THE CATEGORY OF SIMULATIONS FOR WEIGHTED TREE AUTOMATA*

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Simulations of weighted tree automata (wta) are considered. It is shown how such simulations can be decomposed into simpler functional and dual functional simulations also called forward and backward simulations. In addition, it is shown in several cases (fields, commutative rings, NOETHERIAN semirings, semiring of natural numbers) that all equivalent wta M and N can be joined by a finite chain of simulations. More precisely, in all mentioned cases there is a single wta that simulates both M and N. Those results immediately yield decidability of equivalence provided that the semiring is finitely (and effectively) presented.

1. Introduction

Weighted tree automata are widely used in applications such as model checking [1] and natural language processing [25]. They finitely represent mappings, called tree series, that assign a weight to each tree. For example, a probabilistic parser would return a tree series that assigns to each parse tree its likelihood. Consequently, several toolkits [24, 28, 10] implement weighted tree automata.

The notion of simulation that is used in this paper is a generalization of the simulations for unweighted and weighted (finite) string automata of [6, 16]. The aim

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is to relate structurally equivalent automata. The results of Section 9.7 in [6] and [26] show that two unweighted string automata (i.e., potentially nondeterministic string automata over the BOOLEAN semiring) are equivalent if and only if they can be connected by a finite chain of relational simulations, and that in fact *functional* and *dual functional* simulations are sufficient. An extension to finite semirings can be found in [16]. Simulations for weighted string automata (wsa) are called *conjugacies* in [3, 4], where it is shown for the semiring $\mathbb{N}$ of natural numbers and for certain rings, including (skew) fields and the ring $\mathbb{Z}$ of integers, that two wsa are equivalent if and only if they can be connected by a finite chain of simulations. It is also shown that even a finite chain of functional (*covering*) and dual functional (*co-covering*) simulations is sufficient. The origin of those results can be traced back to the pioneering work of SCHÜTZENBERGER in the early 60's, who proved that every wsa over a field is equivalent to a minimal wsa that is simulated by every *trim* equivalent wsa [5]. Relational simulations of wsa are also studied in [9], where they are used to reduce the size of wsa. The relationship between functional simulations and the MILNER-PARK notion of bisimulation [29, 30] is discussed in [6, 9].

In this contribution, we investigate simulations for weighted (finite) tree automata (wta). SCHÜTZENBERGER's minimization method was extended to wta over fields in [2, 8]. In addition, relational and functional simulations for wta are probably first used in [13, 14, 21]. Moreover, simulations can be generalized to presentations in algebraic theories [6], which seems to cover all mentioned instances. Here, we extend and improve the results of [3, 4] to wta. In particular, we show that two wta over a commutative ring, NOETHERIAN semiring, or the semiring $\mathbb{N}$ are equivalent if and only if they are connected by a finite chain of simulations. Moreover, we discuss when the simulations can be replaced by functional and dual functional simulations, which are efficiently computable [21]. Such results are important because they immediately yield the decidability of equivalence provided that the semiring is finitely and effectively presented.

2. Preliminaries

The set of nonnegative integers is $\mathbb{N}$. For every $k \in \mathbb{N}$, the set $\{i \in \mathbb{N} \mid 1 \leq i \leq k\}$ is simply denoted by $[k]$. We write $|A|$ for the cardinality of the set A. A *semiring* is an algebraic structure $\mathcal{A} = (A, +, \cdot, 0, 1)$ such that $(A, +, 0)$ and $(A, \cdot, 1)$ are monoids, of which the former is commutative, and $\cdot$ distributes both-sided over finite sums (i.e., $a \cdot 0 = 0 = 0 \cdot a$ for every $a \in A$ and $a \cdot (b + c) = ab + ac$ and $(b + c) \cdot a = ba + ca$ for every $a, b, c \in A$). The semiring $\mathcal{A}$ is *commutative* if $(A, \cdot, 1)$ is commutative. It is a *ring* if there exists an element $-1 \in A$ such that $(-1) + 1 = 0$. The set U is the set $\{a \in A \mid \exists b \in A\colon ab = 1 = ba\}$ of *(multiplicative) units*. The semiring $\mathcal{A}$ is a *semifield* if $U = A \setminus \{0\}$; i.e., for every $a \in A$ there exists a *multiplicative inverse* $a^{-1} \in A$ such that $aa^{-1} = 1 = a^{-1}a$. A *field* is a semifield that is also a ring. Let $\langle B \rangle_+ = \{b_1 + \cdots + b_n \mid n \in \mathbb{N}, b_1, \ldots, b_n \in B\}$ for every $B \subseteq A$. If $A = \langle B \rangle_+$, then $\mathcal{A}$ is *additively generated by* B. The semiring $\mathcal{A}$ is *equisubtractive* if

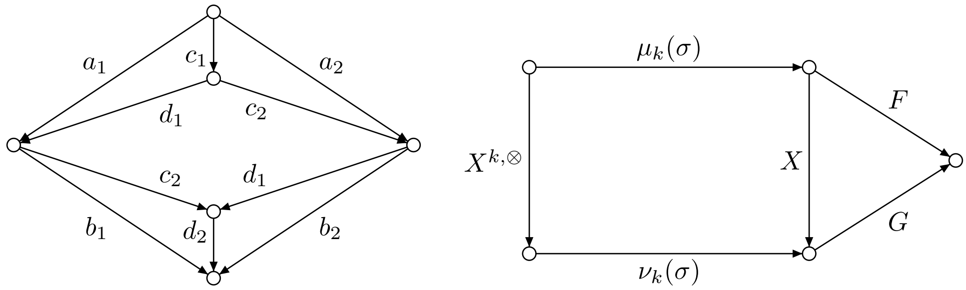

Fig. 1. Illustration of 'equisubtractive' and 'simulation'.

for every $a_1, a_2, b_1, b_2 \in A$ with $a_1 + b_1 = a_2 + b_2$ there exist $c_1, c_2, d_1, d_2 \in A$ such that (i) $a_1 = c_1 + d_1$, (ii) $b_1 = c_2 + d_2$, (iii) $a_2 = c_1 + c_2$, and (iv) $b_2 = d_1 + d_2$. We illustrate equisubtractivity in Fig. 1. The semiring $\mathcal{A}$ is *zero-sum free* (*zero-divisor free*, respectively) if $a + b = 0$ ($a \cdot b = 0$, respectively) implies $0 \in \{a, b\}$ for every $a, b \in A$. Finally, it is *positive* if it is both zero-sum and zero-divisor free. Clearly, any nontrivial (i.e., $0 \neq 1$) ring is not zero-sum free, and any semifield is zero-divisor free. An infinitary sum operation $\sum$ is a family $(\sum_I)_I$, where I is an arbitrary index set, such that $\sum_I \colon A^I \to A$. We generally write $\sum_{i \in I} a_i$ instead of $\sum_I (a_i)_{i \in I}$. The semiring $\mathcal{A}$ together with the infinitary sum operation $\sum$ is *complete* [12, 20, 23] if for all index sets I and $(a_i)_{i \in I} \in A^I$

- $\sum_{i \in I} a_i = a_{j_1} + a_{j_2}$ if $I = \{j_1, j_2\}$ with $j_1 \neq j_2$,
- $\sum_{i \in I} a_i = \sum_{j \in J} \bigl(\sum_{i \in I_j} a_i\bigr)$ for every partition $(I_j)_{j \in J}$ of I, and
- $a \cdot \bigl(\sum_{i \in I} a_i\bigr) = \sum_{i \in I} a a_i$ and $\bigl(\sum_{i \in I} a_i\bigr) \cdot a = \sum_{i \in I} a_i a$ for every $a \in A$.

An $\mathcal{A}$-*semimodule* is a commutative monoid $(B, +, 0)$ together with an action $\cdot \colon A \times B \to B$, written as juxtaposition, such that for every $a, a' \in A$ and $b, b' \in B$ we have (i) $(a + a')b = ab + a'b$, (ii) $a(b + b') = ab + ab'$, (iii) $0b = 0$, (iv) $1b = b$, and (v) $(a \cdot a')b = a(a'b)$. The semiring $\mathcal{A}$ is NOETHERIAN (see Chapter VIII, Section 1 of [22]) if all subsemimodules of every finitely-generated $\mathcal{A}$-semimodule are again finitely-generated.

In the following, we sometimes identify index sets of equal cardinality. Let $X \in A^{I_1 \times J_1}$ and $Y \in A^{I_2 \times J_2}$ for finite sets I_1, I_2, J_1, J_2. We use upper-case letters (like C, D, E, X, Y) for matrices and the corresponding lower-case letters for their entries. The matrix X is *relational* if $X \in \{0,1\}^{I_1 \times J_1}$. Clearly, a relational X corresponds to a relation $\rho_X \subseteq I_1 \times J_1$ (and vice versa) by $(i, j) \in \rho_X$ if and only if $x_{ij} = 1$. Moreover, a relational matrix X is *functional*, *surjective*, or *injective* if ρ_X has this property. As usual, we denote the *transpose* of X by X^{T}, and we call X *non-degenerate* if it has no rows or columns of entirely zeroes. A *diagonal* matrix X is such that $x_{ij} = 0$ for every $i \neq j$. Finally, the matrix X is invertible if there exists a matrix X^{-1} such that $XX^{-1} = I = X^{-1}X$ where I is the unit matrix. The KRONECKER product $X \otimes Y \in A^{(I_1 \times I_2) \times (J_1 \times J_2)}$ is such that $(X \otimes Y)_{i_1 i_2, j_1 j_2} = x_{i_1 j_1} y_{i_2 j_2}$

for every $i_1 \in I_1$, $i_2 \in I_2$, $j_1 \in J_1$, and $j_2 \in J_2$. Clearly, the KRONECKER product is not commutative and $(1) \in A^{[1]}$ acts as neutral element. We let $X^{0,\otimes} = (1)$ and $X^{i+1,\otimes} = X^{i,\otimes} \otimes X$ for every $i \in \mathbb{N}$.

Finally, let us move to trees. A *ranked alphabet* is a finite set Σ together with a mapping $\mathrm{rk}\colon \Sigma \to \mathbb{N}$. We often just write Σ for a ranked alphabet and assume that the mapping rk is implicit. We write $\Sigma_k = \{\sigma \in \Sigma \mid \mathrm{rk}(\sigma) = k\}$ for the set of all k-ary symbols. The set of Σ-*trees* is the smallest set T_Σ such that $\sigma(t_1, \dots, t_k) \in T_\Sigma$ for all $\sigma \in \Sigma_k$ and $t_1, \dots, t_k \in T_\Sigma$. If $\sigma \in \Sigma_0$, then we identify $\sigma()$ with σ. A *tree series* [over Σ and the semiring $\mathcal{A} = (A, +, \cdot, 0, 1)$] is a mapping $\varphi\colon T_\Sigma \to A$. The set of all such tree series is denoted by $A\langle\langle T_\Sigma\rangle\rangle$. For every $\varphi \in A\langle\langle T_\Sigma\rangle\rangle$ and $t \in T_\Sigma$, we often write (φ, t) instead of $\varphi(t)$. Tree series natuarally form an $\mathcal{A}$-semimodule and a Σ-algebra [17].

A *weighted tree automaton (over $\mathcal{A}$)*, for short: wta, is a system (Σ, Q, μ, F) with an input ranked alphabet Σ, a finite set Q of *states*, transitions $\mu = (\mu_k)_{k\in\mathbb{N}}$ such that $\mu_k\colon \Sigma_k \to A^{Q^k \times Q}$ for every $k \in \mathbb{N}$, and a *final weight* vector $F \in A^Q$. Next, let us introduce the semantics $\|M\|$ of the wta M. We first define the mapping $h_\mu\colon T_\Sigma \to A^Q$ so that $h_\mu(\sigma(t_1, \dots, t_k)) = \big(h_\mu(t_1) \otimes \cdots \otimes h_\mu(t_k)\big) \cdot \mu_k(\sigma)$ for every $\sigma \in \Sigma_k$ and $t_1, \dots, t_k \in T_\Sigma$, where the final product $\cdot$ is the classical matrix product. Then $(\|M\|, t) = h_\mu(t)F$ for every $t \in T_\Sigma$, where the product is the usual inner (dot) product. The wta M is *trim* if every state is accessible and co-accessible [17, 19] in the BOOLEAN wta obtained by replacing every nonzero weight by 1.

3. Simulation

Let us introduce the main notion of the paper. From now on, let $M = (\Sigma, Q, \mu, F)$ and $N = (\Sigma, P, \nu, G)$ be wta. Then M *simulates* N (cf., [6, 16], Def. 1 of [3], and Def. 35 of [13]) if there is $X \in A^{Q\times P}$ such that $F = XG$ and $\mu_k(\sigma)X = X^{k,\otimes} \cdot \nu_k(\sigma)$ for every $\sigma \in \Sigma_k$. The matrix X is called *transfer matrix*, and we write $M \to_X N$ if M simulates N with transfer matrix X. Note that $X^{k,\otimes}_{i_1\cdots i_k, j_1\cdots j_k} = \prod_{\ell=1}^k x_{i_\ell, j_\ell}$. We illustrate the notion of simulation in Fig. 1. If $M \to_X M'$ and $M' \to_Y N$, then $M \to_{XY} N$.

Theorem 1. *If M simulates N, then M and N are equivalent.*

Proof. Let $M = (\Sigma, Q, \mu, F)$ and $N = (\Sigma, P, \nu, G)$, and let $X \in A^{Q\times P}$ be a transfer matrix. We claim that $h_\mu(t)X = h_\nu(t)$ for every $t \in T_\Sigma$. We prove this by induction on t. Let $t = \sigma(t_1, \dots, t_k)$ for some $\sigma \in \Sigma_k$ and $t_1, \dots, t_k \in T_\Sigma$.

$$\begin{aligned} &h_\mu(\sigma(t_1, \dots, t_k))X = \big(h_\mu(t_1) \otimes \cdots \otimes h_\mu(t_k)\big) \cdot \mu_k(\sigma)X \\ &= \big(h_\mu(t_1) \otimes \cdots \otimes h_\mu(t_k)\big) \cdot X^{k,\otimes} \cdot \nu_k(\sigma) = \big(h_\mu(t_1)X \otimes \cdots \otimes h_\mu(t_k)X\big) \cdot \nu_k(\sigma) \\ &= \big(h_\nu(t_1) \otimes \cdots \otimes h_\nu(t_k)\big) \cdot \nu_k(\sigma) = h_\nu(\sigma(t_1, \dots, t_k)) \end{aligned}$$

With this claim, the statement can now be proved easily. For every $t \in T_\Sigma$

$$(\|M\|, t) = h_\mu(t)F = h_\mu(t)XG = h_\nu(t)G = (\|N\|, t) \ . \qquad \square$$

Next, we prepare the result for functional simulations. To this end, we first need to prove in which cases the transfer matrix is nondegenerate.

Lemma 2. *Let M and N be trim and $M \to_X N$. If (i) X is functional or (ii) $\mathcal{A}$ is positive, then X is nondegenerate.*

Proof. Let $M = (\Sigma, Q, \mu, F)$ and $N = (\Sigma, P, \nu, G)$. Moreover, let

$$J = \{p \in P \mid \forall q \in Q \colon x_{qp} = 0\} \enspace .$$

Then $\nu_k(\sigma)_{w,j} = 0$ for every $\sigma \in \Sigma_k$, $w \in (P \setminus J)^k$, and $j \in J$. This is seen as follows. Since $\mu_k(\sigma)X = X^{k,\otimes} \cdot \nu_k(\sigma)$ we obtain

$$\sum_{q \in Q} \mu_k(\sigma)_{q_1 \cdots q_k, q} \cdot x_{qj} = 0 = \sum_{p_1, \ldots, p_k \in P} \Big(\prod_{\ell=1}^{k} x_{q_\ell, p_\ell}\Big) \cdot \nu_k(\sigma)_{p_1 \cdots p_k, j} \tag{3}$$

for every $q_1, \ldots, q_k \in Q$ and $j \in J$. If X is functional, then

$$\sum_{p_1, \ldots, p_k \in P} \Big(\prod_{\ell=1}^{k} x_{q_\ell, p_\ell}\Big) \cdot \nu_k(\sigma)_{p_1 \cdots p_k, j} = \nu_k(\sigma)_{\rho_X(q_1) \cdots \rho_X(q_k), j} = 0 \enspace ,$$

which proves the claim. On the other hand, if $\mathcal{A}$ is positive, then (3) implies that $\prod_{\ell=1}^{k} x_{q_\ell, p_\ell} \cdot \nu_k(\sigma)_{p_1 \cdots p_k, j} = 0$ for every $p_1, \ldots, p_k \in P$. Since for every $p_\ell \notin J$, there exists q_ℓ such that $x_{q_\ell, p_\ell} \neq 0$ and $\prod_{\ell=1}^{k} x_{q_\ell, p_\ell} \neq 0$ by zero-divisor freeness, we conclude that $\nu_k(\sigma)_{p_1 \cdots p_k, j} = 0$ for every $p_1, \ldots, p_k \in P \setminus J$, which again proves the claim. Consequently, all states of J are not accessible. Since N is trim, we conclude $J = \emptyset$, and thus, X has no column of zeroes.

If X is functional, then it clearly has no row of zeroes. To prove that X has no row of zeroes in the remaining case, let $I = \{q \in Q \mid \forall p \in P \colon x_{qp} = 0\}$. Then $F_i = 0$ and $\mu_k(\sigma)_{q_1 \cdots q_k, q} = 0$ for every $\sigma \in \Sigma_k$, $q \in Q \setminus I$, $q_1, \ldots, q_k \in Q$, and $i \in I$ such that $q_\ell = i$ for some $\ell \in [k]$. Clearly, $F_i = \sum_{p \in P} x_{ip} G_p = 0$ for every $i \in I$. Moreover, since $\mu_k(\sigma)X = X^{k,\otimes} \cdot \nu_k(\sigma)$ we obtain

$$\sum_{q \in Q} \mu_k(\sigma)_{q_1 \cdots q_k, q} \cdot x_{qp} = \sum_{p_1, \ldots, p_k \in P} \Big(\prod_{\ell=1}^{k} x_{q_\ell, p_\ell}\Big) \cdot \nu_k(\sigma)_{p_1 \cdots p_k, p} = 0 \tag{5}$$

for every $q_1, \ldots, q_k \in Q$, $p \in P$, and $i \in I$ such that $q_\ell = i$ for some $\ell \in [k]$. Since $\mathcal{A}$ is positive, (5) implies that $\mu_k(\sigma)_{q_1 \cdots q_k, q} \cdot x_{qp} = 0$ for every $q \in Q$. However, for all $q \in Q \setminus I$, there exists $p \in P$ such that $x_{qp} \neq 0$ because $q \notin I$. Consequently, $\mu_k(\sigma)_{q_1 \cdots q_k, q} = 0$ by zero-divisor freeness, which yields that all states of I are not accessible. Since M is trim, we have $I = \emptyset$, and thus, X has no row of zeroes. □

Now we relate functional simulation to forward simulation (see Def. 1 of [21]). A surjective mapping $\rho \colon Q \to P$ is a *forward simulation* from M to N if (i) $F_q = G_{\rho(q)}$ for every $q \in Q$ and (ii) $\sum_{q \in \rho^{-1}(p)} \mu_k(\sigma)_{q_1 \cdots q_k, q} = \nu_k(\sigma)_{\rho(q_1) \cdots \rho(q_k), p}$ for every $p \in P$, $\sigma \in \Sigma_k$, and $q_1, \ldots, q_k \in Q$. We say that M *forward simulates* N, written $M \twoheadrightarrow N$, if there exists a forward simulation from M to N. Similarly, we can relate backward simulation (see Def. 16 of [21]) to dual functional simulation. A surjective function $\rho\colon Q \to P$ is a *backward simulation* from M to N if $\sum_{q \in \rho^{-1}(p)} F_q = G_p$ for every $p \in P$ and $\sum_{q_1 \in \rho^{-1}(p_1), \dots, q_k \in \rho^{-1}(p_k)} \mu_k(\sigma)_{q_1 \cdots q_k, q} = \nu_k(\sigma)_{p_1 \cdots p_k, \rho(q)}$ for every $q \in Q$, $\sigma \in \Sigma_k$, and $p_1, \dots, p_k \in P$. We say that M *backward simulates* N, written $M \twoheadleftarrow N$, if there exists a backward simulation from M to N. Using Lemma 2 we obtain the following statement.

Lemma 3. *Let N be trim. Then $M \twoheadrightarrow N$ if and only if there exists a functional transfer matrix X such that $M \to_X N$. Moreover, $M \twoheadleftarrow N$ if and only if there exists a transfer matrix X such that X^{T} is functional and $N \to_X M$.*

Proof. Let $M = (\Sigma, Q, \mu, F)$ and $N = (\Sigma, P, \nu, G)$. First suppose that $M \xrightarrow{X} N$ with functional $X \in A^{Q \times P}$. Then $\rho_X : Q \to P$ is a surjective function by Lemma 2. Conversely, if $M \twoheadrightarrow N$ with the forward simulation $\rho\colon Q \to P$, then ρ induces a surjective functional matrix $X \in A^{Q \times P}$ such that $\rho_X = \rho$. Let $X \in A^{Q \times P}$ be a surjective, functional matrix. It remains to prove that the conditions that (1) "X is a transfer matrix" and (2) "ρ_X is a forward simulation" are equivalent.

(i) $F = XG$ if and only if $F_q = G_{\rho(q)}$ for every $q \in Q$.
(ii) for every $\sigma \in \Sigma_k$, $q_1, \dots, q_k \in Q$, and $p \in P$

$$(\mu_k(\sigma)X)_{q_1 \cdots q_k, p} = \sum_{q \in Q\colon \rho_X(q) = p} \mu_k(\sigma)_{q_1 \cdots q_k, q}$$
$$(X^{k,\otimes} \cdot \nu_k(\sigma))_{q_1 \cdots q_k, p} = \nu_k(\sigma)_{\rho_X(q_1) \cdots \rho_X(q_k), p} \quad .$$

Thus, X is a transfer matrix if and only if ρ_X is a forward simulation, which proves the first part of the lemma.

Second, suppose that $N \xrightarrow{X} M$ with the transfer matrix $X \in A^{P \times Q}$ such that X^{T} is functional. Let $Y = X^{\mathrm{T}}$. Then $\rho_Y : Q \to P$ is a surjective function by Lemma 2. Conversely, if $M \twoheadleftarrow N$ with the backward simulation $\rho\colon Q \to P$, then ρ again induces a surjective, functional matrix $X \in A^{Q \times P}$ such that $\rho_X = \rho$. Let $X \in A^{Q \times P}$ be a surjective, functional matrix. It remains to prove that the conditions that (1) "X^{T} is a transfer matrix" and (2) "ρ_X is a backward simulation" are equivalent.

(i) $G = X^{\mathrm{T}}F$ if and only if $G_p = \sum_{q \in Q\colon \rho_X(q) = p} F_q$ for every $p \in P$.
(ii) for every $\sigma \in \Sigma_k$, $p_1, \dots, p_k \in P$, and $q \in Q$

$$(\nu_k(\sigma)X^{\mathrm{T}})_{p_1 \cdots p_k, q} = \nu_k(\sigma)_{p_1 \cdots p_k, \rho_X(q)}$$
$$((X^{\mathrm{T}})^{k,\otimes} \cdot \mu_k(\sigma))_{p_1 \cdots p_k, q} = \sum_{\substack{q_1, \dots, q_k \in Q \\ \rho_X(q_1) = p_1, \dots, \rho_X(q_k) = p_k}} \mu_k(\sigma)_{q_1 \cdots q_k, q} \quad .$$

Thus, X^{T} is a transfer matrix if and only if ρ_X is a backward simulation. □

Next, we recall two important matrix decomposition results of [3], for which we provide complete proof details for completeness' sake.

Lemma 4. *If $A = \langle U \rangle_+$, then for every $X \in A^{Q \times P}$ there exist matrices C, E, D such that (i) $X = CED$, (ii) C^{T} and D are functional, and (iii) E is an invertible diagonal matrix. If (a) X is nondegenerate or (b) $\mathcal{A}$ has (nontrivial) zero-sums, then C^{T} and D can be chosen to be surjective.*

Proof. For every $q \in Q$ and $p \in P$, let $\ell_{qp} \in \mathbb{N}$ and $u_{qp1}, \ldots, u_{qp\ell_{qp}} \in U$ be such that $x_{qp} = \sum_{i=1}^{\ell_{qp}} u_{qpi}$. In addition, let

$$J = \{ (q, i, p) \mid q \in Q, p \in P, i \in [\ell_{qp}] \} \enspace .$$

Finally, let $\pi_1 \colon J \to Q$ and $\pi_3 \colon J \to P$ be such that $\pi_1(\langle q, i, p \rangle) = q$ and $\pi_3(\langle q, i, p \rangle) = p$ for every $\langle q, i, p \rangle \in J$. Then we set C^{T} and D to the functional matrices represented by π_1 and π_3, respectively. Together with the diagonal matrix E such that $e_{\langle q,i,p \rangle, \langle q,i,p \rangle} = u_{qpi}$ for every $\langle q, i, p \rangle \in J$, we obtain $X = CED$. For every $q \in Q$ and $p \in P$ we have

$$\sum_{j_1, j_2 \in J} c_{q, j_1} e_{j_1, j_2} d_{j_2, p} = \sum_{i=1}^{\ell_{qp}} e_{\langle q,i,p \rangle, \langle q,i,p \rangle} = \sum_{i=1}^{\ell_{qp}} u_{qpi} = x_{qp} \enspace .$$

It is clear that C^{T} and D are functional matrices. Moreover, E is an invertible diagonal matrix because $EE^{-1} = I = E^{-1}E$ where E^{-1} is the matrix obtained from E by inverting each nonzero element. If X is nondegenerate, then C^{T} and D are surjective. Finally, if there are zero-sums, then for every $q \in Q$ and $p \in P$ there exist $u, v \in U$ such that $x_{qp} = 0 = u + v$, which yields that we can choose $\ell_{qp} > 0$. This completes the proof. □

Lemma 5. *Let $\mathcal{A}$ be equisubtractive. Moreover, let $R \in A^Q$ and $C \in A^P$ be such that $\sum_{q \in Q} r_q = \sum_{p \in P} c_p$. Then there exists a matrix $X \in A^{Q \times P}$ with row sums R and column sums C.*

Proof. If $|Q| \leq 1$ or $|P| \leq 1$, then the statement is trivially true. Otherwise, select $i \in Q$ and $j \in P$, and let $Q' = Q \setminus \{i\}$ and $P' = P \setminus \{j\}$. By assumption

$$\sum_{q \in Q'} r_q + r_i = \sum_{p \in P'} c_p + c_j \enspace .$$

Thus, by equisubtractivity there exist $a, c'_j, r'_i, x_{ij} \in A$ such that

$$\sum_{q \in Q'} r_q = a + c'_j \qquad r_i = r'_i + x_{ij} \qquad \sum_{p \in P'} c_p = a + r'_i \qquad c_j = c'_j + x_{ij} \enspace .$$

Continuing the row decomposition, we obtain $Y \in A^{Q'}$ and $R' \in A^{Q'}$ such that $r_q = r'_q + y_q$ for every $q \in Q'$ and $\sum_{q \in Q'} r'_q = a$. In a similar manner we perform column decomposition to obtain $Y' \in A^{P'}$ and $C' \in A^{P'}$ such that $c_p = c'_p + y'_p$ for every $p \in P'$ and $\sum_{p \in P'} c'_p = a$. Thus, by the induction hypothesis, there exists a matrix $X' \in A^{Q' \times P'}$ with row sums R' and column sums C' because $\sum_{q \in Q'} r'_q = \sum_{p \in P'} c'_p$. Then the matrix

$$X = \begin{pmatrix} X' & Y \\ (Y')^{\mathrm{T}} & x_{ij} \end{pmatrix}$$

obviously has the required row and column sums R and C, respectively. □

The following lemma observes a simple property that we use without further mention. Its proof is simple and left as an easy exercise.

Lemma 6. *If $X \in A^{Q \times P}$ is functional (respectively, invertible diagonal), then $X^{k,\otimes}$ is functional (respectively, invertible diagonal) for every $k \in \mathbb{N}$.*

We can now obtain the main result of this section, which shows how we can decompose a simulation into functional and dual functional simulations (or: forward and backward simulations, respectively) and an invertable diagonal simulation.

Theorem 7. *Let $\mathcal{A}$ be equisubtractive with $A = \langle U \rangle_+$. Then $M \to_X N$ if and only if there exist two wta M' and N' such that (i) $M \to_C M'$ where C^{T} is functional, (ii) $M' \to_E N'$ where E is an invertible diagonal matrix, and (iii) $N' \to_D N$ where D is functional. If M and N are trim, then $M' \twoheadleftarrow M$ and $N' \twoheadrightarrow N$.*

Proof. Clearly, $M \to_C M' \to_E N' \to_D N$, which proves that $M \to_{CED} N$. For the converse, Lemma 4 shows that there are matrices C, E, and D such that $X = CED$, C^{T} and D are functional matrices, and $E \in A^{I \times I}$ is an invertible diagonal matrix. Finally, let $\varphi \colon I \to Q$ and $\psi \colon I \to P$ be the functions associated to C^{T} and D.

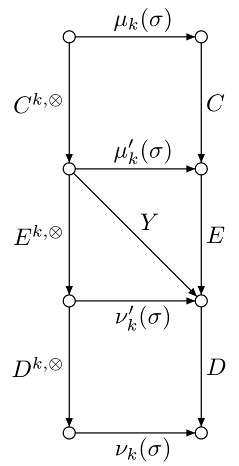

Fig. 2. Relating the matrices.

It remains to determine the wta M' and N'. Let $M' = (\Sigma, I, \mu', F')$ and $N' = (\Sigma, I, \nu', G')$ with $G' = DG$ and $F' = EDG$. Then we have $CF' = CEDG = XG = F$. Thus, it remains to specify $\mu'_k(\sigma)$ and $\nu'_k(\sigma)$ for every $\sigma \in \Sigma_k$. To this end, we determine a matrix $Y \in A^{I^k \times I}$ such that we have (1) $C^{k,\otimes} \cdot Y = \mu_k(\sigma) CE$ and (2) $YD = E^{k,\otimes} \cdot D^{k,\otimes} \cdot \nu_k(\sigma)$. Let $\mu'_k(\sigma) = YE^{-1}$ and $\nu'_k(\sigma) = (E^{k,\otimes})^{-1} \cdot Y$. Consequently, we have $\mu_k(\sigma)C = C^{k,\otimes} \cdot \mu'_k(\sigma)$, $\mu'_k(\sigma)E = E^{k,\otimes} \cdot \nu'_k(\sigma)$, and $\nu'_k(\sigma)D = D^{k,\otimes} \cdot \nu_k(\sigma)$. These equalities are displayed in Fig. 2.

Finally, we need to specify the matrix Y. For every $q \in Q$ and $p \in P$, let $I_q = \varphi^{-1}(q)$ and $J_p = \psi^{-1}(p)$. Obviously, Y can be decomposed into disjoint (not necessarily contiguous) submatrices $Y_{q_1 \cdots q_k, p} \in A^{(I_{q_1} \times \cdots \times I_{q_k}) \times J_p}$ with $q_1, \ldots, q_k \in Q$

and $p \in P$. Then properties (1) and (2) hold if and only if for every $q_1, \dots, q_k \in Q$ and $p \in P$ the following two conditions hold:

(1) For every $i \in I$ such that $\psi(i) = p$, the sum of the i-column of $Y_{q_1 \cdots q_k, p}$ is $\mu_k(\sigma)_{q_1 \cdots q_k, \varphi(i)} \cdot e_{i,i}$.
(2) For all $i_1, \dots, i_k \in I$ such that $\varphi(i_j) = q_j$ for every $j \in [k]$, the sum of the $(i_1, \dots, i_k)$-row of $Y_{q_1 \cdots q_k, p}$ is $\prod_{j=1}^{k} e_{i_j, i_j} \cdot \nu_k(\sigma)_{\psi(i_1) \cdots \psi(i_k), p}$.

Those two conditions are compatible because

$$\sum_{\substack{i \in I \\ \psi(i) = p}} \mu_k(\sigma)_{q_1 \cdots q_k, \varphi(i)} \cdot e_{i,i} = \big(\mu_k(\sigma) CED\big)_{q_1 \cdots q_k, p} = \big(\mu_k(\sigma) X\big)_{q_1 \cdots q_k, p}$$
$$= \big(X^{k,\otimes} \cdot \nu_k(\sigma)\big)_{q_1 \cdots q_k, p} = \big(C^{k,\otimes} \cdot E^{k,\otimes} \cdot D^{k,\otimes} \cdot \nu_k(\sigma)\big)_{q_1 \cdots q_k, p}$$
$$= \sum_{\substack{i_1, \dots, i_k \in I \\ \forall j \in [k] \colon \varphi(i_j) = q_j}} \Big(\prod_{j=1}^{k} e_{i_j, i_j}\Big) \cdot \nu_k(\sigma)_{\psi(i_1) \cdots \psi(i_k), p} \enspace .$$

Consequently, the row and column sums of the submatrices $Y_{q_1 \cdots q_k, p}$ are consistent, so that we can determine all submatrices (and the whole matrix) using Lemma 5. If M and N are trim, then either (a) $\mathcal{A}$ is zero-sum free (and thus positive because it is additively generated by its units), in which case X is nondegenerate by Lemma 2, or (b) $\mathcal{A}$ has nontrivial zero-sums. In both cases, Lemma 4 shows that the matrices C^{T} and D are surjective, which yields the additional statement by Lemma 3. □

The decomposition of simulations into forward and backward simulation is effective and offers computational benefits because forward and backward simulations can be efficiently computed [21]. To keep the presentation simple, we will continue to deal with simulation in the following, although we could decompose them in many of the following cases.

4. Category of simulations

In this section our aim is to show that several well-known constructions of wta are *functorial*: they may be extended to simulations in a functorial way. Below we will only deal with the sum, Hadamard product, σ_0-product, and σ_0-iteration (cf. [17, 15, 19]). Scalar OI-substition, † (the dagger operation) [7], homomorphism, quotient, and top-concatenation [15, 19] may be covered in a similar fashion.

In this section, let $\mathcal{A}$ be commutative. Moreover, let $M = (\Sigma, Q, \mu, F)$, $M' = (\Sigma, Q', \mu', F')$, and $M'' = (\Sigma, Q'', \mu'', F'')$ be wta. We already remarked that, if $M \to_X M'$ and $M' \to_Y M''$, then $M \to_{XY} M''$. Moreover, $M \to_I M$ with the unit matrix $I \in A^{Q \times Q}$. Thus, wta over the alphabet Σ form a category $\mathbf{Sim}_\Sigma$.

In the following, let $M = (\Sigma, Q, \mu, F)$ and $N = (\Sigma, P, \nu, G)$ be wta such that $Q \cap P = \emptyset$. The sum $M \oplus N$ of M and N is the wta $(\Sigma, Q \cup P, \kappa, H)$ where

$H = \langle F, G\rangle = \left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$ and

$$\kappa_k(\sigma)_{q_1\cdots q_k,q} = (\mu_k(\sigma) \bowtie \nu_k(\sigma))_{q_1\cdots q_k,q} = \begin{cases} \mu_k(\sigma)_{q_1\cdots q_k,q} & \text{if } q, q_1, \ldots, q_k \in Q \\ \nu_k(\sigma)_{q_1\cdots q_k,q} & \text{if } q, q_1, \ldots, q_k \in P \\ 0 & \text{otherwise.} \end{cases}$$

for all $\sigma \in \Sigma_k$ and $q, q_1, \ldots, q_k \in Q \cup P$. It is known that $\|M \oplus N\| = \|M\| + \|N\|$. Next, we extend the sum construction to simulations. To this end, let $M \to_X M'$ and $N \to_Y N'$ with $N' = (\Sigma, P', \nu', G')$. The sum $X \oplus Y \in A^{(Q\cup P)\times(Q'\cup P')}$ of the transfer matrices X and Y is $X \oplus Y = \left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$.

Proposition 8. *We have* $(M \oplus N) \stackrel{X\oplus Y}{\longrightarrow} (M' \oplus N')$.

Proof. For every $\sigma \in \Sigma_k$ we have

$$\begin{aligned}\big(\mu_k(\sigma) \bowtie \nu_k(\sigma)\big) \cdot (X \oplus Y) &= \mu_k(\sigma)X \bowtie \nu_k(\sigma)Y \\ &= X^{k,\otimes} \cdot \mu'_k(\sigma) \bowtie Y^{k,\otimes} \cdot \nu'_k(\sigma) = (X \oplus Y)^{k,\otimes} \cdot \big(\mu'_k(\sigma) \bowtie \nu'_k(\sigma)\big)\end{aligned}$$

and $\langle F, G\rangle = \langle XF', YG'\rangle = (X \oplus Y) \cdot \langle F', G'\rangle$, which completes the proof. □

Proposition 9. *The function $\oplus$, which is defined on wta and transfer matrices, is a functor $\boldsymbol{Sim}_\Sigma^2 \to \boldsymbol{Sim}_\Sigma$.*

Proof. It is a routine matter to verify that identity transfer matrices are preserved and the interchange rule $(X \oplus Y) \cdot (X' \oplus Y') = XX' \oplus YY'$ holds for all composable transfer matrices X, X', Y, Y'. □

Next, we treat the remaining operations. Let σ_0 be a distinguished symbol in Σ_0. The σ_0-product $M \cdot_{\sigma_0} N$ of M with N is the wta $(\Sigma, Q\cup P, \kappa, H)$ such that $H = \langle F, 0\rangle$ and for each $\sigma \in \Sigma_k$ with $\sigma \neq \sigma_0$,

$$\kappa_k(\sigma)_{q_1\cdots q_k,q} = \begin{cases} \mu_k(\sigma)_{q_1\cdots q_k,q} & \text{if } q, q_1, \ldots, q_k \in Q \\ \mu_0(\sigma_0)_q \cdot \nu_k(\sigma)_{q_1\cdots q_k} G & \text{if } q \in Q \text{ and } q_1, \ldots, q_k \in P \\ \nu_k(\sigma)_{q_1\cdots q_k,q} & \text{if } q, q_1, \ldots, q_k \in P \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$\kappa_0(\sigma_0)_q = \begin{cases} \mu_0(\sigma_0)_q \cdot \nu_0(\sigma_0)G & \text{if } q \in Q \\ \nu_0(\sigma_0)_q & \text{if } q \in P. \end{cases}$$

It is known that $\|M \cdot_{\sigma_0} N\| = \|M\| \cdot_{\sigma_0} \|N\|$. Let $M \to_X M'$ and $N \to_Y N'$. We define $X \cdot_{\sigma_0} Y = X + Y$. The Hadamard product $M \cdot_{\mathrm{H}} N$ is the wta $(\Sigma, Q \times P, \kappa, H)$ where $H = F \otimes G$ and $\kappa_k(\sigma) = \mu_k(\sigma) \otimes \nu_k(\sigma)$ for all $\sigma \in \Sigma_k$. If $M \to_X M'$ and $N \to_Y N'$, then we define $X \cdot_{\mathrm{H}} Y = X \otimes Y$. Finally, let $\mathcal{A}$ be complete. Thus,

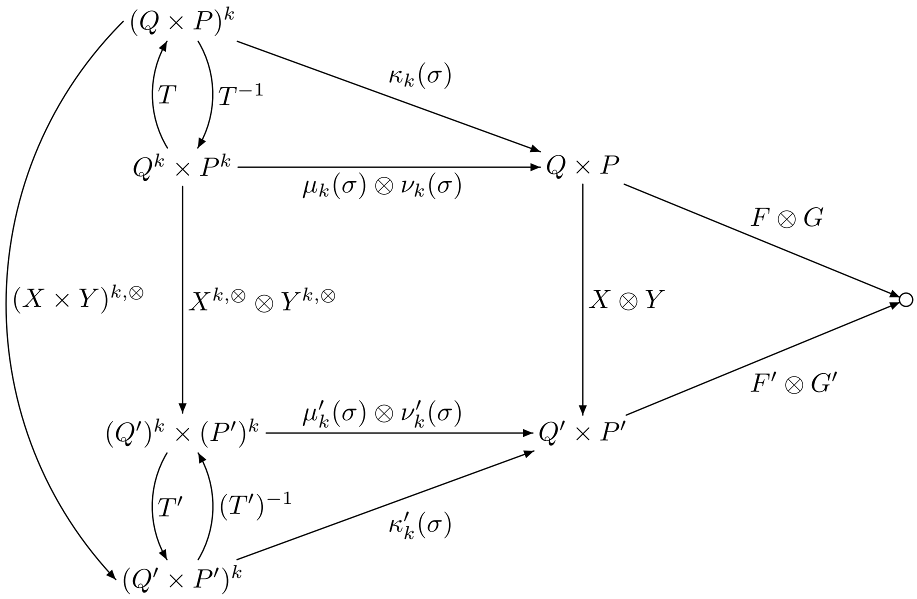

Fig. 3. Commutative diagram used in the proof of Proposition 10.

$\mathcal{A}$ allows the definition of the star operation $a^* = \sum_{n \in \mathbb{N}} a^n$ for every $a \in A$. The σ_0-iteration $M^{*\sigma_0}$ of M is the wta (Σ, Q, κ, F) where

$$\kappa_k(\sigma)_{q_1 \cdots q_k, q} = \mu_k(\sigma)_{q_1 \cdots q_k, q} + \big(\|M\|(\sigma_0)\big)^* \cdot \big(\mu_k(\sigma) F\big)_{q_1 \cdots q_k}$$

for all $\sigma \in \Sigma_k \setminus \{\sigma_0\}$ and $\kappa_0(\sigma_0) = \mu_0(\sigma_0)$. If $M \to_X M'$, then we define $X^{*\sigma_0} = X$.

Proposition 10. *The functions $\cdot_{\sigma_0}$ and $\cdot_{\mathrm{H}}$, which are defined on wta and transfer matrices, are functors $\boldsymbol{Sim}_\Sigma^2 \to \boldsymbol{Sim}_\Sigma$. Moreover, σ_0-iteration is a functor $\boldsymbol{Sim}_\Sigma \to \boldsymbol{Sim}_\Sigma$ if $\mathcal{A}$ is complete.*

Proof. We only prove the first two statements. For the σ_0-product, we claim that

$$\big(\kappa_k(\sigma) \cdot (X \oplus Y)\big)_{q_1 \cdots q_k, q} = \big((X \oplus Y)^{k,\otimes} \cdot \kappa'_k(\sigma)\big)_{q_1 \cdots q_k, q} \tag{15}$$

for every $\sigma \in \Sigma_k$ and $q, q_1, \ldots, q_k \in Q' \cup P'$. We distinguish 4 cases:

- First, suppose that $q, q_1, \ldots, q_k \in Q'$. Then both sides of (15) are equal to $(\mu_k(\sigma) X)_{q_1 \cdots q_k, q} = (X^{k,\otimes} \cdot \mu'_k(\sigma))_{q_1 \cdots q_k, q}$.
- Similarly, in case $q, q_1, \ldots, q_k \in P'$, we have that both sides of (15) are equal to $(\nu_k(\sigma) Y)_{q_1 \cdots q_k, q} = (Y^{k,\otimes} \cdot \nu'_k(\sigma))_{q_1 \cdots q_k, q}$.
- If $q \in Q'$ and $q_1, \ldots, q_k \in P'$, then

$$\begin{aligned} \big((X \oplus Y)^{k,\otimes} \cdot \kappa'_k(\sigma)\big)_{q_1 \cdots q_k, q} &= \mu'_0(\sigma_0)_q \cdot \big(Y^{k,\otimes} \cdot \nu'_k(\sigma) \cdot G'\big)_{q_1 \cdots q_k} \\ = \mu'_0(\sigma_0)_q \cdot \big(\nu_k(\sigma) \cdot Y \cdot G'\big)_{q_1 \cdots q_k} &= \mu'_0(\sigma_0)_q \cdot \big(\nu_k(\sigma) \cdot G\big)_{q_1 \cdots q_k} \end{aligned}$$

$$= \big(\mu_0(\sigma_0)X\big)_q \cdot \big(\nu_k(\sigma) \cdot G\big)_{q_1\cdots q_k} = \big(\kappa_k(\sigma) \cdot (X \oplus Y)\big)_{q_1\cdots q_k, q}$$

- Finally, in all remaining cases, both sides of (15) are 0.

This establishes (15). Next, we observe that $(X \oplus Y) \cdot \langle F, 0\rangle = \langle XF, 0\rangle = \langle F', 0\rangle$. Finally, the functorial property follows again from the interchange rule (see proof of Proposition 9).

The proof for the HADAMARD product is indicated in the commutative diagram of Fig. 3. It can be proved to be correct using the simulation properties, the interchange rule, and the properties of the permutation matrices T and T'. □

5. Joint reduction

In this section, we will establish equivalence results using an improved version of the approach called *joint reduction* in [4]. Let $V \subseteq A^I$ be a set of vectors for a finite set I. The $\mathcal{A}$-semimodule generated by V is denoted by $\langle V\rangle$. Given two wta $M = (\Sigma, Q, \mu, F)$ and $N = (\Sigma, P, \nu, G)$ with $Q \cap P = \emptyset$, we first compute $M \oplus N = (\Sigma, Q \cup P, \mu', F')$ as defined in Section 4. The aim is to compute a finite set $V \subseteq A^{Q\cup P}$ such that

(i) $(v_1 \otimes \cdots \otimes v_k) \cdot \mu'_k(\sigma) \in \langle V\rangle$ for every $\sigma \in \Sigma_k$ and $v_1, \ldots, v_k \in V$, and
(ii) $v_1F = v_2G$ for every $(v_1, v_2) \in V$ such that $v_1 \in A^Q$ and $v_2 \in A^P$.

With such a finite set V we can now construct a wta $M' = (\Sigma, V, \nu', G')$ with $G'_v = vF'$ for every $v \in V$ and $\sum_{v\in V} \nu'_k(\sigma)_{v_1\cdots v_k, v} \cdot v = (v_1 \otimes \cdots \otimes v_k) \cdot \mu'_k(\sigma)$ for every $\sigma \in \Sigma_k$ and $v_1, \ldots, v_k \in V$. It remains to prove that M' simulates $M \oplus N$. To this end, let $X = (v)_{v\in V}$ where each $v \in V$ is a row vector. Then for every $\sigma \in \Sigma_k$, $v_1, \ldots, v_k \in V$, and $q \in Q \cup P$, we have

$$\begin{aligned}
(\nu'_k(\sigma)X)_{v_1\cdots v_k, q} &= \sum_{v\in V} \nu'_k(\sigma)_{v_1\cdots v_k, v} \cdot v_q = \Big(\sum_{v\in V} \nu'_k(\sigma)_{v_1\cdots v_k, v} \cdot v\Big)_q \\
&= \big((v_1 \otimes \cdots \otimes v_k) \cdot \mu'_k(\sigma)\big)_q = \sum_{q_1,\ldots,q_k \in Q\cup P} (v_1)_{q_1} \cdot \ldots \cdot (v_k)_{q_k} \cdot \mu'_k(\sigma)_{q_1\cdots q_k, q} \\
&= \big(X^{k,\otimes} \cdot \mu'_k(\sigma)\big)_{v_1\cdots v_k, q} \enspace .
\end{aligned}$$

Moreover, if we let X_1 and X_2 be the restrictions of X to the entries of Q and P, respectively, then we have $\nu'_k(\sigma)X_1 = X_1^{k,\otimes} \cdot \mu_k(\sigma)$ and $\nu'_k(\sigma)X_2 = X_2^{k,\otimes} \cdot \nu_k(\sigma)$. In addition, $G'_v = vF' = \sum_{q\in Q\cup P} v_q F'_q = (XF')_v$ for every $v \in V$, which proves that $M' \to_X (M \oplus N)$. Since $v_1F = v_2G$ for every $(v_1, v_2) \in V$, we have $G'_{(v_1,v_2)} = (v_1, v_2)F' = v_1F + v_2G = (1 + 1)v_1F = (1 + 1)v_2G$. Now, let $G''_{(v_1,v_2)} = v_1F = v_2G$ for every $(v_1, v_2) \in V$. Then

$$(X_2G)_v = \sum_{p\in P} v_pG_p = v_2G = G''_v = v_1F = \sum_{q\in Q} v_qF_q = (X_1F)_v$$

for every $v = (v_1, v_2) \in V$. Consequently, $M'' \to_{X_1} M$ and $M'' \to_{X_2} N$, where $M'' = (\Sigma, V, \nu', G'')$. This proves the next theorem.

Theorem 11. *Let M and N be equivalent. If there exists a finite set $V \subseteq A^{Q \cup P}$ with properties (i) and (ii), then a finite chain of simulations joins M and N. In fact, there exists a single wta that simulates both M and N.*

Let us first recall a known result [2] for fields. Note that, in comparison to our results, the single wta can be chosen to be a minimal wta.

Theorem 12 (see p. 453 of [2]) *Every two equivalent trim wta M and N over a field $\mathcal{A}$ can be joined by a finite chain of simulations. Moreover, there exists a minimal wta that simulates both M and N.*

We can obtain a similar theorem with the help of Theorem 11 as follows. Let $\mathcal{A}$ be a NOETHERIAN semiring. Let $V_0 = \{\mu'_0(\alpha) \mid \alpha \in \Sigma_0\}$ and

$$V_{i+1} = V_i \cup \big(\{(v_1 \otimes \cdots \otimes v_k) \cdot \mu'_k(\sigma) \mid \sigma \in \Sigma_k, v_1, \ldots, v_k \in V_i\} \setminus \langle V_i \rangle\big)$$

for every $i \in \mathbb{N}$. Then $\{0\} \subseteq \langle V_0 \rangle \subseteq \langle V_1 \rangle \subseteq \cdots \subseteq \langle V_k \rangle \subseteq \cdots$ is stationary after finitely many steps because $\mathcal{A}$ is NOETHERIAN. Thus, let $V = V_k$ for some $k \in \mathbb{N}$ such that $\langle V_k \rangle = \langle V_{k+1} \rangle$. Clearly, V is finite and has property (i). Trivially, $V \subseteq \{h_{\mu'}(t) \mid t \in T_\Sigma\}$, so let $v \in V$ be such that $v = \sum_{i \in I}(h_\mu(t_i), h_\nu(t_i))$ for some finite index set I and $t_i \in T_\Sigma$ for every $i \in I$. Then

$$\Big(\sum_{i \in I} h_\mu(t_i)\Big) F = \sum_{i \in I}(\|M\|, t_i) = \sum_{i \in I}(\|N\|, t_i) = \Big(\sum_{i \in I} h_\nu(t_i)\Big) G$$

because $\|M\| = \|N\|$, which proves property (ii).

In fact, since $M \oplus N$ uses only finitely many semiring coefficients, it is sufficient that every finitely generated subsemiring of $\mathcal{A}$ is contained in a NOETHERIAN subsemiring of $\mathcal{A}$. Then the following theorem follows from Theorem 11.

Theorem 13. *Let $\mathcal{A}$ be such that every finitely generated subsemiring is contained in a* NOETHERIAN *subsemiring of $\mathcal{A}$. For all equivalent wta M and N over $\mathcal{A}$, there exists a finite chain of simulations that join M and N. In fact, there exists a single wta that simulates both M and N.*

Note that $\mathbb{Z}$ is a NOETHERIAN ring. More generally, every finitely generated commutative ring is NOETHERIAN (see Cor. IV.2.4 and Prop. X.1.4 of [27]).

Corollary 14 (of Theorem 13) *For all equivalent wta M and N over a commutative ring $\mathcal{A}$, there exists a finite chain of simulations that join M and N. In fact, there exists a single wta that simulates both M and N.*

Finally, let $\mathcal{A} = \mathbb{N}$ be the semiring of natural numbers. We compute the finite set $V \subseteq \mathbb{N}^{Q \cup P}$ as follows:

(1) Let $V_0 = \{\mu'_0(\alpha) \mid \alpha \in \Sigma_0\}$ and $i = 0$.
(2) For every $v, v' \in V_i$ such that $v \leq v'$, replace v' by $v' - v$.
(3) Set $V_{i+1} = V_i \cup \big(\{(v_1 \otimes \cdots \otimes v_k) \cdot \mu'_k(\sigma) \mid \sigma \in \Sigma_k, v_1, \ldots, v_k \in V_i\} \setminus \langle V_i \rangle\big)$.
(4) Until $V_{i+1} = V_i$, increase i and repeat step 2.

Clearly, this algorithm terminates since every vector can only be replaced by a smaller vector in step 2 and step 3 only adds a finite number of vectors, which after the reduction in step 2 are pairwise incomparable. Moreover, property (i) trivially holds because at termination $V_{i+1} = V_i$ after step 3. Consequently, we only need to prove property (ii). To this end, we first prove that $V \subseteq \langle \{h_{\mu'}(t) \mid t \in T_\Sigma\}\rangle_{+,-}$. This is trivially true after step 1 because $\mu'_0(\alpha) = h_{\mu'}(\alpha)$ for every $\alpha \in \Sigma_0$. Clearly, the property is preserved in steps 2 and 3. Finally, property (ii) can now be proved as follows. Let $v \in V$ be such that $v = \sum_{i\in I_1}(h_\mu(t_i), h_\nu(t_i)) - \sum_{i\in I_2}(h_\mu(t_i), h_\nu(t_i))$ for some finite index sets I_1 and I_2 and $t_i \in T_\Sigma$ for every $i \in I_1 \cup I_2$. Then by $\|M\| = \|N\|$ we obtain

$$\begin{aligned}
&\Big(\sum_{i\in I_1} h_\mu(t_i) - \sum_{i\in I_2} h_\mu(t_i)\Big)F = \sum_{i\in I_1} h_\mu(t_i)F - \sum_{i\in I_2} h_\mu(t_i)F \\
&= \sum_{i\in I_1}(\|M\|, t_i) - \sum_{i\in I_2}(\|M\|, t_i) = \sum_{i\in I_1}(\|N\|, t_i) - \sum_{i\in I_2}(\|N\|, t_i) \\
&= \sum_{i\in I_1} h_\nu(t_i)G - \sum_{i\in I_2} h_\nu(t_i)G = \Big(\sum_{i\in I_1} h_\nu(t_i) - \sum_{i\in I_2} h_\nu(t_i)\Big)G \enspace .
\end{aligned}$$

Corollary 15 (of Theorem 11) *For all equivalent wta M and N over $\mathbb{N}$, there exists a finite chain of simulations that join M and N. In fact, there exists a single wta that simulates both M and N.*

For all finitely and effectively presented semirings, Theorems 12 and 13 and Corollaries 14 and 15 also yield decidability of equivalence for M and N. Essentially, we run the trivial semi-decidability test for inequality and a search for the wta that simulates both M and N in parallel. We know that either test will eventually return, thus deciding whether M and N are equivalent. Conversely, if equivalence is undecidable, then simulation cannot capture equivalence [18].

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