Proof Appendix of “The Substitution Vanishes”

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Abstract. This is the proof appendix of “The Substitution Vanishes”. It contains the proofs of Lemma 13 (see Lemma 3), Theorem 14 (see Theorem 4), Lemma 20 (see Lemma 8), and Theorem 21 (see Theorem 9).

Appendix

Lemma 1. Let $M = (F, \{sub^{(n+1)}\}, C, \Pi, R_1, R_2, r_{in})$ be an n-sntt.

$$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ e_0\ \dots\ e_n)) = \underline{tree}(e_0)[\Pi_i/\underline{tree}(e_i)]$$

for every $e_0, \dots, e_n \in E_{C,n}$ such that if there is a subtree of the shape $e'_0[z_i \rightsquigarrow e'_i]$ for some $e'_0 \in E_{C,n}(Z_n)$ and $e'_1, \dots, e'_n \in E_{C,n}$ in e_0, then e'_0 does not contain any constructor of Π.

Proof. We just write $nf(r)$ when we mean $nf(\Rightarrow_{R_1 \cup R_2}, r)$.
(i) First let $e_0 = \Pi_j$ for some $j \in [n]$.

$$\begin{aligned} & \underline{tree}(nf(sub\ \Pi_j\ e_1\ \dots\ e_n)) \\ = \quad & \text{(by reduction and because } e_j \text{ is in normal form)} \\ & \underline{tree}(e_j) \\ = \quad & \text{(by definition of } \underline{tree} \text{ and substitution)} \\ & \underline{tree}(\Pi_j)[\Pi_i/\underline{tree}(e_i)] \end{aligned}$$

(ii) Let $e_0 = c\ e'_1\ \dots\ e'_m$ for some $m \in \mathbb{N}$, $c \in (C - \Pi)^{(m)}$, and $e'_1, \dots, e'_m \in E_{C,n}$.

$$\begin{aligned} & \underline{tree}(nf(sub\ (c\ e'_1\ \dots\ e'_m)\ e_1\ \dots\ e_n)) \\ = \quad & \text{(by reduction)} \\ & \underline{tree}(nf((c\ (sub\ e'_1\ z_1\ \dots\ z_n)\ \dots\ (sub\ e'_m\ z_1\ \dots\ z_n))[z_i \rightsquigarrow e_i])) \\ = \quad & \text{(by normal form and definition of } \underline{tree} \text{ because } e_i \text{ is in normal form)} \\ & (c\ \underline{tree}(nf(sub\ e'_1\ z_1\ \dots\ z_n))\ \dots \\ & \quad \underline{tree}(nf(sub\ e'_m\ z_1\ \dots\ z_n)))[z_i/\underline{tree}(e_i)] \\ = \quad & \text{(by induction hypothesis [} m \text{ times])} \end{aligned}$$

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$$
\begin{aligned}
& (c\ \underline{tree}(e'_1)[\Pi_i/\underline{tree}(z_i)]\ \ldots\ \underline{tree}(e'_m)[\Pi_i/\underline{tree}(z_i)])[z_i/\underline{tree}(e_i)] \\
= \quad & \text{(by substitution and definition of } \underline{tree}\text{)} \\
& \underline{tree}(c\ e'_1\ \ldots\ e'_m)[\Pi_i/z_i][z_i/\underline{tree}(e_i)] \\
= \quad & \text{(by substitution because } \underline{tree}(c\ e'_1\ \ldots\ e'_m) \in T_C\text{)} \\
& \underline{tree}(c\ e'_1\ \ldots\ e'_m)[\Pi_i/\underline{tree}(e_i)]
\end{aligned}
$$

(iii) Finally, let $e_0 = e'_0[z_i \rightsquigarrow e'_i]$ for some $e'_0 \in E_{C,n}(Z_n)$ and $e'_1, \ldots, e'_n \in E_{C,n}$.

$$
\begin{aligned}
& \underline{tree}(nf(sub\ e'_0[z_i \rightsquigarrow e'_i]\ e_1\ \ldots\ e_n)) \\
= \quad & \text{(by reduction)} \\
& \underline{tree}(nf(e'_0[z_i \rightsquigarrow sub\ e'_i\ z_1\ \ldots\ z_n]\,[z_i \rightsquigarrow e_i])) \\
= \quad & \text{(by normal form because } e_i \text{ and } e'_0 \text{ are in normal form)} \\
& \underline{tree}(e'_0[z_i \rightsquigarrow nf(sub\ e'_i\ z_1\ \ldots\ z_n)]\,[z_i \rightsquigarrow e_i]) \\
= \quad & \text{(by definition of } \underline{tree}\text{)} \\
& \underline{tree}(e'_0)[z_i/\underline{tree}(nf(sub\ e'_i\ z_1\ \ldots\ z_n))]\,[z_i/\underline{tree}(e_i)] \\
= \quad & \text{(by induction hypothesis } [n \text{ times}]\text{)} \\
& \underline{tree}(e'_0)[z_i/\underline{tree}(e'_i)[\Pi_i/\underline{tree}(z_i)]\,]\,[z_i/\underline{tree}(e_i)] \\
= \quad & \text{(by associativity of substitution} \\
& \quad \text{because } \underline{tree}(e'_0) \text{ does not contain constructors of } \Pi\text{)} \\
& \underline{tree}(e'_0)[z_i/\underline{tree}(e'_i)]\,[\Pi_i/\underline{tree}(z_i)]\,[z_i/\underline{tree}(e_i)] \\
= \quad & \text{(by definition of } \underline{tree}\text{)} \\
& \underline{tree}(e'_0[z_i \rightsquigarrow e'_i])[\Pi_i/z_i]\,[z_i/\underline{tree}(e_i)] \\
= \quad & \text{(by substitution because } \underline{tree}(e'_0[z_i \rightsquigarrow e'_i]) \in T_C\text{)} \\
& \underline{tree}(e'_0[z_i \rightsquigarrow e'_i])[\Pi_i/\underline{tree}(e_i)]
\end{aligned}
$$

This completes the proof of the lemma. □

Lemma 2. Let $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$ with $Sub = \{sub^{(n+1)}\}$ be an n-sntt.

$$
\begin{aligned}
& \underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, (sub\ r_0\ \ldots\ r_n)[x_i/t_i])) \\
= \ & \underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_0[x_i/t_i]))[\Pi_j/\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_j[x_i/t_i]))]
\end{aligned}
$$

for every $m \in \mathbb{N}$, $t_1, \ldots, t_m \in T_C$, and $r_0, \ldots, r_n \in RHS(F, Sub, C, X_m)$.

Proof. We just write $nf(r)$ when we mean $nf(\Rightarrow_{R_1 \cup R_2}, r)$.

$$
\begin{aligned}
& \underline{tree}(nf((sub\ r_0\ \ldots\ r_n)[x_i/t_i])) \\
= \quad & \text{(by substitution)} \\
& \underline{tree}(nf(sub\ r_0[x_i/t_i]\ \ldots\ r_n[x_i/t_i])) \\
= \quad & \text{(by normal form)} \\
& \underline{tree}(nf(sub\ nf(r_0[x_i/t_i])\ \ldots\ nf(r_n[x_i/t_i])))
\end{aligned}
$$

Let $e_i = nf(r_i[x_i/t_i])$ for every $0 \leq i \leq n$. Clearly, $e_i \in E_{C,n}$. Note that whenever there is a subtree of the shape $e'_0[z_i \rightsquigarrow e'_i]$ for some $e'_0 \in E_{C,n}(Z_n)$ and $e'_1, \ldots, e'_n \in E_{C,n}$ in e_0, then e'_0 does not contain any constructor of Π. This property can easily be observed by considering the rules for *sub*. It remains to prove

$$\underline{tree}(nf(sub\ e_0\ \ldots\ e_n)) = \underline{tree}(e_0)[\Pi_i/\underline{tree}(e_i)]\ ,$$

which holds by Lemma 1. □

Lemma 3 (see Lemma 13). Let $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$ with $Sub = \{sub^{(n+1)}\}$ be an n-sntt, and let $acc(M) = (acc(F), C, acc(R_1), acc(r_{in}))$ be the n-satt constructed from M by accumulation.

(a) For every $f \in F$ and $t \in T_C$:

$$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ (f\ t)\ z_1\ \ldots\ z_n)) = \underline{tree}(nf(\Rightarrow_{acc(R_1)}, f\ t\ z_1\ \ldots\ z_n))\ .$$

(b) For every $k \in \mathbb{N}$, $r \in RHS(F, Sub, C, X_k)$, and $t_1, \ldots, t_k \in T_C$:

$$\begin{aligned} &\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ r[x_i/t_i]\ z_1\ \ldots\ z_n)) \\ = &\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r\ z_1\ \ldots\ z_n)[x_i/t_i]))\ . \end{aligned}$$

Proof. We prove the two statements by simultaneous induction. Statement (a) is proved by structural induction on t, so let $t = c\ t_1\ \ldots\ t_k$ for some $k \in \mathbb{N}$, $c \in C^{(k)}$, and $t_1, \ldots, t_k \in T_C$.

$$\begin{aligned} &\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ (f\ (c\, t_1\ \ldots\ t_k))\ z_1\ \ldots\ z_n)) \\ = &\quad \text{(by reduction)} \\ &\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ rhs_{R_1,f,c}[x_i/t_i]\ z_1\ \ldots\ z_n)) \\ = &\quad \text{(by Statement (b))} \\ &\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ rhs_{R_1,f,c}\ z_1\ \ldots\ z_n)[x_i/t_i])) \\ = &\quad \text{(by substitution)} \\ &\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ rhs_{R_1,f,c}\ y_1\ \ldots\ y_n)[x_i/t_i]\,[y_i/z_i])) \\ = &\quad \text{(by definition of } acc(R_1) \text{ and reduction)} \\ &\underline{tree}(nf(\Rightarrow_{acc(R_1)}, f\ (c\, t_1\ \ldots\ t_k)\ z_1\ \ldots\ z_n)) \end{aligned}$$

Statement (b) is proved by induction on r.

(i) First let $r = \Pi_j$ for some $j \in [n]$.

$$\begin{aligned} &\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ \Pi_j[x_i/t_i]\ z_1\ \ldots\ z_n)) \\ = &\quad \text{(by substitution and reduction)} \\ &\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, z_j)) \\ = &\quad \text{(by } nf(\Rightarrow_{R_1 \cup R_2}, \cdot)\text{)} \end{aligned}$$

$\underline{tree}(z_j)$

$= \quad$ (by $nf(\Rightarrow_{acc(R_1)}, \cdot)$ and substitution)

$\underline{tree}(nf(\Rightarrow_{acc(R_1)}, z_j\,[x_i/t_i]))$

$= \quad$ (by reduction)

$\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ \Pi_j\ z_1\ \ldots\ z_n)[x_i/t_i]))$

(ii) Now let $r = f\ x_j$ for some $f \in F$ and $j \in [k]$.

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ (f\ x_j)[x_i/t_i]\ z_1\ \ldots\ z_n))$

$= \quad$ (by substitution)

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ (f\ t_j)\ z_1\ \ldots\ z_n))$

$= \quad$ (by Statement (a))

$\underline{tree}(nf(\Rightarrow_{acc(R_1)}, f\ t_j\ z_1\ \ldots\ z_n))$

$= \quad$ (by substitution)

$\underline{tree}(nf(\Rightarrow_{acc(R_1)}, (f\ x_j\ z_1\ \ldots\ z_n)[x_i/t_i]))$

$= \quad$ (by reduction)

$\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ (f\ x_j)\ z_1\ \ldots\ z_n)[x_i/t_i]))$

(iii) Let $r = c\ r_1\ \ldots\ r_m$ for some $m \in \mathbb{N}$, $c \in (C - \Pi)^{(m)}$, and $r_1, \ldots, r_m \in RHS(F, Sub, C, X_k)$.

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ (c\, r_1\ \ldots\ r_m)[x_i/t_i]\ z_1\ \ldots\ z_n))$

$= \quad$ (by substitution)

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ (c\, r_1[x_i/t_i]\ \ldots\ r_m[x_i/t_i])\ z_1\ \ldots\ z_n))$

$= \quad$ (by reduction)

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, (c\ (sub\ r_1[x_i/t_i]\ z_1\ \ldots\ z_n)\ \ldots$
$\qquad (sub\ r_m[x_i/t_i]\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow z_i]))$

$= \quad$ (by $nf(\Rightarrow_{R_1 \cup R_2}, \cdot)$, definition of $\underline{tree}$, and substitution)

$c\ \underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ r_1[x_i/t_i]\ z_1\ \ldots\ z_n))\ \ldots$
$\ \underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ r_m[x_i/t_i]\ z_1\ \ldots\ z_n))$

$= \quad$ (by induction hypothesis [m times])

$c\ \underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_1\ z_1\ \ldots\ z_n)[x_i/t_i]))\ \ldots$
$\ \underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_m\ z_1\ \ldots\ z_n)[x_i/t_i]))$

$= \quad$ (by $nf(\Rightarrow_{acc(R_1)}, \cdot)$, $nf(\Rightarrow_{\overline{R_2}}, \cdot)$, definition of $\underline{tree}$, and substitution)

$\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}},$
$\qquad c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i]))$

$= \quad$ (by substitution)

$\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}},$
$\qquad c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))\,[x_i/t_i]))[z_i/z_i]$

$= \quad$ (by definition of $\underline{tree}$ and $nf(\Rightarrow_{acc(R_1)}, \cdot)$)

$\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}},$
$c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))\,[x_i/t_i]\,[z_i \rightsquigarrow z_i]))$

$= \quad$ (by substitution and $nf(\Rightarrow_{\overline{R_2}}, \cdot)$)

$\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}},$
$(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow z_i])\,[x_i/t_i]))$

$= \quad$ (by reduction)

$\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ (c\ r_1\ \ldots\ r_m)\ z_1\ \ldots\ z_n)[x_i/t_i]))$

(iv) Finally, let $r = sub\ r_0\ \ldots\ r_n$ for some $r_0, \ldots, r_n \in RHS(F, Sub, C, X_k)$.

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ (sub\ r_0\ \ldots\ r_n)[x_i/t_i]\ z_1\ \ldots\ z_n))$

$= \quad$ (by substitution)

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ (sub\ r_0[x_i/t_i]\ \ldots\ r_n[x_i/t_i])\ z_1\ \ldots\ z_n))$

$= \quad$ (by Lemma 2)

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ r_0[x_i/t_i]\ \ldots\ r_n[x_i/t_i]))[\Pi_i/\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, z_i))]$

$= \quad$ (by Lemma 2)

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_0[x_i/t_i]))$
$[\Pi_i/\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_i[x_i/t_i]))\,]\,[\Pi_i/\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, z_i))]$

$= \quad$ (by associativity of substitution)

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_0[x_i/t_i]))$
$[\Pi_i/\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_i[x_i/t_i]))[\Pi_j/\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, z_j))]\,]$

$= \quad$ (by Lemma 2 [n times])

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_0[x_i/t_i]))$
$[\Pi_i/\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ r_i[x_i/t_i]\ z_1\ \ldots\ z_n))]$

$= \quad$ (by induction hypothesis [n times])

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_0[x_i/t_i]))$
$[\Pi_i/\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_i\ z_1\ \ldots\ z_n)[x_i/t_i]))]$

$= \quad$ (by substitution because $\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_0[x_i/t_i])) \in T_C$)

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_0[x_i/t_i]))[\Pi_i/z_i]$
$[z_i/\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_i\ z_1\ \ldots\ z_n)[x_i/t_i]))]$

$= \quad$ (by $nf(\Rightarrow_{R_1 \cup R_2}, \cdot)$ and definition of $\underline{tree}$)

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, r_0[x_i/t_i]))[\Pi_i/\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, z_i))]$
$[z_i/\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_i\ z_1\ \ldots\ z_n)[x_i/t_i]))]$

$= \quad$ (by Lemma 2)

$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub\ r_0[x_i/t_i]\ z_1\ \ldots\ z_n))$
$[z_i/\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_i\ z_1\ \ldots\ z_n)[x_i/t_i]))]$

$= \quad$ (by induction hypothesis)

$$\begin{aligned}
&\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\ z_1\ \dots\ z_n)[x_i/t_i]))\\
&\qquad [z_i/\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_i\ z_1\ \dots\ z_n)[x_i/t_i]))]\\
= &\quad \text{(since } \underline{tree} \text{ distributes over substitutions)}\\
&\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\ z_1\ \dots\ z_n)[x_i/t_i])\\
&\qquad [z_i/nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_i\ z_1\ \dots\ z_n)[x_i/t_i])])\\
= &\quad \text{(since } nf(\Rightarrow_{acc(R_1)}, \cdot) \text{ distributes over substitutions)}\\
&\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\ z_1\ \dots\ z_n)[x_i/t_i]\\
&\qquad [z_i/nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_i\ z_1\ \dots\ z_n)[x_i/t_i]\,]))\\
= &\quad \text{(by } nf(\Rightarrow_{\overline{R_2}}, \cdot) \text{ and substitution)}\\
&\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\\
&\qquad (\underline{sub}\ r_1\ z_1\ \dots\ z_n)\ \dots\ (\underline{sub}\ r_n\ z_1\ \dots\ z_n))[x_i/t_i]))\\
= &\quad \text{(by substitution)}\\
&\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\\
&\qquad (\underline{sub}\ r_1\ z_1\ \dots\ z_n)\ \dots\ (\underline{sub}\ r_n\ z_1\ \dots\ z_n))[x_i/t_i]))[z_i/z_i]\\
= &\quad \text{(by definition of } \underline{tree},\ nf(\Rightarrow_{acc(R_1)}, \cdot), \text{ and substitution)}\\
&\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\\
&\qquad (\underline{sub}\ r_1\ z_1\ \dots\ z_n)\ \dots\ (\underline{sub}\ r_n\ z_1\ \dots\ z_n))[z_i \rightsquigarrow z_i]\,[x_i/t_i]))\\
= &\quad \text{(by } nf(\Rightarrow_{\overline{R_2}}, \cdot)\,)\\
&\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, (\underline{sub}\ r_0\\
&\qquad (\underline{sub}\ r_1\ z_1\ \dots\ z_n)\ \cdots\ (\underline{sub}\ r_n\ z_1\ \dots\ z_n))[z_i \rightsquigarrow z_i])[x_i/t_i]))\\
= &\quad \text{(by reduction)}\\
&\underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ (sub\ r_0\ \dots\ r_n)\ z_1\ \dots\ z_n)[x_i/t_i]))
\end{aligned}$$

This completes the proof. □

Theorem 4 (see Theorem 14). Let M be an n-sntt. Then, $\tau_M = \tau_{acc(M)}$.

Proof. Let $M = (F, \{sub^{(n+1)}\}, C, \Pi, R_1, R_2, r_{in})$ be an n-sntt and $acc(M) = (acc(F), C, acc(R_1), acc(r_{in}))$. Then, for every $t \in T_C$:

$$\begin{aligned}
&\tau_M(t)\\
&= \underline{tree}(nf(\Rightarrow_{R_1\cup R_2}, r_{in}[x_1/t])) && \text{(Def. } \tau_M)\\
&= \underline{tree}(nf(\Rightarrow_{R_1\cup R_2}, sub\ nf(\Rightarrow_{R_1\cup R_2}, r_{in}[x_1/t])\ \Pi_1 \dots \Pi_n)) && (*)\\
&= \underline{tree}(nf(\Rightarrow_{R_1\cup R_2}, sub\ nf(\Rightarrow_{R_1\cup R_2}, r_{in}[x_1/t])\ z_1 \dots z_n))[z_i/\Pi_i]\\
&= \underline{tree}(nf(\Rightarrow_{R_1\cup R_2}, sub\ r_{in}[x_1/t]\ z_1 \dots z_n))[z_i/\Pi_i] && \text{(confluence)}\\
&= \underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_{in}\ z_1 \dots z_n)[x_1/t]))[z_i/\Pi_i] && \text{(Lm. 3(b))}\\
&= \underline{tree}(nf(\Rightarrow_{acc(R_1)}, nf(\Rightarrow_{\overline{R_2}}, sub\ r_{in}\ \Pi_1 \dots \Pi_n)[x_1/t]))\\
&= \underline{tree}(nf(\Rightarrow_{acc(R_1)}, acc(r_{in})[x_1/t])) && \text{(Def. } acc(r_{in}))\\
&= \tau_{acc(M)}(t) && \text{(Def. } \tau_{acc(M)})
\end{aligned}$$

The statement $(*)$ holds, because every occurrence of a Π_j is substituted by Π_j. $(*)$ can be formally proved by induction on expressions $e_0 \in E_{C,n}$ such that if there is a subtree of the shape $e'_0[z_i \rightsquigarrow e'_i]$ for some $e'_0 \in E_{C,n}(Z_n)$ and $e'_1, \ldots, e'_n \in E_{C,n}$ in e_0, then e'_0 does not contain any constructor of Π. Note that $(*)$ does not hold without the outer application of $\underline{tree}$. $\square$

Lemma 5. Let C be a ranked alphabet, and let $e_1, e_2 \in E_{C,n}(Z_n)$ and $z \in Z_n$. If $\underline{tree}(e_1) = \underline{tree}(e_2)$, then $\underline{rel}(z, e_1) = \underline{rel}(z, e_2)$.

Proof. The proof is trivial and hence omitted. $\square$

Lemma 6. Let $M = (F, \{sub^{(n+1)}\}, C, \Pi, R_1, R_2, r_{in})$ be an n-sntt, and let $e_0, \ldots, e_n \in E_{C^\circ,n}$.

$$\begin{aligned}&\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ e_0\ \ldots\ e_n))\\ =\ &\underline{step}(e_0) + \textstyle\sum_{i=1}^{n} \underline{rel}(z_i, e_0[\Pi_l/z_l]) \cdot \underline{step}(e_i)\end{aligned}$$

Proof. For brevity, we abbreviate $nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r)$ by just $nf(r)$. We prove the statement by induction on e_0.

(i) First let $e_0 = \Pi_j$ for some $j \in [n]$.

$$\begin{aligned}&\underline{step}(nf(sub\ \Pi_j\ e_1\ \ldots\ e_n))\\ =\ &\quad \text{(by reduction and normal form because } e_j \text{ is in normal form)}\\ &\underline{step}(e_j)\\ =\ &\quad \text{(by definition of } \underline{step} \text{ and } \underline{rel} \text{ and substitution)}\\ &\underline{step}(\Pi_j) + \textstyle\sum_{i=1}^{n} \underline{rel}(z_i, \Pi_j[\Pi_l/z_l]) \cdot \underline{step}(e_i)\end{aligned}$$

(ii) Now let $e_0 = c\ e'_1\ \ldots\ e'_m$ for some $m \in \mathbb{N}$, $c \in (C - \Pi)^{(m)}$, and $e'_1, \ldots, e'_m \in E_{C^\circ,n}$. Let $a = 1$ if $c = \circ$ and $a = 0$ otherwise.

$$\begin{aligned}&\underline{step}(nf(sub\ (c\ e'_1\ \ldots\ e'_m)\ e_1\ \ldots\ e_n))\\ =\ &\quad \text{(by reduction)}\\ &\underline{step}(nf((c\ (sub\ e'_1\ z_1\ \ldots\ z_n)\ \ldots\ (sub\ e'_m\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow e_i]))\\ =\ &\quad \text{(by normal form because } e_i \text{ is in normal form)}\\ &\underline{step}(nf(c\ (sub\ e'_1\ z_1\ \ldots\ z_n)\ \ldots\ (sub\ e'_m\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow e_i])\\ =\ &\quad \text{(by definition of } \underline{step})\\ &\underline{step}(nf(c\ (sub\ e'_1\ z_1\ \ldots\ z_n)\ \ldots\ (sub\ e'_m\ z_1\ \ldots\ z_n))) +\\ &+ \textstyle\sum_{i=1}^{n} \underline{rel}(z_i, nf(c\ (sub\ e'_1\ z_1\ \ldots\ z_n)\ \ldots\ (sub\ e'_m\ z_1\ \ldots\ z_n))) \cdot \underline{step}(e_i)\\ =\ &\quad \text{(by normal form and definition of } \underline{step})\\ &a + \textstyle\sum_{i=1}^{m} \underline{step}(nf(sub\ e'_i\ z_1\ \ldots\ z_n)) +\\ &\quad + \textstyle\sum_{i=1}^{n} \underline{rel}(z_i, nf(c\ (sub\ e'_1\ z_1\ \ldots\ z_n)\ \ldots\\ &\qquad\qquad (sub\ e'_m\ z_1\ \ldots\ z_n))) \cdot \underline{step}(e_i)\\ =\ &\quad \text{(by Observation 5 because } \underline{tree}(e) = \underline{tree}(e[z_i \rightsquigarrow z_i])\text{)}\end{aligned}$$

$$
\begin{array}{rl}
& a + \sum_{i=1}^{m} \underline{step}(nf(sub\ e_i'\ z_1\ \ldots\ z_n)) + \\
& \quad + \sum_{j=1}^{n} \underline{rel}(z_j, nf((c\ (sub\ e_1'\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad\qquad (sub\ e_m'\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow z_i])) \cdot \underline{step}(e_j) \\
= & \text{(by reduction)} \\
& a + \sum_{i=1}^{m} \underline{step}(nf(sub\ e_i'\ z_1\ \ldots\ z_n)) + \\
& \quad + \sum_{i=1}^{n} \underline{rel}(z_i, nf(sub\ (c\ e_1'\ \ldots\ e_m')\ z_1\ \ldots\ z_n)) \cdot \underline{step}(e_i) \\
= & \text{(by induction hypothesis } [m \text{ times}]) \\
& a + \sum_{j=1}^{m} \Big(\underline{step}(e_j') + \sum_{l=1}^{n} \underline{rel}(z_l, e_j'[\Pi_i/z_i]) \cdot \underline{step}(z_l) \Big) + \\
& \quad + \sum_{i=1}^{n} \underline{rel}(z_i, nf(sub\ (c\ e_1'\ \ldots\ e_m')\ z_1\ \ldots\ z_n)) \cdot \underline{step}(e_i) \\
= & \text{(by definition of } \underline{step} \text{ because } \underline{step}(z_l) = 0) \\
& a + \sum_{i=1}^{m} \underline{step}(e_i') + \\
& \quad + \sum_{i=1}^{n} \underline{rel}(z_i, nf(sub\ (c\ e_1'\ \ldots\ e_m')\ z_1\ \ldots\ z_n)) \cdot \underline{step}(e_i) \\
= & \text{(by Lemma 1, } \underline{tree}(e)[\Pi_i/z_i] = \underline{tree}(e[\Pi_i/z_i]) \text{, and Observation 5)} \\
& a + \sum_{i=1}^{m} \underline{step}(e_i') + \\
& \quad + \sum_{i=1}^{n} \underline{rel}(z_i, (c\ e_1'\ \ldots\ e_m')[\Pi_i/z_i]) \cdot \underline{step}(e_i) \\
= & \text{(by definition of } \underline{step}) \\
& \underline{step}(c\ e_1'\ \ldots\ e_m') + \sum_{i=1}^{n} \underline{rel}(z_i, (c\ e_1'\ \ldots\ e_m')[\Pi_i/z_i]) \cdot \underline{step}(e_i)
\end{array}
$$

(iii) Finally, let $e_0 = e_0'[z_i \rightsquigarrow e_i']$ for some $e_0' \in E_{C^\circ,n}(Z_n)$ and $e_0', \ldots, e_n' \in E_{C^\circ,n}$.

$$
\begin{array}{rl}
& \underline{step}(nf(sub\ e_0'[z_i \rightsquigarrow e_i']\ e_1\ \ldots\ e_n)) \\
= & \text{(by reduction)} \\
& \underline{step}(nf(e_0'[z_i \rightsquigarrow sub\ e_i'\ z_1\ \ldots\ z_n]\,[z_i \rightsquigarrow e_i])) \\
= & \text{(by normal form because } e_i \text{ is in normal form)} \\
& \underline{step}(nf(e_0'[z_i \rightsquigarrow sub\ e_i'\ z_1\ \ldots\ z_n])[z_i \rightsquigarrow e_i]) \\
= & \text{(by definition of } \underline{step}) \\
& \underline{step}(nf(e_0'[z_i \rightsquigarrow sub\ e_i'\ z_1\ \ldots\ z_n])) + \\
& + \sum_{j=1}^{n} \underline{rel}(z_j, nf(e_0'[z_i \rightsquigarrow sub\ e_i'\ z_1\ \ldots\ z_n])) \cdot \underline{step}(e_j) \\
= & \text{(by normal form and definition of } \underline{step} \text{ because } e_0' \text{ is in normal form)} \\
& \underline{step}(e_0') + \sum_{i=1}^{n} \underline{rel}(z_i, e_0') \cdot \underline{step}(nf(sub\ e_i'\ z_1\ \ldots\ z_n)) + \\
& \qquad + \sum_{j=1}^{n} \underline{rel}(z_j, nf(e_0'[z_i \rightsquigarrow sub\ e_i'\ z_1\ \ldots\ z_n])) \cdot \underline{step}(e_j) \\
= & \text{(by Observation 5 because } \underline{tree}(e) = \underline{tree}(e[z_i \rightsquigarrow z_i])\,) \\
& \underline{step}(e_0') + \sum_{i=1}^{n} \underline{rel}(z_i, e_0') \cdot \underline{step}(nf(sub\ e_i'\ z_1\ \ldots\ z_n)) + \\
& \qquad + \sum_{j=1}^{n} \underline{rel}(z_j, nf(e_0'[z_i \rightsquigarrow sub\ e_i'\ z_1\ \ldots\ z_n]\,[z_i \rightsquigarrow z_i])) \cdot \underline{step}(e_j) \\
= & \text{(by reduction)}
\end{array}
$$

$$\underline{step}(e_0') + \sum_{i=1}^{n} \underline{rel}(z_i, e_0') \cdot \underline{step}(nf(sub\ e_i'\ z_1\ \dots\ z_n)) +$$
$$+ \sum_{j=1}^{n} \underline{rel}(z_j, nf(sub\ e_0'[z_i \rightsquigarrow e_i']\ z_1\ \dots\ z_n)) \cdot \underline{step}(e_j)$$
$= \quad$ (by induction hypothesis [n times])
$$\underline{step}(e_0') + \sum_{j=1}^{n} \underline{rel}(z_j, e_0') \cdot$$
$$\cdot \Big(\underline{step}(e_j') + \sum_{l=1}^{n} \underline{rel}(z_l, e_j'[\Pi_i/z_i]) \cdot \underline{step}(z_l)\Big) +$$
$$+ \sum_{j=1}^{n} \underline{rel}(z_j, nf(sub\ e_0'[z_i \rightsquigarrow e_i']\ z_1\ \dots\ z_n)) \cdot \underline{step}(e_j)$$
$= \quad$ (by definition of $\underline{step}$ because $\underline{step}(z_l) = 0$)
$$\underline{step}(e_0') + \sum_{i=1}^{n} \underline{rel}(z_i, e_0') \cdot \underline{step}(e_i') +$$
$$+ \sum_{j=1}^{n} \underline{rel}(z_j, nf(sub\ e_0'[z_i \rightsquigarrow e_i']\ z_1\ \dots\ z_n)) \cdot \underline{step}(e_j)$$
$= \quad$ (by Lemma 1, $\underline{tree}(e)[\Pi_i/z_i] = \underline{tree}(e[\Pi_i/z_i])$, and Observation 5)
$$\underline{step}(e_0') + \sum_{i=1}^{n} \underline{rel}(z_i, e_0') \cdot \underline{step}(e_i') +$$
$$+ \sum_{j=1}^{n} \underline{rel}(z_j, e_0'[z_i \rightsquigarrow e_i']\,[\Pi_i/z_i]) \cdot \underline{step}(e_j)$$
$= \quad$ (by definition of $\underline{step}$)
$$\underline{step}(e_0'[z_i \rightsquigarrow e_i']) + \sum_{j=1}^{n} \underline{rel}(z_j, e_0'[z_i \rightsquigarrow e_i']\,[\Pi_i/z_i]) \cdot \underline{step}(e_j)$$

This ends the proof of the statement. □

Lemma 7. Let $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$ with $Sub = \{sub^{(n+1)}\}$ be an n-sntt, and let $acc(M) = (acc(F), C, acc(R_1), acc(r_{in}))$ be the n-satt constructed from M by accumulation.

(a) Let $t \in T_C$ and $e_1, \dots, e_n \in E_{C^\circ,n}$.

$$\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, f\ t\ e_1\ \dots\ e_n))$$
$$= \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, (f\ t\ z_1\ \dots\ z_n)[z_i \rightsquigarrow e_i]))$$

(b) Let $r \in RHS(F, Sub, C, X_k)$, $t_1, \dots, t_k \in T_C$, and $e_1, \dots, e_n \in E_{C^\circ,n}$ for some $k \in \mathbb{N}$.

$$\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r\ e_1\ \dots\ e_n)[x_i/t_i]))$$
$$= \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r\ z_1\ \dots\ z_n)[x_i/t_i]\,[z_i \rightsquigarrow e_i]))$$

Proof. We prove the two statements by simultaneous induction. We abbreviate $nf(\Rightarrow_{acc(R_1)^\circ}, e)$ simply by $nf_1(e)$ and similarly $nf(\Rightarrow_{\overline{R_2}}, e)$ by $nf_2(e)$. By induction on t we first show Statement (a) with the help of Statement (b). Let $t = c\, t_1\ \dots\ t_k$ for some $k \in \mathbb{N}$, $c \in C^{(k)}$, and $t_1, \dots, t_k \in T_C$.

$$\underline{step}(nf_1(f\ (c\, t_1\ \dots\ t_k)\ e_1\ \dots\ e_n))$$
$= \quad$ (by reduction and definition of $acc(R_1)^\circ$)
$$\underline{step}(nf_1((\circ\ rhs_{acc(R_1),f,c})[x_i/t_i]\,[y_i/e_i]))$$
$= \quad$ (by nf_1, definition of $\underline{step}$ and $acc(R_1)$, and substitution)

$$
\begin{aligned}
& 1 + \underline{step}(nf_1(nf_2(\underline{sub}\ rhs_{R_1,f,c}\ y_1\ \ldots\ y_n)[x_i/t_i]\,[y_i/e_i])) \\
= \quad & \text{(by substitution because } e_i \text{ is in normal form)} \\
& 1 + \underline{step}(nf_1(nf_2(\underline{sub}\ rhs_{R_1,f,c}\ e_1\ \ldots\ e_n)[x_i/t_i])) \\
= \quad & \text{(by Statement (b))} \\
& 1 + \underline{step}(nf_1(nf_2(\underline{sub}\ rhs_{R_1,f,c}\ z_1\ \ldots\ z_n)[x_i/t_i]\,[z_i \rightsquigarrow e_i])) \\
= \quad & \text{(by } nf_1\text{, definition of } \underline{step}\text{, and substitution)} \\
& \underline{step}(nf_1((\circ\ nf_2(\underline{sub}\ rhs_{R_1,f,c}\ y_1\ \ldots\ y_n))[x_i/t_i]\,[y_i/z_i]\,[z_i \rightsquigarrow e_i])) \\
= \quad & \text{(by definition of } acc(R_1)\text{)} \\
& \underline{step}(nf_1((\circ\ rhs_{acc(R_1),f,c})[x_i/t_i]\,[y_i/z_i]\,[z_i \rightsquigarrow e_i])) \\
= \quad & \text{(by reduction and definition of } acc(R_1)^\circ\text{)} \\
& \underline{step}(nf_1((f\ (c\,t_1\ \ldots\ t_k)\ z_1\ \ldots\ z_n)[z_i \rightsquigarrow e_i]))
\end{aligned}
$$

Now we prove Statement (b) by induction on r.

(i) First, let $r = \Pi_j$ for some $j \in [n]$.

$$
\begin{aligned}
& \underline{step}(nf_1(nf_2(\underline{sub}\ \Pi_j\ e_1\ \ldots\ e_n)[x_i/t_i])) \\
= \quad & \text{(by reduction, substitution, and } nf_1 \text{ because } e_j \text{ is in normal form)} \\
& \underline{step}(e_j) \\
= \quad & \text{(by definition of } \underline{step} \text{ and } \underline{rel}\text{)} \\
& \underline{step}(z_j) + \textstyle\sum_{i=1}^{n} \underline{rel}(z_i, z_j) \cdot \underline{step}(e_i) \\
= \quad & \text{(by definition of } \underline{step}\text{)} \\
& \underline{step}(z_j[z_i \rightsquigarrow e_i]) \\
= \quad & \text{(by } nf_1\text{, substitution, and reduction)} \\
& \underline{step}(nf_1(nf_2(\underline{sub}\ \Pi_j\ z_1\ \ldots\ z_n)[x_i/t_i]\,[z_i \rightsquigarrow e_i]))
\end{aligned}
$$

(ii) Second, let $r = f\ x_j$ for some $f \in F$ and $j \in [k]$.

$$
\begin{aligned}
& \underline{step}(nf_1(nf_2(\underline{sub}\ (f\ x_j)\ e_1\ \ldots\ e_n)[x_i/t_i])) \\
= \quad & \text{(by reduction)} \\
& \underline{step}(nf_1((f\ x_j\ e_1\ \ldots\ e_n)[x_i/t_i])) \\
= \quad & \text{(by substitution)} \\
& \underline{step}(nf_1(f\ t_j\ e_1\ \ldots\ e_n)) \\
= \quad & \text{(by Statement (a))} \\
& \underline{step}(nf_1((f\ t_j\ z_1\ \ldots\ z_n)[z_i \rightsquigarrow e_i])) \\
= \quad & \text{(by substitution)} \\
& \underline{step}(nf_1((f\ x_j\ z_1\ \ldots\ z_n)[x_i/t_i]\,[z_i \rightsquigarrow e_i])) \\
= \quad & \text{(by reduction)}
\end{aligned}
$$

$\underline{step}(nf_1(nf_2(\underline{sub}\ (f\ x_j)\ z_1\ \ldots\ z_n)[x_i/t_i]\,[z_i \rightsquigarrow e_i]))$

(iii) Third, let $r = c\, r_1\ \ldots\ r_m$ for some $m \in \mathbb{N}$, $c \in (C - \Pi)^{(m)}$, and $r_1, \ldots, r_m \in RHS(F, Sub, C, X_k)$.

$$
\begin{aligned}
& \underline{step}(nf_1(nf_2(\underline{sub}\ (c\, r_1\ \ldots\ r_m)\ e_1\ \ldots\ e_n)[x_i/t_i])) \\
= \quad & \text{(by reduction)} \\
& \underline{step}(nf_1(nf_2((c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow e_i])[x_i/t_i])) \\
= \quad & \text{(by } nf_1\text{, substitution, and } nf_2\text{)} \\
& \underline{step}(nf_1(nf_2(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])[z_i \rightsquigarrow e_i]) \\
= \quad & \text{(by definition of } \underline{step}\text{)} \\
& \underline{step}(nf_1(nf_2(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])) + \\
& + \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf_1(nf_2(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])) \cdot \underline{step}(e_j) \\
= \quad & \text{(by definition of } \underline{step} \text{ because } \underline{step}(z_j) = 0\text{)} \\
& \underline{step}(nf_1(nf_2(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])) + \\
& + \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf_1(nf_2(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])) \cdot \underline{step}(e_j) + \\
& + \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf_1(nf_2(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])) \cdot \underline{step}(z_j) \\
= \quad & \text{(by definition of } \underline{step} \text{ and Observation 5)} \\
& \underline{step}(nf_1(nf_2(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])[z_i \rightsquigarrow z_i]) + \\
& + \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf_1(nf_2(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])[z_i \rightsquigarrow z_i]) \cdot \underline{step}(e_j) \\
= \quad & \text{(by definition of } \underline{step}\text{)} \\
& \underline{step}(nf_1(nf_2(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])[z_i \rightsquigarrow z_i]\,[z_i \rightsquigarrow e_i]) \\
= \quad & \text{(by } nf_1\text{, substitution, and } nf_2\text{)} \\
& \underline{step}(nf_1(nf_2((c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow z_i])[x_i/t_i])[z_i \rightsquigarrow e_i]) \\
= \quad & \text{(by reduction)} \\
& \underline{step}(nf_1(nf_2(\underline{sub}\ (c\, r_1\ \ldots\ r_m)\ z_1\ \ldots\ z_n)[x_i/t_i])[z_i \rightsquigarrow e_i]) \\
= \quad & \text{(by } nf_1\text{)}
\end{aligned}
$$

$$\underline{step}(nf_1(nf_2(\underline{sub}\ (c\, r_1\ \ldots\ r_m)\ z_1\ \ldots\ z_n)[x_i/t_i]\,[z_i \rightsquigarrow e_i]))$$

(iv) Finally, let $r = sub\ r_0\ \ldots\ r_n$ for some $r_0, \ldots, r_n \in RHS(F, Sub, C, X_k)$.

$$\begin{aligned}
& \underline{step}(nf_1(nf_2(\underline{sub}\ (sub\ r_0\ \ldots\ r_n)\ e_1\ \ldots\ e_n)[x_i/t_i])) \\
= \quad & \text{(by reduction)} \\
& \underline{step}(nf_1(nf_2((\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_n\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow e_i])[x_i/t_i])) \\
= \quad & \text{(by } nf_1\text{, substitution, and } nf_2\text{)} \\
& \underline{step}(nf_1(nf_2(\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])[z_i \rightsquigarrow e_i]) \\
= \quad & \text{(by definition of } \underline{step}\text{)} \\
& \underline{step}(nf_1(nf_2(\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])) + \\
& + \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf_1(nf_2(\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])) \cdot \underline{step}(e_j) \\
= \quad & \text{(by definition of } \underline{step} \text{ because } \underline{step}(z_j) = 0\text{)} \\
& \underline{step}(nf_1(nf_2(\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])) + \\
& + \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf_1(nf_2(\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])) \cdot \underline{step}(e_j) + \\
& + \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf_1(nf_2(\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])) \cdot \underline{step}(z_j) \\
= \quad & \text{(by definition of } \underline{step} \text{ and Observation 5)} \\
& \underline{step}(nf_1(nf_2(\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])[z_i \rightsquigarrow z_i]) + \\
& + \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf_1(nf_2(\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])[z_i \rightsquigarrow z_i]) \cdot \underline{step}(e_j) \\
= \quad & \text{(by definition of } \underline{step}\text{)} \\
& \underline{step}(nf_1(nf_2(\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])[z_i \rightsquigarrow z_i]\,[z_i \rightsquigarrow e_i]) \\
= \quad & \text{(by } nf_1\text{, substitution, and } nf_2\text{)} \\
& \underline{step}(nf_1(nf_2((\underline{sub}\ r_0\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots \\
& \qquad\qquad (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow z_i])[x_i/t_i])[z_i \rightsquigarrow e_i]) \\
= \quad & \text{(by reduction)} \\
& \underline{step}(nf_1(nf_2(\underline{sub}\ (sub\ r_0\ \ldots\ r_n)\ z_1\ \ldots\ z_n)[x_i/t_i])[z_i \rightsquigarrow e_i]) \\
= \quad & \text{(by } nf_1\text{)} \\
& \underline{step}(nf_1(nf_2(\underline{sub}\ (sub\ r_0\ \ldots\ r_n)\ z_1\ \ldots\ z_n)[x_i/t_i]\,[z_i \rightsquigarrow e_i]))
\end{aligned}$$

Thus the proof is complete. □

Lemma 8 (see Lemma 20). Let $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$ with $Sub = \{sub^{(n+1)}\}$ be an n-sntt, and let $acc(M) = (acc(F), C, acc(R_1), acc(r_{in}))$ be the n-satt constructed from M by accumulation.

(a) For every $f \in F$ and $t \in T_C$:

$$\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ (f\ t)\ z_1\ \ldots\ z_n)) = \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, f\ t\ z_1\ \ldots\ z_n)) \ .$$

(b) For every $k \in \mathbb{N}$, $r \in RHS(F, Sub, C, X_k)$, and $t_1, \ldots, t_k \in T_C$:

$$\begin{aligned} &\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r[x_i/t_i]\ z_1\ \ldots\ z_n)) \\ =\ &\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r\ z_1\ \ldots\ z_n)[x_i/t_i])) \ . \end{aligned}$$

Proof. We prove the two statements by simultaneous induction. Statement (a) is proved by structural induction on t, so let $t = c\ t_1\ \ldots\ t_k$ for some $k \in \mathbb{N}$, $c \in C^{(k)}$, and $t_1, \ldots, t_k \in T_C$.

$$\begin{aligned} &\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ (f\ (c\ t_1\ \ldots\ t_k))\ z_1\ \ldots\ z_n)) \\ =\ &\quad \text{(by reduction and definition of } rhs_{R_1^\circ, f, c}) \\ &\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ (\circ\ rhs_{R_1, f, c})[x_i/t_i]\ z_1\ \ldots\ z_n)) \\ =\ &\quad \text{(by substitution and reduction)} \\ &\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, (\circ\ (sub\ rhs_{R_1, f, c}[x_i/t_i]\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow z_i])) \\ =\ &\quad \text{(by } nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, \cdot) \text{ and definition of } \underline{step} \text{ because } \underline{step}(z_i) = 0) \\ &1 + \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ rhs_{R_1, f, c}[x_i/t_i]\ z_1\ \ldots\ z_n)) \\ =\ &\quad \text{(by Statement (b))} \\ &1 + \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ rhs_{R_1, f, c}\ z_1\ \ldots\ z_n)[x_i/t_i])) \\ =\ &\quad \text{(by substitution, } nf(\Rightarrow_{acc(R_1)^\circ}, \cdot) \text{, and definition of } \underline{step}) \\ &\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, (\circ\ nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ rhs_{R_1, f, c}\ y_1\ \ldots\ y_n))[x_i/t_i]\,[y_i/z_i])) \\ =\ &\quad \text{(by reduction and definition of } rhs_{acc(R_1)^\circ, f, c} \text{ and } rhs_{acc(R_1), f, c}) \\ &\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, f\ (c\ t_1\ \ldots\ t_k)\ z_1\ \ldots\ z_n)) \end{aligned}$$

Statement (b) is proved by induction on r.

(i) First let $r = \Pi_j$ for some $j \in [n]$.

$$\begin{aligned} &\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ \Pi_j[x_i/t_i]\ z_1\ \ldots\ z_n)) \\ =\ &\quad \text{(by substitution and reduction)} \\ &\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, z_j)) \\ =\ &\quad \text{(by } nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, \cdot),\ nf(\Rightarrow_{acc(R_1)^\circ}, \cdot) \text{, and substitution)} \\ &\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, z_j[x_i/t_i])) \end{aligned}$$

$$= \quad \text{(by reduction)}$$

$$\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ \Pi_j\ z_1\ \ldots\ z_n)[x_i/t_i]))$$

(ii) Now let $r = f\ x_j$ for some $f \in F$ and $j \in [k]$.

$$\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ (f\ x_j)[x_i/t_i]\ z_1\ \ldots\ z_n))$$

$$= \quad \text{(by substitution)}$$

$$\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ (f\ t_j)\ z_1\ \ldots\ z_n))$$

$$= \quad \text{(by Statement (a))}$$

$$\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, f\ t_j\ z_1\ \ldots\ z_n))$$

$$= \quad \text{(by substitution)}$$

$$\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, (f\ x_j\ z_1\ \ldots\ z_n)[x_i/t_i]))$$

$$= \quad \text{(by reduction)}$$

$$\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ (f\ x_j)\ z_1\ \ldots\ z_n)[x_i/t_i]))$$

(iii) Let $r = c\ r_1\ \ldots\ r_m$ for some $m \in \mathbb{N}$, $c \in (C - \Pi)^{(m)}$, and $r_1, \ldots, r_m \in RHS(F, Sub, C, X_k)$. Moreover, let $a = 1$ if $c = \circ$ and $a = 0$ otherwise.

$$\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ (c\ r_1\ \ldots\ r_m)[x_i/t_i]\ z_1\ \ldots\ z_n))$$

$$= \quad \text{(by substitution)}$$

$$\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ (c\ r_1[x_i/t_i]\ \ldots\ r_m[x_i/t_i])\ z_1\ \ldots\ z_n))$$

$$= \quad \text{(by reduction)}$$

$$\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, (c\ (sub\ r_1[x_i/t_i]\ z_1\ \ldots\ z_n)\ \ldots$$
$$(sub\ r_m[x_i/t_i]\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow z_i]))$$

$$= \quad \text{(by } nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, \cdot) \text{ and definition of } \underline{step} \text{ because } \underline{step}(z_i) = 0)$$

$$a + \textstyle\sum_{j=1}^{m} \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_j[x_i/t_i]\ z_1\ \ldots\ z_n))$$

$$= \quad \text{(by induction hypothesis [} m \text{ times])}$$

$$a + \textstyle\sum_{j=1}^{m} \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_j\ z_1\ \ldots\ z_n)[x_i/t_i]))$$

$$= \quad \text{(by } nf(\Rightarrow_{acc(R_1)^\circ}, \cdot),\ nf(\Rightarrow_{\overline{R_2}}, \cdot)\text{, substitution, and definition of } \underline{step})$$

$$\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}},$$
$$c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i]))$$

$$= \quad \text{(by definition of } \underline{step} \text{ because } \underline{step}(z_i) = 0)$$

$$\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}},$$
$$c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[x_i/t_i])[z_i \rightsquigarrow z_i])$$

$$= \quad \text{(by } nf(\Rightarrow_{acc(R_1)^\circ}, \cdot),\ nf(\Rightarrow_{\overline{R_2}}, \cdot)\text{, and substitution)}$$

$$\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}},$$
$$(c\ (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_m\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow z_i])[x_i/t_i]))$$

$$
\begin{aligned}
= &\quad \text{(by reduction)}\\
&\underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ (c\ r_1\ \dots\ r_m)\ z_1\ \dots\ z_n)[x_i/t_i]))
\end{aligned}
$$

(iv) Finally, let $r = sub\ r_0\ \dots\ r_n$ for some $r_0, \dots, r_n \in RHS(F, Sub, C, X_k)$.

$$
\begin{aligned}
&\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ (sub\ r_0\ \dots\ r_n)[x_i/t_i]\ z_1\ \dots\ z_n))\\
= &\quad \text{(by substitution)}\\
&\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ (sub\ r_0[x_i/t_i]\ \dots\ r_n[x_i/t_i])\ z_1\ \dots\ z_n))\\
= &\quad \text{(by } nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, \cdot)\text{)}\\
&\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ},\\
&\qquad sub\ nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_0[x_i/t_i]\ \dots\ r_n[x_i/t_i])\ z_1\ \dots\ z_n))\\
= &\quad \text{(by Lemma 6)}\\
&\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_0[x_i/t_i]\ \dots\ r_n[x_i/t_i])) +\\
&+ \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_0[x_i/t_i]\ \dots\ r_n[x_i/t_i])[\Pi_i/z_i]) \cdot \underline{step}(z_j)\\
= &\quad \text{(because } \underline{step}(z_j) = 0\text{)}\\
&\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_0[x_i/t_i]\ \dots\ r_n[x_i/t_i]))\\
= &\quad \text{(by } nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, \cdot)\text{)}\\
&\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])\ \dots\\
&\qquad\qquad nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_n[x_i/t_i])))\\
= &\quad \text{(by Lemma 6)}\\
&\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])) +\\
&+ \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])[\Pi_i/z_i]) \cdot\\
&\qquad \cdot\ \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_j[x_i/t_i]))\\
= &\quad \text{(by definition of } \underline{step} \text{ because } \underline{step}(z_i) = 0\text{)}\\
&\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])) +\\
&+ \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])[\Pi_i/z_i]) \cdot\\
&\qquad \cdot \Big(\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_j[x_i/t_i])) +\\
&\qquad\quad + \textstyle\sum_{l=1}^{n} \underline{rel}(z_l, nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_j[x_i/t_i])[\Pi_i/z_i]) \cdot \underline{step}(z_l)\Big)\\
= &\quad \text{(by Lemma 6)}\\
&\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])) +\\
&+ \textstyle\sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])[\Pi_i/z_i]) \cdot\\
&\qquad \cdot\ \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_j[x_i/t_i])\ z_1\ \dots\ z_n))\\
= &\quad \text{(by } nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, \cdot)\text{)}
\end{aligned}
$$

$$
\begin{aligned}
& \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])) + \\
& + \sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])[\Pi_i/z_i]) \cdot \\
& \qquad\qquad \cdot \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_j[x_i/t_i]\ z_1\ \ldots\ z_n)) \\
= & \quad \text{(by Lemma 1 and Observation 5 and } \underline{step}(z_j) = 0\text{)} \\
& \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])) + \\
& + \sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])[\Pi_i/z_i]) \cdot \underline{step}(z_j) + \\
& + \sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_0[x_i/t_i]\ z_1\ \ldots\ z_n)) \cdot \\
& \qquad\qquad \cdot \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_j[x_i/t_i]\ z_1\ \ldots\ z_n)) \\
= & \quad \text{(by induction hypothesis [} n \text{ times])} \\
& \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])) + \\
& + \sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])[\Pi_i/z_i]) \cdot \underline{step}(z_j) + \\
& + \sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_0[x_i/t_i]\ z_1\ \ldots\ z_n)) \cdot \\
& \qquad\qquad \cdot \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_j\ z_1\ \ldots\ z_n)[x_i/t_i])) \\
= & \quad \text{(by Lemmata 6 and 3 and Observation 5)} \\
& \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_0[x_i/t_i])\ z_1\ \ldots\ z_n)) + \\
& + \sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\ z_1\ \ldots\ z_n)[x_i/t_i])) \cdot \\
& \qquad\qquad \cdot \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_j\ z_1\ \ldots\ z_n)[x_i/t_i])) \\
= & \quad \text{(by } nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, \cdot)\text{)} \\
& \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_0[x_i/t_i]\ z_1\ \ldots\ z_n)) + \\
& + \sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\ z_1\ \ldots\ z_n)[x_i/t_i])) \cdot \\
& \qquad\qquad \cdot \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_j\ z_1\ \ldots\ z_n)[x_i/t_i])) \\
= & \quad \text{(by induction hypothesis)} \\
& \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\ z_1\ \ldots\ z_n)[x_i/t_i])) + \\
& + \sum_{j=1}^{n} \underline{rel}(z_j, nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\ z_1\ \ldots\ z_n)[x_i/t_i])) \cdot \\
& \qquad\qquad \cdot \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_j\ z_1\ \ldots\ z_n)[x_i/t_i])) \\
= & \quad \text{(by definition of } \underline{step}\text{)} \\
& \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\ z_1\ \ldots\ z_n)[x_i/t_i]) \\
& \qquad [z_j \rightsquigarrow nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_j\ z_1\ \ldots\ z_n)[x_i/t_i])]) \\
= & \quad \text{(by } nf(\Rightarrow_{acc(R_1)^\circ}, \cdot)\text{)} \\
& \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0\ z_1\ \ldots\ z_n)[x_i/t_i] \\
& \qquad [z_j \rightsquigarrow nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_j\ z_1\ \ldots\ z_n)[x_i/t_i]\,])) \\
= & \quad \text{(by Lemma 7)}
\end{aligned}
$$

$$
\begin{aligned}
& \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0 \\
& \quad nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_1\ z_1\ \ldots\ z_n)[x_i/t_i]\ \ldots \\
& \quad nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_n\ z_1\ \ldots\ z_n)[x_i/t_i])[x_i/t_i])) \\
= \quad & \text{(by } nf(\Rightarrow_{\overline{R_2}}, \cdot) \text{ and substitution)} \\
& \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0 \\
& \quad (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_n\ z_1\ \ldots\ z_n))[x_i/t_i])) \\
= \quad & \text{(by definition of } \underline{step} \text{ because } \underline{step}(z_i) = 0) \\
& \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_0 \\
& \quad (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_n\ z_1\ \ldots\ z_n))[x_i/t_i])[z_i \rightsquigarrow z_i]) \\
= \quad & \text{(by } nf(\Rightarrow_{acc(R_1)^\circ}, \cdot)\text{, substitution, and } nf(\Rightarrow_{\overline{R_2}}, \cdot)\text{)} \\
& \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, (\underline{sub}\ r_0 \\
& \quad (\underline{sub}\ r_1\ z_1\ \ldots\ z_n)\ \ldots\ (\underline{sub}\ r_n\ z_1\ \ldots\ z_n))[z_i \rightsquigarrow z_i])[x_i/t_i])) \\
= \quad & \text{(by reduction)} \\
& \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ (sub\ r_0\ \ldots\ r_n)\ z_1\ \ldots\ z_n)[x_i/t_i]))
\end{aligned}
$$

This concludes the proof. □

Theorem 9 (see Theorem 21). Let $M = (F, \{sub^{(n+1)}\}, C, \Pi, R_1, R_2, r_{in})$ be an n-sntt and $acc(M) = (acc(F), C, acc(R_1), acc(r_{in}))$. Then, for every $t \in T_C$:

$$\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_{in}[x_1/t])) = \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, acc(r_{in})[x_1/t])) \ .$$

Proof.

$$
\begin{aligned}
& \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_{in}[x_1/t])) & \\
= & \ \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_{in}[x_1/t])\ z_1 \ldots z_n)) & (*) \\
= & \ \underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub\ r_{\mathrm{in}}[x_1/t]\ z_1 \ldots z_n)) & \text{(confluence)} \\
= & \ \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_{in}\ z_1 \ldots z_n)[x_1/t])) & \text{(Lm. 8(b))} \\
= & \ \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, nf(\Rightarrow_{\overline{R_2}}, \underline{sub}\ r_{in}\ \Pi_1 \ldots \Pi_n)[x_1/t])) & (**) \\
= & \ \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, acc(r_{in})[x_1/t])) & \text{(Def. } acc(r_{in})\text{)}
\end{aligned}
$$

The statement $(*)$ holds, because *sub* neither deletes nor copies $\circ$-symbols and preserves sharings in its recursion argument (in particular because of the last rule of Definition 6). $(*)$ can be formally proved by induction on $nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_{in}[x_1/t])$. Note that $(*)$ does not hold without the outer application of $\underline{step}$.

The statement $(**)$ holds, because for every $j \in [n]$, both, z_j and Π_j do not contain $\circ$-symbols and are normal forms w.r.t. $\Rightarrow_{\overline{R_2}}$ and $\Rightarrow_{acc(R_1)^\circ}$. $(**)$ can be formally proved by induction on r_{in}. □