is reductively complete and satisfies:*

(3.4) *If $D_1 \subseteq D_2$ and $\varphi \in L(D_1)$, then $D_1 \models_M \varphi$ iff $D_2 \models_M \varphi$,*

then the two truth definitions involving quantifier clauses (2.4) and (2.7) are equivalent on M. That is, where $\models'$ is like $\models$ except that (2.7) replaces (2.4): for any $D \in M$ and φ in $L(D)$, $D \models_M \varphi$ iff $D \models'_M \varphi$.

The intuitive meaning of Lemma 11 can be described as follows: In parametric models that are reductively complete and satisfy (3.4), any object c that is found in any diagram $D' \supseteq D$ is also found in some diagram $D'' \supseteq D$ that is obtained by adding just c to D, and with the same properties and standing in the same relations to objects in D as it does in D'. That is, in such models, the introduction of a new object never depends on the introduction of other objects; whenever it is possible to introduce a set of new objects, it is also possible to introduce them one by one.

Proof Suppose that M is reductively complete and satisfies (3.4). Again we proceed by induction on the complexity of φ and consider only the case where φ has the form $\exists x \psi(x)$. Suppose first that $D \models_M \varphi$. Then there is a $D' \in M$ such that $D \subseteq D'$

and a $c \in Con D'$ such that $D' \models_M \psi(c)$. Since M is reductively complete, there is a $D'' \in M$ such that $D \subseteq_c D''$ and $D'' \subseteq D'$. Since M satisfies (3.4), $D'' \models_M \psi(c)$. By inductive hypothesis, $D'' \models'_M \psi(c)$, so by clause (2.8), $D \models'_M \exists x \psi(x)$. Conversely, suppose that $D \models'_M \exists x \psi(x)$. Then there is a constant c and a $D' \in M$ such that $D \subseteq_c D'$ and $D' \models'_M \psi(c)$. By inductive hypothesis, $D' \models_M \psi(c)$. So, by clause (2.5), $D \models_M \exists x \psi(x)$. □

We have seen that open models of infinite cardinality κ are weak nets, are reductively complete, have an inexhaustible set of constants and have the universal understudy property. So, it follows by Lemmas 6, 7, and 11 that the two truth definitions represented by $\models$ and $\models'$ coincide on such models. Henceforth in this section we write $\models$ even though we will often make use of the truth clause (2.7) rather than (2.4) or (2.5) in arguments that some quantified formula holds at some diagram of some open model.

The main results concerning open models are Theorems 12, 13, and 14 below.

Theorem 12 *Suppose that L does not contain any individual constants, and let O be an open model for L of infinite cardinality κ. Then $Th(O)$ is decidable.*

The notation for the fully general method of the proof of this theorem is cumbersome, but we think that an example should suffice. Consider the simple quantified sentence $\forall x \exists y Rxy$ belonging to the language $\{R\}$. According to the parametric truth definition applied to an open model O for $\{R\}$ of infinite cardinality this is equivalent to

(3.5) $(\forall c)(\forall D \supseteq_c \emptyset)(\exists c')(\exists D' \supseteq_{c'} D)(D' \models_O Rcc')$.[8]

Since O has the universal understudy property, we may replace this by the equivalent

(3.6) $(\forall D \supseteq_{c_1} \emptyset)(\exists c')(\exists D' \supseteq_{c'} D)(D' \models_O Rc_1c')$, where c_1 is some arbitrarily chosen constant from the set W_O.

There are just two diagrams in the language $\{R, c_1\}$, namely D_1 defined by $D_1(Rc_1c_1) = 1$ and D_2 defined by $D_2(Rc_1c_1) = 0$. So, (3.6) is equivalent to a conjunction

(3.7) $(\exists c')(\exists D' \supseteq_{c'} D_1)(D' \models_O Rc_1c') \wedge (\exists c')(\exists D' \supseteq_{c'} D_2)(D' \models_O Rc_1c')$

The first conjunct of (3.7) can be rewritten as a disjunction between the case where the constant c' is different from c_1 and that where it is c_1.

(3.9) $(\exists c')(c' \neq c_1 \wedge (\exists D' \supseteq_{c'} D_1)(D' \models_O Rc_1c')) \vee (\exists D')(D' \supseteq_{c_1} D_1 \wedge D' \models_O Rc_1c_1)$.

Since c_1 belongs to $Con D_1$, the condition '$D' \supseteq_{c_1} D_1$' that is part of the second disjunct of (3.9) says that D' is the same as D_1. So the second disjunct can be

[8]Note that the condition '$c' \neq c$' means that c' and c are distinct symbols. It does not mean that they have distinct denotations.

replaced by $D_1 \models_O Rc_1c_1$, which we know to be the case. So we can conclude that (3.9) as a whole is true.

The second conjunct of (3.7) can be rewritten in analogous way to the first conjunct as in (3.10).

(3.10) $(\exists c')(c' \neq c_1 \wedge (\exists D' \supseteq_{c'} D_{1,2})(D' \models_O Rc_1c')) \vee (\exists D')(D' \supseteq_{c_1} D_2 \wedge D' \models_O Rc_1c_1)$.

Here the second disjunct is false. But the first disjunct can be rewritten as a disjunction of statements about particular small finite diagrams. Note that because of of the universal understudy property the first disjunct of (3.10) can be replaced by (3.11):

(3.11) $(\exists D' \supseteq_{c_2} D_2)(D' \models_O Rc_1c_2)$,

where c_2 is some particular constant distinct from c_1. (3.11) is equivalent to the disjunction in (3.12) of statements about the eight diagrams with Domain $\{c_1, c_2\}$ that extend D_2—let us call them $D_{2.1}$, $D_{2.2}$, ..., $D_{2.8}$—and differ from each other in the truth values they assign to the atomic sentences Rc_1c_2, Rc_2c_1, Rc_2c_2. Since O is an open model, each of these diagrams belongs to O.

(3.12) $\bigvee_{i=1,\dots,8} D_{2.i} \models_O Rc_1c_2$.

It is obvious that (3.12) is decidable: we can obviously write out the finite truth table of each of the diagrams $D_{2.i}$ and check whether it assigns 1 or 0 to the atomic sentence Rc_1c_2 in question. (In this case it is obvious that there is at least one $D_{2.i}$ that verifies Rc_1c_2 and thus that (3.12) is true. So the result of the above procedure is that $\forall x \exists y Rxy$ is true in O.)

The general moral of this example is that the question whether a complex sentence is true or false in an open model O can be rewritten into some kind of Boolean combination of statements of the truth of atomic sentences in finite diagrams that can be identified in terms of the truth values they assign to atomic sentences, so that the truth values of the atomic sentences in question in those diagrams can be simply read off from the tables that identify the diagrams. The decidability of the boolean combination of these statements then follows from the decidability of classical propositional calculus. (It should also be clear that the size of L does not impose an essential restriction. The argument applied to $\forall x \exists y Rxy$ is easily modified so that it applies to sentences of other languages with a finite set of predicates. And even if L is infinite any sentence φ of L will contain only a finite number of predicates and thus be treated as a sentence of the language $L(\varphi)$ which contains just the predicates that occur in φ.)

Theorem 13 *Suppose that L does not contain any individual constants and that O is an open model for L of infinite cardinality κ. Then $Th(O)$ is an $\forall\exists$ theory. More exactly, $Th(O)$ is axiomatized by an axiomatization of first-order logic together with the set $A(O)$ of all sentences of the form*

$$\forall x_1, \dots, \forall x_n \exists y_1, \dots, y_m F(\mathbf{x}, \mathbf{y}),$$

where (i) $n \geq 0$, (ii) $m \geq 1$, (iii) $F(\mathbf{x}, \mathbf{y})$ is a classically consistent conjunction of basic formulae built from predicates of L and variables from the list $x_1, \dots, x_n$,

$y_1, ..., y_m$, and (iv) in each conjunct of $F(\mathbf{x}, \mathbf{y})$ there is at least one occurrence of a variable y_i ($i = 1, ..., m$).

Proof We begin by assuming that L has only finitely many predicates. First we show that all sentences in $A(O)$ are true in O. Let $\forall\mathbf{x}\exists\mathbf{y}F(\mathbf{x}, \mathbf{y})$ be a member of $A(O)$. By assumption, $F(\mathbf{x}, \mathbf{y})$ is a conjunction $\bigwedge_{i=1,...,k} p_i(\beta_i(\mathbf{x}, \mathbf{y}))$ of formulae $p_i(\beta_i(\mathbf{x}, \mathbf{y}))$, where, for each i, β_i is an atomic formula in the predicates of L and the variables $x_1, ..., x_n, y_1, ..., y_m$ and $p_i(\beta_i(\mathbf{x}, \mathbf{y}))$ is either $\beta_i(\mathbf{x}, \mathbf{y})$ or $\neg\beta_i(\mathbf{x}, \mathbf{y})$, and where, moreover, each β_i contains at least one occurrence of a variable y_j ($1 \leq j \leq m$). To establish $\models_O \forall\mathbf{x}\exists\mathbf{y} \bigwedge p_i(\beta_i(\mathbf{x}, \mathbf{y}))$, we must show that, for all $c_1, ..., c_n \in U_O$ and $D \supseteq_{\{c_1,...,c_n\}} \emptyset$, $D \models_O \exists\mathbf{y} \bigwedge(p_i(\beta_i(\mathbf{x}, \mathbf{y})))c_1/x_1...c_n/x_n$. This is the case provided that there are, for each such D, $c'_1, ..., c'_m \in U_O$, a $D' \in O$ such that $D' \supseteq_{\{c'_1,...,c'_m\}} D$ and

(3.23) $\quad D' \models_O \bigwedge(p_i(\beta_i(\mathbf{x}, \mathbf{y})))c_1/x_1...c_n/x_n, c'_1/y_1...c'_m/y_m.$

Suppose D is any diagram in the language $L \cup \{c_1, ..., c_n\}$. We must show that there is a diagram $D' \in O$ in the language $L \cup \{c_1, ..., c_n, c'_1, ..., c'_m\}$ such that (a) $D \subseteq D'$ and (b) for each conjunct $p_i(\beta_i(\mathbf{x}, \mathbf{y}))c_1/x_1 ... c_n/x_n, c'_1/y_1 ... c'_m/y_m$ of the formula in (3.23), $D'((p_i(\beta_i(\mathbf{x}, \mathbf{y}))c_1/x_1 ... c_n/x_n, c'_1/y_1 ... c'_m/y_m) = 1$ if $p_i(\beta_i(\mathbf{x}, \mathbf{y})) = \beta_i(\mathbf{x}, \mathbf{y})$, and $D'((p_i(\beta_i(\mathbf{x}, \mathbf{y})))c_1/x_1 ... c_n/x_n, c'_1/y_1 ... c'_m/y_m) = 0$ if $p_i(\beta_i(\mathbf{x}, \mathbf{y})) = \neg\beta_i(\mathbf{x}, \mathbf{y})$. To see that such a diagram exists, note that each of the formulae $(\beta_i(\mathbf{x}, \mathbf{y}))c_1/x_1 ... c_n/x_n, c'_1/y_1 ... c'_m/y_m$ contains at least one occurrence of one of the constants $c'_1, ..., c'_m$. This means that nothing is said in D about the predication $\beta_i(\mathbf{x}, \mathbf{y})))\mathbf{c}/\mathbf{x}, \mathbf{c}'/\mathbf{y}$. So we can stipulate the value of $D'(\beta_i(\mathbf{x}, \mathbf{y})\mathbf{c}/\mathbf{x}, \mathbf{c}'/\mathbf{y})$ in accordance with the polarity p_i of β_i. Evidently there exists a D' extending D that satisfies all these stipulations. This shows that $\forall\mathbf{x}\exists\mathbf{y}F(\mathbf{x}, \mathbf{y})$ holds in O at its empty core (and thus is true in O).

Therefore the requirements under (ii) are all independent of the condition (a) that $D \subseteq D'$. So, there exists a D' satisfying both (a) and (b). Furthermore, as $\{c_1, ..., c_n, c'_1, ..., c'_m\} \subseteq U_O$ and O is an open model of infinite cardinality κ, $D' \in O$. Note that, since O satisfies (3.4), if φ is an axiom of $A(O)$ and D is any member of O, then $D \models_O \varphi$.

To show that $A(O)$ yields all sentences in $Th(O)$ we argue as follows. For any diagram D in the language $L' = L \cup \{c_1, ..., c_m\}$ and variables $x_1, ..., x_m$, let $\bigwedge D(x_1, ..., x_m)$ be the result of (c) forming the conjunction of all atomic sentences ψ in L' such that $D(\psi) = 1$ and all negations of such sentences ψ such that $D(\psi) = 0$; and (d) substituting in this conjunction x_i for $c_i (i = 1, ..., m)$. In case $D = \emptyset$, we stipulate that $\bigwedge D(x_1, ..., x_m)$ is some fixed tautology $\top$ of the language L. (Note that when L is finite and has no individual constants, then any diagram for $L \bigcup \{c_1, ..., c_n\}$ is finite, so $\bigwedge D$ is a formula of (finitary) first-order logic.)

Similarly, for any sentence φ of L', let $\varphi(x_1, ..., x_m)$ be the result of replacing the c_i by the x_i. Then, for any such D and φ, we claim that

(3.24) $\quad D \models \varphi$ iff $A(O) \vdash \forall\mathbf{x}(\bigwedge D(x_1, ..., x_m) \rightarrow \varphi(x_1, ..., x_m)).$

(3.24) is proved by induction on the complexity of φ. Taking the special case where $m = 0$, so that $L = L'$ and $D = \emptyset$, we get the conclusion of the theorem: $\models_O \varphi$ iff $A(O) \vdash \top \to \varphi$ iff $A(O) \vdash \varphi$.

First observe that for each $D \in O$ the set of sentences φ of $L(D)$ such that $D \models_O \varphi$ is a first-order theory. This follows from the fact that Theorem 10 applies to the model $O \upharpoonright \{D' \in O : D \subseteq D'\}$. Consequently, it will suffice to prove (3.24) for all sentences φ in prenex form. We prove this restricted version of (3.24) by induction on the complexity of φ. First, let φ be quantifier-free. Then, if $D \models \varphi$, $\bigwedge D(x_1, ..., x_m) \to \varphi(x_1, ..., x_m)$ has the form of a classical tautology. So, obviously, since our $\forall\exists$ theory is a theory of classical logic, (3.24) holds in the left-to-right direction. It also holds from right to left. For, as we already saw, all the axioms of $A(O)$ are true in O at D. Moreover, truth in O at D is preserved by the inference rules of first-order logic. So, if $A(O) \vdash \forall \mathbf{x}(\bigwedge D(x_1, ..., x_m) \to \varphi(x_1, ..., x_m))$, then $D \models_O \forall \mathbf{x}(\bigwedge D(x_1, ..., x_m) \to \varphi(x_1, ..., x_m))$. So, $D \models_O \bigwedge D \to \varphi$. But evidently $D \models_O \bigwedge D$. Therefore, $D \models_O \varphi$.

The inductive step we shall consider is that where φ is $\exists y \psi(c_1, ..., c_m, y)$. Assume first that $D(c_1, ..., c_m) \models_O \varphi$. Then there is a constant c_{m+1} and a D' such that $D \subseteq_{c_{m+1}} D'$ and $D' \models_O \psi(c_1, ..., c_m, c_{m+1})$. So, by induction hypothesis and classical quantification theory,

(3.25) $\quad A(O) \vdash \forall \mathbf{x}, y(\bigwedge D'(x_1, ..., x_m, y) \to \psi(x_1, ..., x_m, y))$.

$\bigwedge D'(x_1, ..., x_m, y)$ can be written as a conjunction $\delta_1(x_1, ..., x_m) \wedge \delta_2(x_1, ..., x_m, y)$, where each conjunct of δ_2 contains at least one occurrence of y. Clearly δ_2 is a consistent conjunction of basic formulae. So, $\forall \mathbf{x} \exists y \delta_2(\mathbf{x}, y)$ is an axiom of our axiomatization; hence, $A(O) \vdash \forall \mathbf{x} \exists y \delta_2(\mathbf{x}, y)$. Thus, $A(O) \vdash \forall \mathbf{x}(\delta_1(\mathbf{x}) \to (\delta_1(\mathbf{x}) \wedge \exists y \delta_2(\mathbf{x}, y)))$. But $\delta_1(\mathbf{x})$ is just $\bigwedge D(x_1, ..., x_m)$. So,

(3.26) $\quad A(O) \vdash \forall \mathbf{x}(\bigwedge D(\mathbf{x}) \to \exists y \bigwedge D'(\mathbf{x}, y))$.

(3.25) and (3.26) give

(3.27) $\quad A(O) \vdash \forall \mathbf{x}(\bigwedge D(\mathbf{x}) \to \exists y \psi(\mathbf{x}, y))$.

Conversely, suppose that $D \not\models_O \varphi(c_1, ..., c_m)$. Then $D \models_O \forall y \psi^*(c_1, ..., c_m, y)$, where ψ^* is the prenex formula equivalent to $\neg\psi$ (obtained by, say, working the outer negation sign all the way in). So, for each D_i such that $D \subseteq_{c_{m+1}} D_i$, $D_i \models_O \psi^*(c_1, ..., c_m, c_{m+1})$. Therefore, by induction hypothesis, $A(O) \vdash \forall \mathbf{x}, y(\bigwedge D_i(x_1, ..., x_m, y) \to \psi^*(x_1, ..., x_m, y))$ for each such D_i. But clearly

(3.28) $\quad A(O) \vdash \forall \mathbf{x}, y(\bigwedge D(\mathbf{x}) \to \bigvee_i \bigwedge D_i(\mathbf{x}, y))$[9]

So, $A(O) \vdash \forall \mathbf{x}, y(\bigwedge D(\mathbf{x}) \to \psi^*(\mathbf{x}, y))$, and consequently $A(O) \vdash \forall \mathbf{x}(\bigwedge D(\mathbf{x}) \to \neg\exists y \psi(\mathbf{x}, y))$. Suppose that $A(O) \vdash \forall \mathbf{x}(\bigwedge D(\mathbf{x}) \to \exists y \psi(\mathbf{x}, y))$. Then by classical logic $A(O) \vdash \forall \mathbf{x}(\bigwedge D(\mathbf{x}) \to (\exists y \psi(\mathbf{x}, y) \wedge \neg\exists y \psi(\mathbf{x}, y)))$. So $A(O) \vdash \forall \mathbf{x} \neg \bigwedge D(\mathbf{x})$

[9]In fact this is a theorem of ordinary predicate logic. Note that the formula is meaningful, since there are only finitely many extensions D_i that are diagrams for the language $L \bigcup \{c_1, ..., c_m\}$. So $\bigvee_i \bigwedge D_i(\mathbf{x}, y)$ is a finite disjunction.

and thus $A(O) \vdash \neg\exists\ \mathbf{x} \bigwedge D(\mathbf{x})$. But, by the first half of the proof, for every sentence ϑ derivable from $A(O)$ and every diagram D' of O, $D' \models_O \vartheta$. So, we conclude that $D \models_O \neg\exists\ \mathbf{x} \bigwedge D(\mathbf{x})$. But evidently $D \models_O \exists \mathbf{x} \bigwedge D(\mathbf{x})$, which entails that $D \not\models_O \neg\exists\ \mathbf{x} \bigwedge D(\mathbf{x})$. So by reductio we conclude that $A(O) \nvdash \forall \mathbf{x}(\bigwedge D(\mathbf{x}) \rightarrow \exists y \psi(\mathbf{x}, y))$.

This completes the proof of (3.25) for the case that L has finitely many predicates. When L has a denumerably infinite number of predicates, then L is the union $\bigcup_n L_n$ of languages L_n each of which has finitely many predicates and where, for each n, $L_n \subset L_{n+1}$. It is easy to see that the union of the axiom sets $A(L_n)$ for $n = 1, 2, ...$ axiomatizes $Th(O)$ where O is an open model for L. Evidently the axioms in this union are all of the required form. This concludes the proof of Theorem 13. □

Let L be a language with finitely many predicates and no individual constants and suppose that M is any parametric model for L that shares some of the chief properties of open models: in particular, suppose M is a weak net, has an inexhaustible set of constants, is reductively complete and has the universal understudy property. Then, according to Theorem 8, $Th(M)$ contains all of first-order logic. Moreover, it is easy to see that $Th(M)$ is axiomatized by the set $A(M)$, consisting of

(e) all axioms of the form $\forall \mathbf{x}(\bigwedge D(\mathbf{x}) \rightarrow \exists y \bigwedge D'(\mathbf{x}, y))$, where for some $c_1, ..., c_m, c_{m+1}$, $D(c_1, ..., c_m)$, $D'(c_1, ..., c_m, c_{m+1}) \in M$ and $D \subseteq_{c_{m+1}} D'$; and
(f) all axioms of the form $\forall \mathbf{x}, y(\bigwedge D(\mathbf{x}) \rightarrow \bigvee_i \bigwedge D_i(\mathbf{x}, y))$, where for some $c_1, ..., c_m, c_{m+1}$, $D(c_1, ..., c_m) \in M$ and $D_1(c_1, ..., c_m, c_{m+1})$, ..., $D_n(c_1, ..., c_m, c_{m+1})$ are all the $D_i \in M$ such that $D \subseteq_{c_{m+1}} D_i$.[10]

We summarize this result as

Theorem 14 *Suppose that M is a parametric model for a language L with finitely many predicates and no individual constants and that M is a weak net, has an inexhaustible set of constants, is reductively complete and has the universal understudy property. Then $Th(M)$ is the first-order theory axiomatized by the axioms of forms (e) and (f) above.*

The proof of Theorem 14 is completely analogous to that of Theorem 13. We remark that, although $A(M)$ will in general be more comprehensive than the special set $A(O)$ we defined in connection with Theorem 13, it remains true that $A(M)$ is a set of axioms all of which are of ∀∃ form. (Purely universal formulas can also be regarded as (degenerate) instances of this form.) Note also that Theorem 14 entails that the axiomatization we have proposed for the theory of open models with empty core can be simplified somewhat: we can make do with those axioms of the form $\forall x_1, ..., \forall x_n \exists y_1, ..., y_m F(\mathbf{x}, \mathbf{y})$ in which $m = 1$ (i.e. those in which the initial universal quantifiers are followed by a single existential quantifier). Inspection of the proof of Theorem 13 shows that it goes through also when we impose this restriction on the form of the axioms of $A(O)$.

[10] Note that because of the properties of M the choice of constants here is immaterial.

If M is not reductively complete but satisfies the other conditions mentioned in Theorem 14, then in general no simple axiomatization results for $Th(M)$ seem to be forthcoming. However, an axiomatization similar to the one given in Theorem 14 can be formulated for the case where reductive completeness is replaced by the following weaker condition: there is a fixed natural number k such that, whenever $D, D' \in M$ and $D \subseteq D'$, there is for each $c \in ConD' - ConD$ a $D'' \in M$ such that $D \subseteq D'' \subseteq D'$, $c \in ConD''$, and $|ConD''| \leq |ConD| + k$. The quantifier over y in the axioms $\forall\mathbf{x}, y(D(\mathbf{x}) \rightarrow \bigvee_i D_i(\mathbf{x}, y))$ then has to be replaced by a string of k quantifiers over distinct variables $y_1, ..., y_k$.

Returning to the preconditions of Theorem 14, note that even if $Th(M)$ is axiomatized by a set of $\forall\exists$-formulae, this does not entail that the theory is decidable or even that it is recursively enumerable. (Even a theory axiomatized by atomic sentences need not be r.e.; it will be if and only if the specified axiom set is.) Suppose that M satisfies the hypothesis of Theorem 14. Since $Th(M)$ is a complete first-order theory, it will be decidable iff $A(M)$ is recursively enumerable. The recursive enumerability of $A(M)$ can be restated as follows. We can associate with M a function f_M defined on the finite diagrams $D(c_1, ..., c_m)$ in M, which assigns to each such diagram the set of diagrams $D'(c_1, ..., c_m, c_{m+1})$ such that for some constant $c'_{m+1} D'(c_1, ..., c_m, c'_{m+1}) \in M$. Because of the special properties of M, this set is finite and independent of the choice of constants. (That is, if $D(c_1, ..., c_m) \in M$ and $c'_1, ..., c'_m$ is some other m-place sequence of constants, then $D(c'_1, ..., c'_m)$ will also belong to M; and, likewise, the set of diagrams $D'(c'_1, ..., c'_m, c_{m+1})$ that makes up the f-value $f(D(c'_1, ..., c'_m))$ is independent of the choice of $c'_1, ..., c'_m, c_{m+1}$.) That the f-values are finite sets follows from the fact that the arguments of f are finite diagrams and that the diagrams in f-values extend the f-arguments with just one constant.

The connection between the decidability of $Th(M)$ and f is stated in Theorem 15.

Theorem 15 *$Th(M)$ is decidable iff the graph of f is recursively enumerable.*

Proof It is convenient to assume that $x_1, ..., x_m, x_{m+1}$ is a fixed enumeration of the variables of L and to recast f in the form of a function f' which is defined on the set of those finite 'diagrams' $D(x_1, ..., x_m)$, such that for some $c_1, ..., c_m D(c_1, ..., c_m) \in M$ and which assigns to each such 'diagram' the finite set of 'diagrams' $D'(x_1, ..., x_m, x_{m+1})$ such that $D'(c'_1, ..., c'_m, c'_{m+1}) \in M$ for some c'_1, ..., c'_m, c'_{m+1}. (The remarks above about invariance of M under replacement of constants guarantee that f' is well-defined and that its graph is r.e. iff the graph of f is.)

First, assume that the graph of f' is recursively enumerable. Suppose φ is a sentence of the form (e) specified in the paragraph above Theorem 14; that is, φ is of the form $\forall\mathbf{x}(\bigwedge D(\mathbf{x}) \rightarrow \exists y \bigwedge D'(\mathbf{x}, y))$. The set of axioms of $Th(M)$ of this form is enumerated by f' in the following way: whenever f' generates a pair $< D(x_1, ...x_m), \{D_1(x_1, ...x_m, x_{m+1}), ..., D_n(x_1, ...x_m, x_{m+1})\} >$ and D' is among the $D_1, ..., D_n$, then φ is registered as one of the axioms; formulas φ such that $D' \notin \{D_1(x_1, ...x_m, x_{m+1}), ..., D_n(x_1, ...x_m, x_{m+1})\}$ will be registered at no point and thus not be included in the list. As regards formulae of the form (f): these are included in the enumeration of the axioms of $Th(M)$ iff and when the enumeration of the graph of f' yields the pair $< D(x_1, ...x_m), \{D_1(x_1, ... x_m, x_{m+1}), ..., D_n(x_1, ... x_m, x_{m+1})\} >$. Since $Th(M)$ is a complete theory, the recursive enumerability of an axiomatization of it entails its decidability.

For the converse assume that $Th(M)$ is decidable. To generate the graph of f' we proceed as follows. Let $D(\mathbf{x})$ be any finite 'diagram' of L. We consider all sentences of the form (f) $\forall \mathbf{x}, y(\bigwedge D(\mathbf{x}) \to \bigvee_i \bigwedge D_i(\mathbf{x}, y))$ for the given D and arbitrary disjunctions $\bigvee_i \bigwedge D_i(\mathbf{x}, y)$, in which the D_i are 'diagrams' for L. Since $Th(M)$ is decidable, we can generate it recursively. Suppose that in the course of generation we hit upon a sentence of the form (f). Then we know that $f'(D)$ is included in the set of D' that occur among the disjuncts of $\bigvee_i \bigwedge D_i(\mathbf{x}, y))$. But since this last set is finite, there is only a finite number of sentences of form (f) such that the set of conjunctions $\bigwedge D'(\mathbf{x}, y)$ that occur as disjuncts of their consequents is included within the set of disjuncts $\bigwedge D_i(\mathbf{x}, y)$ in the sentence that has just been generated. (Since that set is finite, it has only finitely many subsets.) We can check for each of these finitely many sentences whether or not it also belongs to $Th(M)$. Among these sentences there will be one for which the set of 'diagrams' D' occurring as disjuncts in its consequent is minimal (i.e., is included in the sets of disjuncts that make up the consequents of all the other sentences of this form that are true in M; we leave the proof of this to the reader). This sentence will define the value of $f'(D)$: $f'(D)$ is the set of 'diagrams' $D'(\mathbf{x}, y)$ occurring as disjuncts in its consequent. In this way we can generate the graph of f', and from that the graph of f. □

An obvious consequence of Theorem 15 is that when f is a recursive function, then $Th(M)$ is decidable. (If f is recursive, then its graph is r.e.)

All the results of this section show (in their different ways) the simplicity of the first-order theories of parametric models that satisfy enough closure and uniformity conditions.

4 The Class of all Parametric Models

In this section we deal with the notions of validity and logical consequence with respect to the class of all parametric models. As we remarked earlier, these notions are weaker than their classical counterparts, in the sense that fewer formulae are valid, and (with only marginal exceptions) fewer formulae are logical consequences of any given formula set. Since our treatment of the sentential connectives is classical, the parametric and classical notions of validity and consequence coincide for quantifier-free formulae. But in the realm of quantification there are important differences. For instance, none of the following classical theorems are valid parametrically:

(4.1) $\forall x(\varphi \to \psi) \to (\varphi \to \forall x \psi)$, where x is not free in φ. (Quine's axiom schema (3.2) above)

(4.2) $(\varphi \to \forall x \psi) \to \forall x(\varphi \to \psi)$, where x is not free in φ. (The converse of (4.1))

(4.3) $\exists x \forall y \varphi \to \forall y \exists x \varphi$

(4.4) $(\exists x\varphi \wedge \exists y\psi) \rightarrow \exists x\exists y(\varphi \wedge \psi)$, where x does not occur in ψ and y does not occur in φ)

(4.5) $\forall x\forall y(\varphi \vee \psi) \rightarrow (\forall x\varphi \vee \forall y\psi)$, where x does not occur in ψ and y does not occur in φ)

In general, formulas that Quine calls 'rules of passage' are not parametrically valid.

It may be instructive to see *in concreto* why these schemata are not parametrically valid. For example, consider (4.1)'s instance:

(4.6) $\forall x(\exists yPy \rightarrow Px) \rightarrow (\exists yPy \rightarrow \forall xPx)$

Let us assume that the language L, for which we define a parametric model in which (4.6) fails, is as small as it can be if it is to have (4.6) as one of its sentences; in other words, the only nonlogical constant of L is the 1-place predicate P. Let M be the parametric model $\{D_0, D_1, D_2\}$ where $D_0 = \emptyset$, D_1 is a diagram such that $ConD_1 = \{c_1\}$ and $D_1(Pc_1) = 0$, and D_2 is a diagram such that $ConD_2 = \{c_2\}$ and $D_2(Pc_2) = 1$. Then $[\![\forall x(\exists yPy \rightarrow Px)]\!]_{D_0,M} = 1$. For it is true for $D_i\,(i = 0, 1, 2)$ that, for any constant c' in $ConD_i$, $[\![\exists yPy \rightarrow Pc']\!]_{D_i,M} = 1$. (For $i = 0$ this is true vacuously, since $ConD_0 = \emptyset$; for $i = 1$ it is true since $[\![\exists yPy]\!]_{D_1,M} = 0$; and for $i = 2$ it is true because $[\![Pc_2]\!]_{D_2,M} = 1$.) Also, $[\![\exists yPy]\!]_{D_0,M} = 1$, in view of the fact that $D_0 \subseteq D_2$, and that $[\![Pc_2]\!]_{D_2,M} = 1$. But $[\![\forall xPx]\!]_{D_0,M} = 0$ in view of the fact that $[\![Pc_1]\!]_{D_1,M} = 0$. So $[\![(4.6)]\!]_{D_0,M} = 0$. We leave it to the reader to construct counterexamples to (4.2)-(4.5).

An axiomatization of parametric validity should involve a combination of

– Some complete axiomatization of classical sentential logic, and
– Some set of axioms and rules that, when joined with that axiomatization, give a weaker system of proof than classical quantification theory, and which, in particular, do not yield any of (4.1)-(4.5) as theorems.

Our choice of axiomatization has been guided, first, by the advantages of genuinely axiomatic systems (i.e., systems which consist mostly of axioms and have only a few simple inference rules) when it comes to proving soundness and completeness, and, second, by the desire that it should be easy to compare our system to familiar axiomatizations of classical predicate logic. This has led us to adopt for our present purpose a system the inference rules of which are Modus Ponens (MP) and Universal Generalization (UG).

Including UG entails that proofs in general consist not only of sentences (i.e., closed formulae) but also of formulae with free variables; this happens even in cases where the theorem to be proved itself is without free variables. We thus need a definition of validity that applies to open as well as closed formulae. To this end we generalize the definition of parametric validity we gave earlier in the way suggested by UG: the validity of an open formula is equivalent to that of its universal closure. A formula φ is *parametrically valid* iff for every parametric model M, diagram $D \in M$ and function f such that $\mathrm{Dom}f$ includes the set of variables with free occurrences in φ and $\mathrm{Ran}f$ is a set of constants included in $ConD$, $[\![f(\varphi)]\!]_{D,M} = 1$ (where $f(\varphi)$ is the result of replacing each free occurrence of any variable x in φ by the constant

$f(x)$). Likewise we say that the formula φ is a *parametric consequence of* a set of formulae Γ iff for every parametric model M, diagram $D \in M$ and function f such that Dom$f$ includes the set of variables with free occurrences in φ and the formulae in Γ, if $[\![f(\psi)]\!]_{D,M} = 1$ for every formula ψ in Γ, then $[\![(f(\varphi)]\!]_{D,M} = 1$. Note that in order to show that a formula φ is parametrically valid it is enough to show that for every model M and $c_1, ..., c_n \in D_0(M)$, $D_0(M) \models_M \varphi(c_1/x_1, ..., c_n/x_n)$. This follows from the fact that if D is any diagram of M, then $M' = \{D' \in M : D \subseteq D'\}$ is a parametric model with core D and, for each $\varphi \in L(D)$, $D \models_{M'} \varphi$ iff $D \models_M \varphi$.

The axiom system we have chosen is formulated, for convenience, for languages containing all the familiar logical operators ($\neg, \rightarrow, \wedge, \vee, \leftrightarrow, \forall, \exists$) as primitives. Let L be such a language.[11] The system, A, has, in addition to the rules

MP

$$\frac{\vdash_A \varphi \qquad \vdash_A \varphi \rightarrow \psi}{\vdash_A \psi}$$

UG

$$\frac{\vdash_A (\varphi)y/x}{\vdash_A \forall x\varphi}, \text{ where } y \text{ does not occur in } \varphi,$$

all formulae (with or without free variables) that instantiate one of the following axiom schemata:

1. The following complete set of schemata for classical sentential logic with $\neg$ and $\rightarrow$ as primitives:
   (a) $\varphi \rightarrow (\psi \rightarrow \varphi)$
   (b) $(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \theta))$
   (c) $(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$
2. The quantificational schemata
   (a) Universal Instantiation (UI)

   $$\forall x\varphi \rightarrow (\varphi)t/x, \text{ where } t \text{ is a term free for } x \text{ in} \varphi$$

   (b) Distributivity (DIS)

   $$\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \forall x\psi)$$

   (c) Restricted Vacuous Quantification (RVQ)
   $\varphi \rightarrow \forall x\varphi$, where x does not occur free in φ and either (a) φ is quantifier-free or (b) φ is a universal formula (i.e. φ is of the form $\forall z\psi$, where ψ can be any formula).

[11] This switch is strictly one of convenience; we could, at some slight cost, have persisted with languages based on the logical vocabulary consisting of just $\neg$, $\rightarrow$ and $\forall$.

3. Schemata that 'define' the logical constants $\leftrightarrow$, $\wedge$, $\vee$, and $\exists$ in terms of $\neg$, $\rightarrow$ and $\forall$ in the standard way.

Later on the following facts about A will be useful:

(T1) $\vdash_A \forall x(\varphi \wedge \psi) \leftrightarrow (\forall x\varphi \wedge \forall x\psi)$

(from (2.b) and the definition of $\wedge$ in terms of $\rightarrow$ and $\neg$)

(T2) $\vdash_A \forall x(\varphi \rightarrow \psi) \rightarrow (\exists x\varphi \rightarrow \exists x\psi)$

(from (2.b) and the definition of $\exists$ in terms of $\forall$ and $\neg$)

(T3) $\vdash_A \exists x(\varphi \vee \psi) \rightarrow (\exists x\varphi \vee \exists x\psi)$

(from (2.b) and the definitions of $\exists$ and $\vee$ in terms of $\forall$, $\rightarrow$ and $\neg$)

(T4) $\vdash_A (\forall x\varphi \wedge \exists x\psi) \rightarrow \exists x(\varphi \wedge \psi)$

(from (2.b) and the definitions of $\exists$ and $\wedge$ in terms of $\forall$, $\rightarrow$ and $\neg$)

(T5) (Change of Bound Variables)
$\vdash_A \forall x\varphi \rightarrow \forall y(\varphi)y/x$, where x is free for y in φ and y has no free occurrences in φ

First we show this for the special case where y does not occur in φ at all. (Then y has no free occurrences in φ and x is free for y in φ.) For this special case the argument is simple: $\vdash_A \forall x\varphi \rightarrow [\varphi]y/x$ by (2.a); so $\vdash_A \forall y(\forall x\varphi \rightarrow [\varphi]y/x)$ by UG; so $\vdash_A \forall y\forall x\varphi \rightarrow \forall y[\varphi]y/x$; but $\vdash_A \forall x\varphi \rightarrow \forall y\forall x\varphi$ by (2.c); the transitivity of $\rightarrow$ gives us what we want.

To prove the general case on the basis of this partial result is more work. In general when y has no free occurrences in φ and x is free for y in φ, y may nevertheless have bound occurrences inside φ (so long as the quantifiers that bind these occurrences do not have free occurrences of x within their scope). To reduce this case to the restricted case we have proved involves a fair amount of formula transformation. We do not spell this out, for one thing because it is only the restricted form that we will be using later on in the completeness proof.

(T6) $\vdash_A (\forall x\varphi \wedge \forall y\psi) \rightarrow \forall x\forall y(\varphi \wedge \psi)$, provided x does not occur free in ψ

From (2c) we get: $\vdash_A \forall x\varphi \rightarrow \forall y\forall x\varphi$ and, from this by propositional logic and (T1,) $\vdash_A (\forall x\varphi \wedge \forall y\psi) \rightarrow \forall y(\forall x\varphi \wedge \psi)$. By U.G. and (2.b), $\vdash_A \forall x(\forall x\varphi \wedge \forall y\psi) \rightarrow \forall x\forall y(\forall x\varphi \wedge \psi)$. By more applications of (2.c) and (T1), $\vdash_A (\forall x\varphi \wedge \forall y\psi) \rightarrow \forall x(\forall x\varphi \wedge \forall y\psi)$. So, combining, we get $\vdash_A (\forall x\varphi \wedge \forall y\psi) \rightarrow \forall x\forall y(\forall x\varphi \wedge \psi)$. But, by (2.a), $\vdash_A \forall x\varphi \rightarrow \varphi$, so, combining this with the previous line and by further applications of (2.b) and propositional calculus, $\vdash_A (\forall x\varphi \wedge \forall y\psi) \rightarrow \forall x\forall y(\varphi \wedge \psi)$.

As is typical of axiomatic characterizations of provability, A accounts directly only for the provability of logical theorems: $\vdash_A \varphi$ iff there is a finite sequence of formulae containing φ such that each formula in the sequence is either an axiom or comes from one or two formulae earlier in the sequence by Universal Generalization or Modus Ponens. The notion can be extended in the usual way to cover provability from nonlogical premisses: We say that φ is *provable in A from* the set of formulae

Γ iff there are $\gamma_1, ..., \gamma_n \in \Gamma$ such that $\vdash_A (\gamma_1 \wedge ... \wedge \gamma_n) \rightarrow \varphi$. A set Γ of formulae is called A-inconsistent iff $\psi \wedge \neg\psi$ is provable from Γ for some formula ψ, and is called A-consistent otherwise.

Theorem 16 *A sentence φ of a given language L is a parametric consequence of the set Γ of sentences of L iff $\Gamma \vdash_A \varphi$.*

Proof The proof of A's soundness is straightforward. We have to show (i) for each axiom schema Ψ of A, if M is a parametric model for L, D a diagram of M, ψ is any formula of L that instantiates the schema Ψ and f any function that maps the free variables of ψ to constants belonging to $ConD$, then $[\![f(\varphi)]\!]_{D,M} = 1$; and (ii) both inference rules (MP and UI) preserve validity.

The least problematic are the rule and schemata relating to sentential logic (i.e. MP, (A.1.a)–(A.1.c) and the 'definitions' for $\wedge$, $\vee$ and $\leftrightarrow$ in terms of $\rightarrow$ and $\neg$ alluded to in (A.3)). The cases that deserve attention are the axioms (A.2.a)–(A.2.c) and the rule UG. Here we explicitly consider only (A.2.c) and UG. (UI is just (3.1) above.)

RVQ (= A.2.c): First suppose that φ is an atomic formula, that x does not occur free in φ and that $y_1, .., y_k$ are the free variables of $\varphi \rightarrow \forall x\varphi$. Suppose that M is a parametric model for L, D a diagram of L and $c_1, .., c_k$ constants belonging to $ConD$ such that $[\![[\varphi]c_1/y_1...c_k/y_k]\!]_{D,M} = 1$. Then, since $[\varphi]c_1/y_1...c_k/y_k$ is an atomic sentence, $D([\varphi]c_1/y_1...c_k/y_k) = 1$. So, for all $D' \in M$ such that $D \subseteq D'$, $D'([\varphi]c_1/y_1 ... c_k/y_k) = 1$. But since x is not free in φ, we have that for any $c[\varphi]c_1/y_1 ... c_k/y_k$ is the same formula as $[\varphi]c_1/y_1 ... c_k/y_k, c/x$. So, for all $D' \in M$ such that $D \subseteq D'$, $[\![[\varphi]c_1/y_1 ... c_k/y_k, c/x]\!]_{D',M} = 1$. So, $[\![\forall x[\varphi]c_1/y_1 ... c_k/y_k]\!]_{D,M} = 1$. Second, when φ is the negation of an atomic formula, i.e. $\varphi = \neg\psi$ for some atomic formula ψ, then we infer from $[\![[\varphi]c_1/y_1 ... c_k/y_k]\!]_{D,M} = 1$ that $D([\psi]c_1/y_1 ... c_k/y_k) = 0$. In all other respects the argument proceeds in the same way as in the case where φ is atomic. Thirdly, suppose that φ is of the form $\forall z\psi$. We have to show, for any M, D and $c_1, .., c_k$ as above, that if $[\![[\varphi]c_1/y_1 ... c_k/y_k]\!]_{D,M} = 1$, then $[\![\forall x[\varphi]c_1/y_1 ... c_k/y_k]\!]_{D,M} = 1$. Suppose that $[\![[\varphi]c_1/y_1 ... c_k/y_k]\!]_{D,M} = 1$. Then

(*) for all $D' \in M$ that such $D \subseteq D'$ and $c \in ConD'$, $[\![[\psi]c/z, c_1/y_1...c_k/y_k]\!]_{D',M} = 1$.

We have to show that for all $D' \in M$ such that $D \subseteq D'$ and $c' \in ConD'$, $[\![[\varphi]c'/x, c_1/y_1 ... c_k/y_k]\!]_{D',M} = 1$. But since x has no free occurrences in φ, this last condition can be rewritten as $[\![[\varphi]c_1/y_1 ... c_k/y_k]\!]_{D',M} = 1$. This condition holds iff for all $D'' \in M$ such that $D' \subseteq D''$ and $c' \in ConD''[\![[\psi]c'/z, c_1/y_1 ... c_k/y_k]\!]_{D'',M} = 1$. However, $D' \subseteq D''$ and $D \subseteq D'$ entail $D \subseteq D''$. So D'' and c' satisfy the preconditions in (*). So $[\![[\psi]c'/x, c_1/y_1 ... c_k/y_k]\!]_{D'',M} = 1$.

UG: Suppose that the formula $(\varphi)y/x$ of L is valid and that this formula has, in addition to y, the free variables $z_1, ..., z_k$. Let M be any parametric model for L. Then, for all $D \in M$ and all $c, c_1, ..., c_k \in ConD$, $[\![(\varphi)c/x, c_1/z_1 ... c_k/z_k]\!]_{D,M} = 1$. So, for all $D \in M$ and all $c_1, ..., c_k \in ConD$, $[\![(\forall x\varphi)c_1/z_1 ... c_k/z_k]\!]_{D,M} = 1$. Hence, $\forall x\varphi$ holds in M at $D_0(M)$. Since this is true for arbitrary models M, $\forall x\varphi$ is valid.

So much for the soundness part.

The general strategy of our completeness proof is familiar. As in classical logic, $\Gamma \nvdash_A \varphi$ entails that the set $\Gamma \cup \{\neg\varphi\}$ is A-consistent. So it suffices to show that for every A-consistent set Γ of sentences there is a parametric model in which all the sentences in Γ are true. We will describe the construction of such a model using the method of semantic tableaux. We will, that is, build up inductively a 'semantic tableau structure' T from which a model with the desired property can be derived in a straightforward, mechanical manner. The tableau structure will have the structure of a tree, each node of which determines a diagram of the parametric model that can be derived from it. Furthermore, if t, t' are nodes of the tree and $t \leq t'$ in the sense of the tree ordering $\leq$, then the diagram determined by t' extends the diagram determined by t. The finite stages of the construction can be regarded as finite approximations of T in two respects: (i) the set of nodes of T will be the limit of the finite sets of nodes that can be found at the successive construction stages and (ii) for each node t in T the values assigned to t by decoration functions d_n at successive construction stages n are finite approximations of the decorations assigned to t in T.

We represent our trees as sets of finite sequences of natural numbers; that is, the nodes of our trees are finite number sequences. $<>$ is the empty sequence, and $< n_1, ..., n_k>$ is the sequence of the numbers $n_1, ..., n_k$. The ordering relation $\leq$ of such a tree T is the relation that holds between two nodes t and t' iff t is an initial segment of t'. $\frown$ is the operation of concatenation on sequences. In particular, if t is a sequence and k a natural number, then $t \frown < k >$ is the sequence obtained by appending k to t. In this case we also write '$t \frown k$'.

By a *(canonical) tree* we understand any nonempty set T of finite sequences of natural numbers such that, if $t^\frown < k > \in T$, then (i) $t \in T$; and (ii) if $k = r + 1$ then $t^\frown < r > \in T$. From here on we only consider canonical trees and we drop the qualification 'canonical'.

In the tableau construction we will often need to distinguish between different 'copies' of the same tree. We therefore index our trees, with finite binary sequences (i.e. finite sequences of 0s and 1s) as indices. Thus, by an *indexed tree* we understand a pair $< r, T >$, where r is a finite binary sequence and T is a tree; r will be called the *index* of $< r, T >$.

Tableau structures are 'decorated' indexed canonical trees —to be precise, they are pairs consisting of an indexed tree $< r, T >$ and a function d from the nodes of T to pairs of sets of formulae. (d will be referred to as the *decoration (function)* of the tree.) The intuitive significance of the decorating pairs is this: The first member of the pair $d(t)$ (to which we refer as $d(t)^+$) assigns as value to a node t of T a set of sentences that should be thought of as appearing under the heading 'TRUE' of the semantic tableau determined by t and the second member of $d(t)$, $d(t)^-$, contains the set of formulae that should be thought of as appearing under the heading *FALSE* of that tableau.

To define the notion of a tableau structure for our given language L formally we need to have at our disposal an infinite sequence of constants disjoint from the symbols of L. Let C be a denumerably infinite set of such constants and let $c_1, c_2, ...$ be some fixed enumeration of it.

We can now define the notion of a *tableau structure for L* as follows:

A *tableau structure for* L is a pair $<< r, T >, d>$, where $< r, T >$ is an indexed tree and d is a function from T to pairs of sets of sentences of $L \cup C$. Where $<< r, T >, d >$ is a tableau structure, we refer to r as the *index of* $<< r, T >, d>$ and to the nodes of T as the *nodes of* $<< r, T >, d>$. We also refer to the tree of the tableau structure $<< r, T >, d>$ as 'T_r' and to its decoration function as 'd_r'.

A node t of a tableau structure $<< r, T >, d >$ is *closed in* $<< r, T >, d >$ iff $d(t)^+ \cap d(t)^- \neq \emptyset$; otherwise t is *open in* $<< r, T >, d >$. A tableau structure $<< r, T >, d >$ is *closed* iff there is at least one node t in T such that t is closed in $<< r, T >, d >$; otherwise $<< r, T >, d >$ is *open*.[12]

Tableau structures for L emerge in the course of *tableau constructions* for sets Γ of sentences of L. The initial stage of this tableau construction is the indexed tree $<<<>, \{<>\}>, d>$, where $d(<>)^+ = \Gamma$ and $d(<>)^- = \emptyset$.

As the tableau construction proceeds from this starting point, new constants from the set C may be introduced, so that for nodes t of the unfolding tableau structure the sentences that make up $d(t)$ will all belong to some extension $L \cup \{c_1, ..., c_n\}$ of L, where $< c_1, ..., c_n >$ is an initial segment of the fixed enumeration of C. There will always be a minimal segment $< c_1, ..., c_n >$ for which this is true—i.e. the segment which consists of just those constants from C that make up the set Con(t) = $\{c : c$ occurs in some sentence belonging to $d(<>)^+ \cup d(<>)^-\}$. We will refer to the language $L \cup \{c_1, ..., c_n\}$, where $< c_1, ..., c_n >$ is fixed in this way, as the 'language of t (in the given tableau structure)', or, more succinctly, when it is clear which tableau structure is at stake, as '$L(t)$'.

In general tableau constructions will not produce single tableau structures, but 'disjunctive' sets of such structures. We refer to these sets as *tableau structure sets*. Tableau structure sets consisting of more than one element are the result of certain tableau construction operations that 'split' a given tableau structure into two (so-called 'splitting operations'). Since at any stage n of the construction only finitely many operations have been performed, and so a fortiori only finitely many operations can have been performed that produce splitting, the tableau structure set will at each finite stage of the construction be a finite set.

Tableau structure sets too can be distinguished into 'open' and 'closed'. A tableau structure set is *closed* iff each of the tableau structures belonging to it is closed; otherwise the set is *open*.

When models for sentence sets Γ are constructed by the method of tableau construction, it is crucial that every possible application of any one of the construction operations that define the construction procedure be carried out at some stage. Otherwise the construction might fail to identify the inconsistency of Γ in cases were it is inconsistent. So, the construction procedure has to be set up in such a way that no applications are missed. Here is one way in which we can make sure of this. Let $stage$ be an enumeration in which every quadruple $< r, t, \varphi, p >$, where r is a finite binary sequence, t is a finite sequence of natural numbers, φ is a sentence of $L \cup C$

[12] Note that this differs from the usual definition of tableau closure. Here, what corresponds to a model is not a branch but an entire tableau structure set.

and p is a 'polarity' (i.e. $p \in \{+, -\}$), occurs infinitely often. For each starting stage $\{<<>, \{<>\}, < \Gamma, \emptyset >>\}$, $stage$ determines a unique tableau construction, in which the order of rule applications is completely fixed, as follows: For each construction stage n of the construction let $\mathcal{S}_n$ be the tableau structure set constructed at stage n. Suppose that $\mathcal{S}_n$ has been constructed. Let m be the smallest integer such that $m = stage(< r, t, \varphi, p >)$ and the quadruple $< r, t, \varphi, p >$ determines a possible operation execution on $\mathcal{S}_n$; that is, the construction operation determined by φ and p can be executed at the node t that is part of the tableau structure in $\mathcal{S}_n$ with index r, because φ belongs to the set $d(t)^p$. (The construction operations are defined below.) Thus $< r, t, \varphi, p >$ determines a possible operation on $\mathcal{S}_n$ only if (i) $\mathcal{T}_n$ contains a tableau structure $<< r, T_r >, d_r >$ whose index is r; (ii) $<< r, T_r >, d_r >$ contains the node t; and (iii) $\varphi \in d_r(t)^p$. When no such m can be found, then the construction comes to a halt and is complete.

When $< r, t, \varphi, p >$ determines a possible operation execution on $\mathcal{S}_n$, then there is, as we already indicated, a unique operation that can be applied to φ as a member of $d_r(t)^p$. This operation is determined by the form of φ and the value of p. Our next task is to state these operations in detail. The relevant aspects of the form of φ are (i) whether or not φ is atomic; and (ii) in case φ is not atomic, its main logical operator (i.e. the one with widest scope). We state the operations only for the cases where φ is atomic or where its main operator is one of $\neg$, $\rightarrow$, and $\forall$—from these operations those for the remaining logical operators are easily inferred, given their 'classical definitions' in terms of $\neg$, $\rightarrow$ and $\forall$—and for each of these four cases we separately consider the sub-cases where $p = +$ and where $p = -$. We first consider the cases in which φ is of the form $\neg\psi$, $\psi_1 \rightarrow \psi_2$ and $\forall x\psi$, respectively, and only then the one where φ is atomic.

1. Suppose φ is the sentence $\neg\psi$ and that $p = -$. Then we add ψ to the set $d_r(t)^+$, while removing φ from $d_r(t)^-$.

The over-all effect on $\mathcal{S}_n$ (i.e. on the transition from $\mathcal{S}_n$ to $\mathcal{S}_{n+1}$) is the same for all operations and we state it only once: The tableau structure containing the node t at which the operation is executed gets replaced by the tableau structure (or, in the case of operation 4, the two tableau structures) into which the operation transforms it. In all other respects $\mathcal{S}_{n+1}$ is like $\mathcal{S}_n$.

Notation: The change brought about by executions of operation 1 concerns the tableau structure T_r, and, more specifically, the function d_r. Since after the operation T_r and d_r are different from what they were before, it will often be useful to have an unambiguous way of referring to these distinct entities. To this end we introduce the following convention: when the operation is executed at stage n, we refer to the tableau structure T_r as it exists at stage n (before the operation has been executed) as '$T_{r,n}$' and to the result of executing the operation on $T_{r,n}$ as '$T_{r,n+1}$'; and likewise we refer to the decoration function of $T_{r,n}$ as '$d_{r,n}$' and to that of $T_{r,n+1}$ as '$d_{r,n+1}$'.

2. Suppose that φ is $\neg\psi$ and $p = +$. As operation 1, but with the superscripts $+$ and $-$ on d interchanged.
3. Suppose that φ is the sentence $\psi_1 \rightarrow \psi_2$ and that $p = -$. Then we add ψ_1 to $d_r(t)^+$ and ψ_2 to $d_r(t)^-$ and remove φ from $d_r(t)^-$.

4. Suppose that φ is the sentence $\psi_1 \rightarrow \psi_2$ and that $p = +$. This is a case of splitting. We form two new tableau structures $<< r_0, T_{r_0} >, d_{r_0} >$ and $<< r_1, T_{r_1} >, d_{r_1} >$, where $r_0 = r \frown 0$ and $r_1 = r \frown 1$. $<< r_0, T_{r_0} >, d_{r_0} >$ is like $<< r, T_r >, d_r >$ except that $d_{r_0}(t)^- = (d_r(t)^- \setminus \{\varphi\}) \cup \{\psi_1\}$ and $<<r_1, T_{r_1} >, d_{r_1} >$ is like $<<r, T_r >, d_r >$ except that $d_{r_1}(t)^+ = d_r(t)^+ \cup \{\psi_2\}$. $\mathcal{T}_m$ is obtained from $\mathcal{T}_n$ by (a) removing $<< r, T_r >, d_r >$ and replacing it by $<< r_0, T_{r_0} >, d_{r_0} >$ and $<< r_1, T_{r_1} >, d_{r_1} >$. (In this case φ is removed from $d_r(t)^+$.)
5. φ is of the form $\forall x \psi$ and $p = -$. In this perhaps most interesting case we introduce a new 'tableau', i.e., a new decorated node t', into $<<r, T_{r,n} >, d_{r,n} >$. t' is to be an immediate successor to t and so must be of the form $t \frown m$ for some m. In fact, in order that the set $T \cup \{t \frown < m >\}$ be a tree m must be the smallest natural number such that $\{t \frown < m >\} \notin T$. The function $d_{r,n}$ must be extended to a function $d_{r,n+1}$ whose domain is obtained by adding t' to the domain of $d_{r,n}$. And the new tableau that $d_{r,n+1}$ associates with t' should contain a 'witness' $[\psi]c/x$ for the formula $\forall x \psi$ occurring in $d_{r,n}(t)^-$. This witness should involve a new constant c; we choose c to be the first constant in the fixed enumeration of C which does not occur anywhere in $\mathcal{S}_n$. Since $[\psi]c/x$ is to bear witness to the falsity of $\forall x \psi$, it must itself play the role of a false sentence; so it must become a member of $d_{r,n+1}(t')^-$. In order to make sure that every atomic sentence of the language $L(t')$ associated with t' is decided in the tableau determined by t' as it will emerge from the construction: We add to $d_{r,n+1}(t')^+$ all sentences of the form $\psi \rightarrow \psi$, where ψ is an atomic sentence of the language $L(t')$ of t'. So $d_{r,n+1}(t')^- = \{(\psi)c/x\}$ and $d_{r,n+1}(t')^+ = \{\psi \rightarrow \psi : \psi \in L(t') \text{ and } \psi \text{ atomic}\}$. Moreover, as in all previous operations, φ is removed (from $d_{r,n}(t)^-$).

Comment: By adding all the tautologies $\psi \rightarrow \psi$ for atomic ψ from the language $L(t')$ to the set $d_{r,n+1}(t')^+$ eventually each atomic sentence from $L(t')$ will be decided at some later stage in the construction as part of the decoration of t' in some tableau structure $<< r', T_{r'} >, d_{r'} >$ with r' a (proper or improper) extension of the index r. Note that since all sentences $\psi \rightarrow \psi$ are valid (and, as is easily seen, provable as theorems from the axiom system A), there is no danger that by adding these formulae to the 'TRUE' side of the tableau associated with t' an inconsistency will be introduced that would not be there otherwise: if the tableau construction involving these additions to $d_{r,n+1}(t')^+$ leads to closure, then so would the construction when these sentences would not be added.

6. φ is of the form $\forall x \psi$ and $p = +$. In this case we must add to $d_{r,n}(t)^+$ all formulae of the form $(\psi)c/x$ for c a constant of $L(t)$. But this is not all. According to the parametric definition of truth, a universal sentence is true in a parametric model M at a diagram D iff all its instances are true in M at all $D' \in M$ such that $D \subseteq D'$. In the model M that is to be extracted from the tableau construction in the absence of closure the diagrams will be those associated with the nodes of some tableau structure, and the partial order of the tableau structure will determine the inclusion relation between the diagrams of M. So we must make sure

that all instances of φ will be verified at all $t' \geq t$. One way to achieve this is to add, for each immediate successor t' of t in $T_{r,n}$, to $d_{r,n}(t')^+$ (a) every formula $(\psi)c'/x$ for c' a constant of $L(t')$ and (b) the formula φ itself. Note well that in this case we do *not* remove φ from $d_{r,n}(t)^+$.

At last we come to the operations for atomic sentences. The operations involving atomic sentences can obviously not be 'reducing operations', in which a given sentence is reduced to one or more syntactically simpler sentences. In fact, these operations are 'transfer operations,' in which copies of atomic sentences belonging to the decoration of a node t are added to the decorations of the successors of t. Adding atomic φ to nodes $t' > t$ is needed to guarantee that the diagrams $D(t)$ of the model M that we will extract from the tableau construction in case it does not close (i.e. when no finite construction stage is a closed tableau structure set) stand in the right inclusion relation to each other: if $t < t'$ then $D(t)$ should be the restriction of $D(t')$ to the language $L(t)$ of $D(t)$. Since the information about $D(t)$ will in general not be completely available at the points where new nodes $t' > t$ are introduced into the given tableau structure, information about the truth values of atomic sentences (which is determinate of the diagram $D(t)$ of the model M if and when it is constructed) must be passed up to higher nodes to make sure that the diagrams determined by those nodes are extensions of $D(t)$. As in the case of rule 6, we can achieve eventual transfer to arbitrary successors by specifying the operation as transferring φ just to the immediate successors of t. Because of the way the tableau construction has been set up, later applications of the rule to the occurrences of φ in the immediate successors to t will transfer φ to their immediate successors, and so on.

In the light of these considerations we can state the operations as in 7a and b.

7a. Suppose that φ is atomic and that $p = -$. Then, for each immediate successor $t \frown <m>$ of t in $<<r, T_r>, d_r>$, we add φ to $d_r(t \frown <m>)^-$.

7b. Suppose that φ is atomic and that $p = +$. Then, for each immediate successor $t \frown <m>$ of t in $<<r, T_r>, d_r>$, we add φ to $d_r(t \frown <m>)^+$.

In both 7a and 7b φ is retained as member of $d_r(t)^{+/-}$.

For reasons analogous to those explained in our description of operation 5, we start the tableau construction not from $<<>, \{<>\}, <\Gamma, \emptyset>>$ but from $<<>, \{<>\}, <\Gamma', \emptyset>>$, where $\Gamma' = \Gamma \cup \{\psi \to \psi : \psi$ is an atomic sentence of $L\}$.

There are two possible outcomes to the tableau construction for Γ':

(i) at some stage n the set $\mathcal{S}_n$ is closed.
(ii) S_n is closed for no n.

When the tableau construction reaches a stage n such that $\mathcal{S}_n$ is closed, the construction terminates at that point (by stipulation). On the other hand, if there is no closure, then the construction may go on forever, i.e. it runs through an infinity of stages $\mathcal{S}_n$, for $n = 1, 2, \ldots$. Strictly speaking this is not always so; but it is the rule rather than the exception. It is also the more challenging possibility. So this is the case on which we will focus. (Constructions that come to a halt after a finite number of steps without closing are easier to handle. Readers who have gone through the argument for the non-terminating constructions will have no problem adapting it to the terminating ones.)

The two possibilities (i) and (ii) are each handled using a strategy familiar from other completeness proofs in which the tableau method has been used. For possibility (i) we show that the tableau construction can be reduced to one that also closes but involves only a finite number of sentences as decorations of finitely many nodes of finitely many tableau structures, and assign Representing Formulas to the successive construction stages. The RF of the final stage can be shown to be refutable in the axiom system A and this refutability can then be shown to transfer from the RFs of later to those of earlier construction stages, leading to the conclusion that the set Γ itself is A-inconsistent; in case (ii) we construct a parametric model in which all sentences of Γ are true.

We first consider the second possibility. Suppose that no $\mathcal{S}_n$ is closed. Then each $\mathcal{S}_n$ has at least one member $<< r_n, T_{r_n} >, d_{r_n} >$ which does not close. So, for each n the set R_n of indices r which label open members of $\mathcal{S}_n$ is non-empty. Consider the union $R = \bigcup_{n=1,2,\ldots} R_n$ of all these sets. R is a binary branching tree with the 'initial-segment' relation as tree ordering. This tree is either finite or infinite. In either case it will contain a maximal branch of open tableau structures. (When the tree is infinite, this follows from König's Lemma; when the tree is finite, it is obvious.) When the tree is finite, a maximal open branch will be finite too, and thus will have a 'maximal' index. If the tree is infinite an infinite open branch may take the form of an infinite sequence of growing finite sequences. It is this possibility we pursue further. The case where the branch has a maximal index is similar but simpler.

Suppose that b is an infinite open branch of R, i.e. b is a function from the natural numbers to finite binary sequences such that if $k \leq m$, then $b(k)$ is an initial segment of $b(m)$, and furthermore there is no largest sequence among the values of b. We can use b to define a parametric model M as follows. For each construction stage n there will be exactly one tableau structure $<< r, T_r >, d_r >$ in $\mathcal{S}_n$ such that r belongs to the range of b. We refer to this tableau structure as $<< r_n, T_{r_n} >, d_{r_n} >$, or simply as T_{r_n}. The infinite sequence $\{<< r_n, T_{r_n} >, d_{r_n} >\}_{n\in\omega}$ is a chain in the following sense: if $n < m$, then the set of nodes of T_{r_n} is a subset of the set of nodes of T_{r_m} and for each node t of T_{r_n}, $d_{r_n}(t)^+ \subseteq d_{r_m}(t')^+$ and $d_{r_n}(t)^- \subseteq d_{r_m}(t')^-$.

The diagrams of M from $< T_{r_n} >_{n\in\omega}$ are defined from the nodes occurring in $\{<< r_n, T_{r_n} >, d_{r_n} >\}_{n\in\omega}$ (i.e. the nodes t such that for some n t occurs in T_{r_n}). We want to associate with each such t a diagram D_t in the language $L(t)$. Unfortunately we cannot simply use the pairs $d_{r_n}(t)$ for this, where T_{r_n} is some tableau structure to which t belongs, e.g. by defining, for arbitrary atomic sentences of $L(t)$, $D_t(Pc_1, ..c_m) = 1$ iff $Pc_1, ..c_m \in d_{r_n}(t)^+$ and $D_t(Pc_1, ..c_m) = 0$ iff $Pc_1, ..c_m \in d_{r_n}(t)^-$; for in general the pairs $< d_{r_n}(t)^+, d_{r_n}(t)^- >$ do not decide all atomic sentences of $L(t)$. However, because our tableau construction is for the set Γ' rather than for the set Γ and because we throw in all conditionals $\psi \to \psi$ for atomic sentences ψ of $L(t)$ each time a new node t is created, each atomic sentence is eventually decided at each node

t, in the sense that there will either be an m such that $\psi \in d_{r_m}(t)^+$ or there will be an m such that $\psi \in d_{r_m}(t)^-$. On the other hand, since all tableau structures T_{r_n} are open, no sentence can both belong to a set $d_{r_m}(t)^+$ for one $m > n$ and at the same time belong to a set $d_{r_{m'}}(t)^-$ for some other $m' > n$. Putting these two observations together, and defining $d_\omega(t)^{+/-} = \bigcup_{n\in\omega} d_{r_n}(t)^{+/-}$, we conclude that d_ω partitions the sets of atomic sentences of $L(t)$ into two halves, the 'true' atomic sentences of $L(t)$ and the 'false' ones. So, the following definition of the diagram D_t determined by t in $\{<<r_n, T_{r_n}>, d_{r_n}>\}_{n\in\omega}$ is coherent:

For any atomic sentence $Pc_1, ..c_m$ of $L(t)$, $D_t(Pc_1, ..c_m) = 1$ iff $Pc_1, ..c_m \in d_\omega(t')^+$.

It should be clear that this definition makes D_t into a diagram for the language $L(t)$, and also, in virtue of the repeated applications of the operations 7a,b, that if $t < t'$, then $D_t \subseteq D_{t'}$.

The model M that we obtain from the tableau construction is the set of all diagrams D_t for all nodes t that occur in at least one of the tableau structures T_{r_n}. The partial order of M is given by the set-theoretical inclusion of its diagrams (which as we have seen is in its turn induced by the partial order relation between the nodes). That M is a model of the kind we are looking for is captured by (4.11).

(4.11) All sentences $\varphi \in \Gamma$ are true in M at $D_{<>}$.

(4.11) is established by proving, via induction on complexity of formulae, the stronger (4.12).

(4.12) If t is a node of T_ω and φ is a sentence of the language $L(\mathcal{T}_\omega, t)$ then

(i) if $\varphi \in d_\omega(t)^+$, then $[\![\varphi]\!]_{D_t,M} = 1$;
(ii) if $\varphi \in d_\omega(t)^-$, then $[\![\varphi]\!]_{D_t,M} = 0$.

The proof of (4.12) is straightforward and we omit it.

The second main task in the proof of Theorem 16 is to show that if the tableau construction for the set Γ' closes after some finite number of steps, then the set Γ is A-inconsistent. Suppose that the tableau construction for the set Γ' closes at stage N. Then $\mathcal{S}_N$ is a closed tableau structure set. This means that each of the tableau structures $<< r, T >, d >$ in $\mathcal{S}_N$ is closed, which implies that each of these tableau structures has at least one closed node. Since the construction of $\mathcal{S}_N$ involves only a finite number of construction steps, there will have been only finitely many splittings. So $\mathcal{S}_N$ is a finite set. Moreover, there can have been only finitely many introductions of new nodes in each of the tableau structures that make up this set; so in that sense each of the tableau structures is finite. But for all we have said so far, the decorations of the nodes could still be infinite. However, from the fact that $\mathcal{S}_N$ is closed we can conclude that there is also a tableau construction that ends with a closing stage $\mathcal{S}_M$ and in which all decorations are pairs of finite sets. To see this, first note that since each of the finitely many tableau structures in $\mathcal{S}_N$ is closed, there will be at least one closing node t in each such structure $T_{r,N}$, and the closure of each such t is determined by the occurrence of at least one sentence ψ which belongs to both $d_{r,N}(t)^+$ and $d_{r,N}(t)^-$. For each $T_{r,N}$ select one closing node t_r from this structure and for each of these nodes t_r choose one sentence ψ_r that belongs to both $d_{r,N}(t_r)^+$ and $d_{r,N}(t_r)^-$. And consider for each node t of each of the tableau structures $T_{r,N}$ in $\mathcal{S}_N$ those finite subsets $d'_{r,N}(t_r)^{+/-}$ of $d_{r,N}(t_r)^{+/-}$ which consist of (a) those sentences that are subjected to an operation in the course of constructing $\mathcal{S}_N$ and (b) the sentence ψ_r in case t is the selected closing node t_r of $T_{r,N}$. Now consider the following tableau construction: (i) the starting structure is $\{<<>, \{<>\}>, <\Gamma'', \emptyset>\}$, where Γ'' consists of those sentences of Γ' that belong to the set $d'_{r,N}(<>)^+$. (ii) whenever an application of operation 5 introduces a new node t of a tableau structure $T_{r,n}$ at some stage n of the construction, then we include in $d_{r,n+1}(t)^+$ only that subset of the set of sentences prescribed for inclusion by operation 5, which also belong to the set $d'_{r,N}(t_r)^+$. It is not hard to see that this tableau construction will close just as the original construction of $\mathcal{S}_N$ does, since exactly the same operations will be performed in the two constructions at the same nodes of the same tableau structures. For the remainder of the proof we deal with this tableau construction for the set Γ'' and its successive stages $\mathcal{S}_0$, $\mathcal{S}_1$, ..., $\mathcal{S}_N$.

Because the tableau structure sets $\mathcal{S}_n$ are all finite in the sense just described (i.e. finite sets of finite tableau structures whose node decorations are finite sets), it is possible to associate with them Representing Formulae $RF(\mathcal{S}_n)$ in such a way that the following three conditions are fulfilled.:

(4.13) (i) $RF(\mathcal{S}_0)$ is derivable in A from Γ;

(ii) $\vdash_A \neg RF(\mathcal{S}_N)$

(iii) for all $n < N$, $\vdash_A RF(\mathcal{S}_n) \to RF(\mathcal{S}_{n+1})$.

To define the Representing Formulae some more notation will be helpful. Recall that when a new node is introduced, this is always through application of operation 5 on a sentence $\forall x\psi$ belonging to the negative decoration of some node t. And this operation always involves the introduction of exactly one new constant from C. Since these constants are uniquely determined by the nodes t' as part of whose creation they are introduced, we can denote them as $c_{t'}$. Let us also choose, corresponding to the different $c_{t'}$, distinct variables $x_{t'}$, none of which occur in $\mathcal{S}_N$.

The formulae $RF(\mathcal{S}_n)$ are defined as disjunctions of representing formulae $RF(T_{r,n})$ for the members $T_{r,n}$ of $\mathcal{S}_n$. The formulae $RF(T_{r,n})$, in their turn, are defined as the special cases $RF(<>, T_{r,n})$, of formulae $RF(t, T_{r,n})$ for arbitrary nodes t of $T_{r,n}$. $RF(t, T_{r,n})$ is defined by inverse induction on the indexed tree $T_{r,n}$, i.e., the recursion starts from the leaves and works its way back to the root. Before defining $RF(t, T_{r,n})$, we associate with each node t of $T_{r,n}$ its local representing formula $LRF(t, T_{r,n})$:

(4.14) $LRF(t, T_{r,n}) = \bigwedge d(t)^+ \wedge \bigwedge\{\neg\psi : \psi \in d(t)^-\}$,

Next we define, for each node $t \in T_{r,n}$, $RF(t, T_{r,n})$ as

(4.15) $RF(t, T_{r,n}) = LRF(t, T_{r,n}) \wedge \bigwedge \exists x(RF(t', T_{r,n}))x'_t/c'_t$: t' is an immediate successor of t in $T_{r,n}$.

Finally $RF(\mathcal{S}_n)$ is defined by

(4.16) $RF(\mathcal{S}_n) = \bigvee\{RF(<>, T_{r,n}) : T_{r,n} \in \mathcal{S}_n\}$

From the definition of the formulas $RF(\mathcal{S}_n)$, (4.13.i) follows immediately. (Since $\mathcal{S}_0$ is a singleton set, $RF(\mathcal{S}_0)$ just is the conjunction of some finite subset Γ'' of Γ'. Γ'' consists of (a) a finite set of sentences in Γ and (b) a finite set of sentences of the form $\psi \rightarrow \psi$. But the latter are all theorems of A. So clearly each of the conjuncts of $RF(\mathcal{S}_0)$ is provable from Γ and so the conjunction is, since A includes all of classical propositional logic.)

Second, (4.13.ii) is also easily established, but the details are worth paying attention to, since they give us a first taste of how the quantificational schemata of A enter into the arguments that are needed to establish (4.13.iii). To show that $\vdash_A \neg RF(\mathcal{S}_N)$ it is enough to show that $\vdash_A \neg RF(T_{r,N})$ for each tableau structure $T_{r,N}$ belonging to $\mathcal{S}_N$. To show this we recall that each $T_{r,N}$ contains a closing node t_r. (The closure of t_r in the new construction is brought about in particular by the sentence ψ_r, since the selection of the sentences entered into the different nodes t when they are introduced (at the start of the construction as members of Γ'' or later on) guarantee that ψ_r will belong to both the positive and the negative decoration of t as part of $T_{r,N}$.) So $LRF(t, T_{r,N})$ will be a conjunction of sentences that has both ψ_r and $\neg\psi_r$ among its conjuncts. Let us abbreviate the conjunction $(\psi_r \wedge \neg\psi_r)x_t/c_t$ as $\bot_t$. Evidently $\vdash_A \neg \bot_t$ and $\vdash_A LRF(t, T_{r,N}) \rightarrow \bot_t$. Since $LRF(t, T_{r,N})$ is among the conjuncts of $RF(t, T_{r,N})$, we also have $\vdash_A \neg RF(t, T_{r,N})$. Now suppose that t' is the immediate predecessor of t. (If t has no immediate predecessor, then $t = <>$ and we are done.) Then one of the conjuncts of $RF(t', T_{r,N})$ is the formula $(\exists x_t)(RF(t, T_{r,N})x_t/c_t)$. Since $\vdash_A RF(t, T_{r,N}) \rightarrow \bot_t$,

(4.17) $\vdash_A (\exists x_t)RF(t, T_{r,N})x_t/c_t \rightarrow (\exists x_t) \bot_t$ (by UG and (T2))

On the other hand, since $\vdash_A \neg \bot_t$, $\vdash_A (\exists x_t) \bot_t \rightarrow \bot_t$. So

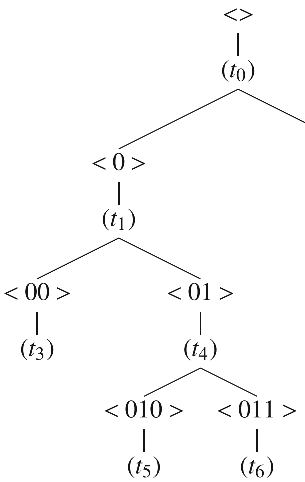
(4.18) $\vdash_A LRF(t', T_{r,N}) \rightarrow \bot_t$

From this we infer that $\vdash_A RF(t', T_{r,N}) \rightarrow \bot_t$. If $t' = <>$ we are done. If not, and t'' is the immediate predecessor of t', then the same argument shows that $\vdash_A LRF(t'', T_{r,N}) \rightarrow \bot_t$ and so on until $<>$ is reached.

Most of the work that has to be done to prove (4.13) goes into proving (4.13.iii). As we have seen in the argument for (4.13.ii), arguments concerning the proof-theoretic properties of the Representing Formulae of tableau structures typically involve inverse induction. Nevertheless the inductive argument we gave for (4.13.ii) was not formalized as a proof by induction in the proper sense of the word. A proper proof would have to proceed by induction on the length (as sequence) of the node at which the operation that leads from $\mathcal{S}_n$ to $\mathcal{S}_{n+1}$ is carried out. We do not see, however, that anything of substance would be gained by such a formal execution of the argument, whereas on the other hand a good deal of perspicuity would be lost.

So, in our arguments for (4.13.iii) we will not proceed by a formal general induction but by example. The example we have chosen—of a tableau structure, shown in (4.19), with some particular node t to which the tableau construction operations are applied in the transition from $\mathcal{S}_n$ to $\mathcal{S}_{n+1}$—is just complex enough to show where the arguments that establish (4.13.iii) must make use of properties of the axiom system A and which properties are needed. In the example the node t is the one labeled 't_4'.

(4.19)



Note that the Representing Formula of this tableau structure has the form shown in (4.20.i) (using the canonical notions for our tree nodes) and in (4.20.ii) (using the labels t_0, t_1 etc.).Here and in what follows, we abridge, for any tableau structure node t', $\bigwedge d(t')^+ \wedge \bigwedge\{\neg\psi : \psi \in d(t')^-\}$ as $\bigwedge d(t')$.

(4.20) (i)

$$\bigwedge d(<>) \wedge$$

$$(\exists x_{<0>})(\bigwedge[d(<0>)]x_{<0>}/c_{<0>} \wedge$$

$$(\exists x_{<00>})\bigwedge[d(<00>)]x_{<00>}/c_{<00>} \wedge$$

$$(\exists x_{<01>})(\bigwedge[d(<01>)]x_{<01>}/c_{<01>} \wedge$$

$$(\exists x_{<010>}) \bigwedge[d(<010>)]x_{<010>}/c_{<010>} \wedge$$

$$(\exists x_{<011>}) \bigwedge[d(<011>)]x_{<011>}/c_{<011>}) \wedge$$

$$(\exists x_{<1>}) \bigwedge[d(<1>)]x_{<1>}/c_{<1>}$$

(ii)

$$\begin{aligned}&\bigwedge d(t_0) \wedge\\ &\quad (\exists x_{t_0})(\bigwedge[d(t_1)]x_{t_0}/c_{t_1} \wedge (\exists x_{t_3}) \bigwedge[d(t_3)]x_{t_3}/c_{t_3} \wedge\\ &\qquad (\exists x_t)(\bigwedge[d(t)]x_t/c_t \wedge\\ &\qquad\quad (\exists x_{t_5}) \bigwedge[d(t_5)]x_{t_5}/c_{t_5} \wedge\\ &\qquad\quad (\exists x_{t_6}) \bigwedge[d(t_6)]x_{t_6}/c_{t_6}) \wedge\\ &\quad (\exists x_{t_2}) \bigwedge[d(t_2)]x_{t_2}/c_{t_2}\end{aligned}$$

We assume that the operation which leads from $\mathcal{S}_n$ to $\mathcal{S}_{n+1}$ involves a sentence that is part of the decoration of node t_4 of the tableau structure displayed in (4.19). In what follows we will refer to t_4 also simply as t. (4.19) should be thought of as an indexed tableau structure $<< r, T >, d >$ belonging to $\mathcal{S}_n$. We will for the remainder of the proof refer to this tableau structure as $T_{r,n}$ and to the result of applying the operation that leads to $\mathcal{S}_{n+1}$ as $T_{r,n+1}$.

We proceed by cases, one for each of the reduction operations that can lead from $\mathcal{S}_n$ to $\mathcal{S}_{n+1}$. Those cases where the reduction step concerns a sentential connective are handled in the same way they are when semantic tableaux are used to prove completeness for the classical propositional or predicate calculus [2]. We consider here just one case, that where the formula φ has the form $\psi_1 \to \psi_2$ and belongs to the set $d(t)^+$. In this case S_{n+1} differs from S_n in that $T_{r,n}$ is replaced by two new tableau structures, $T_{r\frown 0,n+1}$ and $T_{r\frown 1,n+1}$, in each of which the decoration of the node t has changed: in $T_{r\frown 0,n+1}\psi_1$ has been added to $d(t)^-$ and in $T_{r\frown 0,n+1}\psi_2$ has been added to $d(t)^+$. (Also $\psi_1 \to \psi_2$ has been removed from the positive parts of the decorations of the new copies of the node t. But this has no bearing on the argument and we ignore it.) To show that $\vdash_A RT(\mathcal{S}_n) \to RT(\mathcal{S}_{n+1})$ we need to show that $\vdash_A RT(T_{r,n}) \to (RT(T_{r\frown 0,n+1}) \vee RT(T_{r\frown 1,n+1}))$. First, we show that

(4.21) $\vdash_A RF(t, T_{r,n}) \to (RF(t, T_{r\frown 0,n+1}) \vee RF(t, T_{r\frown 1,n+1}))$

Note that (4.21) instantiates the general distribution principle

(4.22) $\vdash_A (\theta \wedge (\psi_1 \vee \psi_2)) \to ((\theta \wedge \psi_1) \vee (\theta \wedge \psi_2))$

which is a classical tautology and so is provable in A.

Next we observe that (4.23) follows from (4.21) by UG and the A-theorems T2 and T4.

(4.23) $\vdash_A (\exists x_t)[RF(t, T_{r,n})]x_t/c_t \to (\exists x_t)([RF(t, T_{r\frown 0,n+1})]x_t/c_t \vee (\exists x_t)[RF(t, T_{r\frown 1,n+1})]x_t/c_t)$

From (4.23) we can deduce that $\vdash_A RF(t_1, T_{r,n}) \to (RF(t_1, T_{r\frown 0,n+1}) \vee RF(t_1, T_{r\frown 1,n+1}))$. (This just involves classical propositional logic.) The same quantificational principles then enable us to conclude that $\vdash_A RF(t_0, T_{r,n}) \rightarrow (RF(t_0, T_{r\frown 0,n+1}) \vee RF(t_0, T_{r\frown 1,n+1}))$ and thus that $\vdash_A RF(T_{r,n}) \rightarrow (RF(T_{r\frown 0,n+1}) \vee RF(T_{r\frown 1,n+1}))$.

Our next case is that where the operation leading from $\mathcal{S}_n$ to $\mathcal{S}_{n+1}$ involves a sentence φ of the form $\forall x\vartheta$ and is reduced as a member of the set $d(t_{r,n})^-$. Recall that this reduction involves the introduction of a new successor node t' to t into $T_{r,n}$. (Given the form of our tableau structure $T_{r,n}$, t' is of the form $t \frown 2$.) The new decoration function $d_{r,n+1}$ in $T_{r,n+1}$ is the same as the old decoration function $d_{r,n}$ for $T_{r,n}$ for all nodes of $T_{r,n+1}$ other than t' (except that φ is removed from $d_{r,n}{}^-$); and for t' it is given in (4.24).

(4.24) (i) $d_{r,n+1}(t')^+ = d_{r,n}(t)^+ \cup \Psi$, where Ψ is some finite subset of $\{\psi \rightarrow \psi : \psi$ is an atomic sentence of $L(t')\}$; (ii) $d_{r,n+1}(t')^- = d_{r,n}(t)^- \cup \{[\vartheta]c_{t'}/x\}$.

Here $c_{t'}$ is the constant introduced in the reduction of φ; so $L(t')$ is the language $L(t) \cup \{c_{t'}\}$.

We first show

(4.25) $\vdash_A RF(t, T_{r,n}) \rightarrow RF(t, T_{r,n+1})$

Comparing $RF(t, T_{r,n+1})$ with $RF(t, T_{n,r})$ we see (from the definition of RF) that $RF(t, T_{r,n+1})$ is like $RF(t, T_{r,n})$ except for having an additional conjunct of the form

(4.26) $(\exists x_{t'})[\bigwedge d_{r,n+1}(t')]x_{t'}/c_{t'}$

The conjuncts of (4.26) are (a) the sentences $\psi \rightarrow \psi$ from the set Ψ, all of which are theorems of A, and (b) the sentence $[\neg\vartheta)]c_{t'}/x$. Since we have $\vdash_A \psi \rightarrow \psi$ for each of the sentences of type (a), we also have, by UG, $\vdash_A (\forall x_{t'})([\psi \rightarrow \psi]x_{t'}/c_{t'})$. So by repeated applications of (T4) we obtain

(4.27) $\vdash_A (\exists x_{t'})[\neg\vartheta]c_{t'}/x \rightarrow (\exists x_{t'})[\bigwedge d_{r,n+1}(t')]x_{t'}/c_{t'}$

In view of (4.27), showing $\vdash_A RF(t, T_{r,n}) \rightarrow RF(t, T_{r,n+1})$ only requires that we show $\vdash_A RF(t, T_{r,n}) \rightarrow (\exists x_{t'})[[\neg\vartheta]c_{t'}/x]x_{t'}/c_{t'}$. But this follows from the fact that one of the conjuncts of the formula $RF(t, T_{r,n})$ is $\neg\forall x\vartheta$. We also have $\vdash_A \neg\forall x\vartheta \rightarrow \exists x\neg\vartheta$ (because of the definition of $\exists$ in terms of $\forall$ and $\neg$) and $\vdash_A \exists x\neg\vartheta \rightarrow (\exists x_{t'})[[\neg\vartheta]c_{t'}/x]x_{t'}/c_{t'}$ (by (T5) and the fact that x is free for $x_{t'}$ in ϑ). The remainder of the argument that $\vdash_A RF(\mathcal{S}_n) \rightarrow RF(\mathcal{S}_{n+1})$ is as in the previous case.

Next we consider the case where $\forall x\vartheta$ is reduced as part of $d_{r,n}(t)^+$. In this case $RF(t, T_{r,n+1})$ differs from $RF(t, T_{r,n})$ in that for each of the nodes t_i (i = 4, 5, 6) $d_{r,n+1}(t_i)^+$ may contain additional conjuncts of the form $[\vartheta]c/x$, where c is some constant from the language $L(t_i)$. Moreover, each of these successor nodes now contains the sentence $\forall x\vartheta$. To see that in this case (4.25) (i.e., $\vdash_A RF(t, T_{r,n}) \rightarrow RF(t, T_{r,n+1})$) holds, we need to show that $RF(t, T_{r,n})$ entails each of the contributions that are made to $RF(t, T_{r,n+1})$ by the new conjunctions $\bigwedge d_{r,n+1}(t_i)$ for i = 4, 5, 6. We distinguish between case (a) where $i = 4$ and case (b) where $i \neq 4$. In case (a) t_i is the node t, so $[\vartheta]c/x$ occurs as a conjunct of a conjunction which also contains $\forall x\vartheta$. So in this case it follows from axiom schema (2.a) that the new conjunction $\bigwedge d_{r,n+1}(t)$ is entailed by the old conjunction $\bigwedge d_{r,n}(t)$. (The argument goes through of course also when several conjuncts of the form $[\vartheta]c/x$ have been added to $d_{r,n}(t)^+$.) In case (b) the new constituents are conjuncts of existentially quantified conjunctions which in their turn are conjuncts of $RF(t, T_{r,n})$. Let us focus on the case where $i = 5$ and where $d_{r,n+1}(t_5)^+$ differs from $d_{r,n}(t)^+$ by just one conjunct $[\vartheta]c/x$ as well as the sentence $\forall x\vartheta$ itself. (Again the case where more than one conjunct $[\vartheta]c/x$ has been added to $d_{r,n}(t)^+$ is similar.) The relevant conjunct of $RF(t, T_{r,n+1})$ is now $(\exists x_{t_5}) \bigwedge[d_{r,n+1}(t_5)]x_{t_5}/c_{t_5}$, where $[\bigwedge d_{n+1,r}(t_5)]x_{t_5}/c_{t_5}$ differs from $[\bigwedge d_{r,n}(t_5)]x_{t_5}/c_{t_5}$ in containing the additional conjuncts $[\vartheta]c/x$ and $\forall x\vartheta$. Note that since $\forall x\vartheta$ and $(\exists x_{t_5})[\bigwedge d_{r,n}(t_5)]x_{t_5}/c_{t_5}$ are both conjuncts of $RF(t, T_{r,n})$,

(4.28) $\vdash_A \ \forall x\vartheta \wedge (\exists x_{t_5})[\bigwedge d_{r,n}(t_5)]x_{t_5}/c_{t_5} \rightarrow (\exists x_{t_5})([\bigwedge d_{r,n}(t_5)]x_{t_5}/c_{t_5} \wedge [[\vartheta]c/x]x_{t_5}/c \wedge \forall x\vartheta)$

This follows from (T4) and (T5), the fact that t_5 does not occur in ϑ and the fact that $\vdash_A \forall x\vartheta \rightarrow \forall x_{t_5}\forall x\vartheta$ by Restricted Vacuous Quantification (principle (2.c)). This concludes in essence the demonstration that (4.25) holds in this case. The remainder of the argument runs once again as before.

Finally, we consider the case where φ is atomic and treated as part of $d_{n,r}(t)^+$ (operation 7a). (The case where φ is atomic and treated as part of $d_{n,r}(t)^-$ is completely analogous, and will be skipped). In an application of (7a) $RF(t, T_{r,n+1})$ will differ from $RF(t, T_{r,n})$ in having conjuncts of the form $(\exists x_{t_j})([\bigwedge d_{r,n}(t_j)]x_{t_j}/c_{t_j} \wedge \varphi)$ where $RF(t, T_{r,n})$ has a corresponding conjunct $(\exists x_{t_j})[\bigwedge d_{r,n}(t_j)]x_{t_j}/c_{t_j}$. To show that in A the latter formula entails the former, note that since the variable x_j has no free occurrences in φ and φ is atomic, $\vdash_A \varphi \rightarrow \forall x_j\varphi$ by axiom schema (2c). But in conjunction with the conjunct $(\exists x_{t_j})[\bigwedge d_{r,n}(t_j)]x_{t_j}/c_{t_j}$ of $RF(t, T_{r,n})\forall x_j\varphi$ will entail in A the existentially quantified conjunction $(\exists x_{t_j}) \bigwedge ([d_{r,n}(t_j)]x_{t_j}/c_{t_j} \wedge \varphi)$ by the same principles that we appealed to in the previous case.

Together the three statements (4.13.i–iii) show that if the tableau construction closes, then Γ is inconsistent. For starting from (4.13.ii) we arrive by a finite number of applications of Modus Tollens (with conditionals instantiating (4.13.iii)) at $\vdash_A \neg RF(\mathcal{S}_0)$. Together with (4.13.i) this leads to the conclusion that both $RF(\mathcal{S}_0)$ and $\neg RF(\mathcal{S}_0)$ are provable from Γ in A. So Γ is A-inconsistent. Conversely, if Γ is A-consistent, then the tableau construction for Γ' does not close, and in that case we can construct a parametric model M in which all sentences of Γ are true. This concludes the proof of Theorem 16. □

5 Conclusion

We have tried to develop a theory of quantification broad enough to incorporate many distinct conceptions of the nature of the objects of quantification and of the ways in which those objects can be referred to or described. The different conceptions of objects and reference that find representation within it, however, suggest different constraints on the parametric structures that are at the center of the theory. These different constraints lead in turn to different quantificational logics. Since different conceptions yield different parametric theories or logics, each appropriate to its proper contexts of application, it might seem misleading to speak of *the* parametric treatment of quantification.

Even so, one might have hoped that the logic of our approach is unified at least in the weaker sense that all these different parametric logics and theories could be seen as extensions of a single 'minimal' parametric logic. It is not clear to us, however, that there is a single, maximally general parametric semantics that defines this minimal logic and from which these various extensions can be obtained by restriction to special classes of models. For instance, if we abandon the assumption that diagrams are total, it is no longer clear how to deal with the sentential connectives; and, for all we know, there will be no single treatment that encompasses all the defensible options.

When we restrict ourselves to bivalent parametric structures, however, the matter appears to be clearer, at least from a formal point of view. For this case Section 4 presents the most general logic, that common to all bivalent parametric structures. But even when the restriction to bivalence is adopted there remain strong reasons for observing the greatest caution in the interpretation of our results. One reason relates to identity, a concept we have not treated explicitly in this paper, but which is evidently crucial for the conceptual issues which we offered as motivations for our approach. Treating identity as a logical notion within the present framework raises both the question of identity within a single diagram and the question of cross-stage identification, i.e., the question when an entity referred to as c at stage s is to be regarded as identical with an entity referred to as c or as c' at some other stage s'.

Another reason for caution involves modality. We have designed our approach to analyze quantifiers in contexts in which existence may be identified with constructibility at some stage of a process of construction. Where this identification is implausible, so is our analysis. Our semantics can be interpreted as giving the quantifiers modal force; its connection with modal logic seems to us another subject that merits further investigation. In particular, it would be of interest to know more about the various ways in which predicate logic with the semantics developed for it here can be interpreted within the expressively richer language of modal predicate logic when the latter is given a semantics in which parametric structures act as Kripke models.

Even within the much narrower limits of this study there are many formal questions we have not solved. One such question, strongly suggested by the results we have obtained, is to find conditions on parametric structures M which are necessary and sufficient for subsumption of classical logic—i.e., conditions C such that M has C iff $Th(M)$ is a classical first-order theory. But more important than such fairly specific technical questions is the larger task of arriving at a clearer and more detailed picture of the full spectrum of conceptually significant parametric theories than we have presented in this paper. We hope to make progress with this larger task in future work. But we realize that it is a task which probably requires much more than one or two people can accomplish on their own.

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