TECHNISCHE UNIVERSITÄT DRESDEN

Fakultät Informatik

TUD-FI03-14 — Dezember 2003

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Institut für Theoretische Informatik

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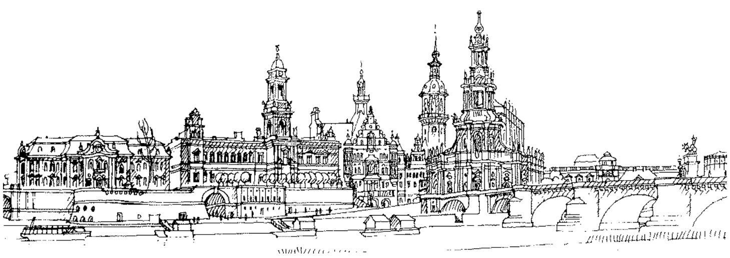
Technische Berichte
Technical Reports
ISSN 1430-211X



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Incomparability Results for Classes of Polynomial Tree Series Transformations

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Abstract

We consider (subclasses of) polynomial bottom-up and top-down tree series transducers over a partially ordered semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$, and we compare the classes of tree-to-tree-series and o-tree-to-tree-series transformations computed by such transducers. Our main result states the following. If, for some $a \in A$ with $\mathbf{1} \preceq a$, the semiring $\mathcal{A}$ is a weak a-growth semiring and either (i) the semiring $\mathcal{A}$ is additively idempotent and $x, y \in \{\text{polynomial}, \text{deterministic}, \text{total}, \text{deterministic and total}, \text{homomorphism}\}$, or (ii) $\mathbf{1} \prec \mathbf{1} \oplus \mathbf{1}$ and $x, y \in \{\text{deterministic}, \text{deterministic and total}, \text{homomorphism}\}$, then the statements $x\text{–BOT}(\mathcal{A}) \bowtie y\text{–BOT}^o(\mathcal{A})$ and $x\text{–BOT}(\mathcal{A}) \bowtie y\text{–TOP}(\mathcal{A})$ hold. Therein $x\text{–BOT}^{\text{mod}}(\mathcal{A})$ for mod $\in \{\varepsilon, o\}$ denotes the class of mod-tree-to-tree-series transformations computed by bottom-up tree series transducers, which have property x, over the semiring $\mathcal{A}$ (the class $y\text{–TOP}(\mathcal{A})$ is defined similarly for top-down tree series transducers). Besides, $\bowtie$ denotes incomparability with respect to set inclusion.

1 Introduction

Tree series transducers [Kui99, EFV02, FGV02, FV03] were introduced as a generalization of tree transducers [Rou70, Tha70, Eng75] and weighted tree automata [Sei94, Kui98, Boz99]. Both historical predecessors of tree series transducers have successfully been motivated from and applied in practice. Specifically, tree transducers are motivated from syntax-directed translations in compilers [Iro61, Eng81, FV98], and they are applied in, e.g., functional program analysis and transformation [Küh98, GKV03, Jür03, VK04], linguistics [MC99, KMM00, MMM01, KMMM03], generation of pictures [Dre00, Dre01], and query languages of XML databases [BMN02, EM03]. Weighted tree automata have been applied to code selection in compilers [FSW94] and tree pattern matching [Sei92]. Moreover, a rich theory of tree transducers was developed (cf. [Eng75, Bak79, Eng82, GS84, NP92, CDG⁺97, GS97, FV98] as seminal or survey papers and monographs) during the seventies, whereas weighted tree automata just recently received some attention (e.g., [Sei92, Sei94, Kui98, Bor03, BV03, DPV03, DV03, ÉK03]).

Roughly speaking, tree series transducers capture both (a) the way of translating input trees into output trees, as it is inherent in bottom-up and top-down tree transducers, and (b) the computation of a weight (or cost) in a semiring, as it is inherent in weighted tree automata. More formally, a (bottom-up or top-down) tree series transducer is a tuple $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$, wherein Q is a finite set of states, Σ and Δ are ranked alphabets of input and output symbols, respectively, $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring, $D \subseteq Q$ is a set of designated states (also called final states if M is bottom-up or initial states if M is top-down), and $\mu = (\mu_k \mid k \in \mathbb{N})$ is a (bottom-up or top-down) tree representation. Such a tree representation consists of mappings $\mu_k : \Sigma^{(k)} \longrightarrow A\langle\!\langle T_\Delta(X)\rangle\!\rangle^{Q \times Q(X_k)^*}$, in which $T_\Delta(X)$ denotes the set of Δ-trees indexed by variables of X, $A\langle\!\langle T_\Delta(X)\rangle\!\rangle$ denotes the set of mappings $\varphi : T_\Delta(X) \longrightarrow A$ (called formal tree series), and $A\langle\!\langle T_\Delta(X)\rangle\!\rangle^{Q \times Q(X_k)^*}$ denotes the set of $(Q \times Q(X_k)^*)$-matrices over $A\langle\!\langle T_\Delta(X)\rangle\!\rangle$. Using mod-substitution of tree series (with mod $\in \{\varepsilon, o\}$; cf. [EFV02, FV03]) in order to substitute tree series into tree series of the form $\mu_k(\sigma)_{q,w}$, we can impose a Σ-algebraic structure on $A\langle\!\langle T_\Delta\rangle\!\rangle^Q$ and thereby obtain the unique Σ-homomorphism $h_\mu^{\text{mod}} : T_\Sigma \longrightarrow A\langle\!\langle T_\Delta\rangle\!\rangle^Q$. Then the mod-tree-to-tree-series transformation computed by M (for short: mod-t-ts transformation) is the mapping

*Supported by German Research Council (DFG, GRK 334/3)

$\tau_M^{\mathrm{mod}} : T_\Sigma \longrightarrow A\langle\langle T_\Delta\rangle\rangle$ defined by $\tau_M^{\mathrm{mod}}(s) = \sum_{q\in D} h_\mu^{\mathrm{mod}}(s)_q$. Thus, for a given input tree $s \in T_\Sigma$, M computes a (possibly infinite) set $\mathrm{supp}(\tau_M^{\mathrm{mod}}(s)) = \{ t \in T_\Delta \mid (\tau_M^{\mathrm{mod}}(s), t) \neq \mathbf{0} \}$ of output trees and associates a coefficient $(\tau_M^{\mathrm{mod}}(s), t) \in A$ to every such output tree $t \in T_\Delta$. Note that $(\tau_M^{\mathrm{mod}}(s), t)$ denotes the application $\varphi(t)$ with $\varphi = \tau_M^{\mathrm{mod}}(s)$. For every so-called polynomial tree series transducer M and input tree $s \in T_\Sigma$, the set $\mathrm{supp}(\tau_M^{\mathrm{mod}}(s))$ of computed and relevant output trees is finite. Polynomial bottom-up and top-down tree series transducers over the boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$ essentially are bottom-up and top-down tree transducers, respectively (cf. Section 4 of [EFV02]).

In the same way as tree transducers, also tree series transducers can have particular properties, e.g., they can be deterministic, total, deterministic and total, or they are homomorphisms (cf., e.g., [Eng75]). The classes of mod-t-ts transformations computed by bottom-up and top-down tree series transducers having the property x (e.g., being deterministic) over a semiring $\mathcal{A}$ are denoted by x–$\mathrm{BOT}^{\mathrm{mod}}(\mathcal{A})$ and x–$\mathrm{TOP}^{\mathrm{mod}}(\mathcal{A})$, respectively.

In [FV03] several classes of the form x–$\mathrm{BOT}^{\mathrm{mod}}(\mathcal{A})$ and x–$\mathrm{TOP}^{\mathrm{mod}}(\mathcal{A})$ have been compared with each other with respect to set inclusion. For instance, it was proved that,

- for every $x \in \{\text{polynomial}, \text{deterministic}, \text{total}, \text{deterministic and total}, \text{homomorphism}\}$ and semiring $\mathcal{A}$, we have x–$\mathrm{TOP}(\mathcal{A}) = x$–$\mathrm{TOP}^o(\mathcal{A})$ (cf. Theorem 5.2 of [FV03]),
- p–$\mathrm{BOT}(\mathbb{N}_\infty) \bowtie$ p–$\mathrm{BOT}^o(\mathbb{N}_\infty)$, where p stands for the property of being polynomial, $\mathbb{N}_\infty = (\mathbb{N} \cup \{+\infty\}, +, \cdot, 0, 1)$ is the semiring of non-negative integers, and $\bowtie$ denotes the usual incomparability with respect to set inclusion (cf. Corollary 5.18 of [FV03]), and
- p–$\mathrm{BOT}(\mathbb{T}) \bowtie$ p–$\mathrm{BOT}^o(\mathbb{T})$, where $\mathbb{T} = (\mathbb{N} \cup \{+\infty\}, \min, +, (+\infty), 0)$ is the tropical semiring (cf. Corollary 5.23 of [FV03]).

The latter two incomparability results motivated us to investigate the question whether this incomparability also holds for semirings different from $\mathbb{N}_\infty$ and $\mathbb{T}$. In this paper we answer this question in the affirmative. Additionally, we compare classes of t-ts transformations which are computed by different types of tree series transducers, i.e., bottom-up and top-down tree series transducers. Our main result is Theorem 5.10 which states the following:

> If $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ is a partially ordered, weak a-growth semiring for some $a \in A$ with $\mathbf{1} \preceq a$, and
>
> - $x, y \in \{\text{polynomial}, \text{deterministic}, \text{total}, \text{deterministic and total}, \text{homomorphism}\}$ and, additionally, $\mathcal{A}$ is additively idempotent, or
> - $x, y \in \{\text{deterministic}, \text{deterministic and total}, \text{homomorphism}\}$ and $\mathbf{1} \preceq \mathbf{1} \oplus \mathbf{1}$,
>
> then the statements x–$\mathrm{BOT}(\mathcal{A}) \bowtie y$–$\mathrm{BOT}^o(\mathcal{A})$ and x–$\mathrm{BOT}(\mathcal{A}) \bowtie y$–$\mathrm{TOP}(\mathcal{A})$ hold.

Let us explain some more details concerning this theorem and then briefly discuss the way to prove it. A partially ordered semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ is a semiring $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ together with a partial order $\preceq \subseteq A \times A$ such that the order is preserved by the operations $\oplus$ and $\odot$. Roughly speaking, a partially ordered semiring is weak a-growth semiring, if (i) there is an element $a \in A$ which has no period (i.e., for every two integers $i, j \in \mathbb{N}$ we have $a^i \neq a^j$, if and only if $i \neq j$), and (ii) whenever $a^n = a_1 \odot b \odot a_2 \oplus d$ for some semiring elements $a_1, a_2 \in A \setminus \{\mathbf{0}\}$ and $b, d \in A$, then $b \preceq a^m$ for some $m \in \mathbb{N}$ (cf. Definition 5.8 for the definition of weak a-growth). The latter condition requires that every element, which might occur in a decomposition of a^n, can be bounded by a power of a. In particular, the following semirings are weak a-growth semirings:

- the semiring $(\mathbb{N} \cup \{+\infty\}, +, \cdot, 0, 1, \leq)$ of non-negative integers is a weak 2-growth semiring,
- the tropical semiring $(\mathbb{N} \cup \{+\infty\}, \min, +, (+\infty), 0, \leq)$ is a weak 1-growth semiring,
- the arctic semiring $(\mathbb{N} \cup \{-\infty\}, \max, +, (-\infty), 0, \leq)$ is a weak 1-growth semiring, and

- the formal language semiring $(\mathcal{P}(S^*), \cup, \circ, \emptyset, \{\varepsilon\}, \subseteq)$ for some alphabet S is a weak s-growth semiring for every $s \in S$.

In order to prove the non-inclusion results (necessary for the incomparabilities) of the main theorem, we use the partial order on the semiring and establish a framework of mappings called *coefficient majorizations*. For a given tree series transducer M, a coefficient majorization is a mapping $f : \mathbb{N}_+ \longrightarrow A$ such that for every $n \in \mathbb{N}_+$ the semiring element $f(n)$ is an upper bound of the set $C_M^{\mathrm{mod}}(n)$, which is the set of all coefficients generated from input trees of height n, i.e.,

$$C_M^{\mathrm{mod}}(n) = \left\{ (h_\mu^{\mathrm{mod}}(s)_q, t) \;\middle|\; q \in Q, s \in T_\Sigma, t \in \mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_q), \mathrm{height}(s) = n \right\}.$$

Now, as usual, given two classes $\mathfrak{T}_1$ and $\mathfrak{T}_2$ of mod_1-t-ts transformations and mod_2-t-ts transformations, respectively, we can prove $\mathfrak{T}_1 \not\subseteq \mathfrak{T}_2$ by (i) proving that some mapping f is a majorization mapping for the class $\mathfrak{T}_2$ and (ii) constructing a tree series transducer M such that $\tau_M^{\mathrm{mod}_1} \in \mathfrak{T}_1$ and $C_M^{\mathrm{mod}_1}(n)$ grows faster than $f(n)$. For the particular classes in which we are interested in this paper, this is achieved in Lemma 5.9.

Coefficient majorizations have been investigated for the specific case in which the coefficient $(h_\mu^{\mathrm{mod}}(s)_q, t)$ is the height or size of the output tree t and n is the height or size of the input tree, e.g., for bottom-up tree transducers (follows in a straightforward manner from Theorem 3.15 in [Eng75]), for top-down tree transducers (cf. Lemma 3.27 of [FV98]), for attributed tree transducers (cf. Lemma 3.3 of [Eng81] and Lemma 5.40 of [FV98]), and for macro tree transducers (cf. Lemma 3.3 of [Eng81] and Lemma 4.23 of [FV98]).

This paper is structured as follows. Section 2 recalls the relevant basic mathematical notions and notations, particularly partially ordered semirings, tree series, and substitution of tree series. Section 3 presents the definition of tree series transducers from [EFV02] in some detail along with the definition of several subclasses of tree series transducers. Section 4 establishes the coefficient majorization for the cost of any relevant output tree computed by a polynomial tree series transducer, which is bottom-up or top-down. Therein we require some properties of the underlying semiring; namely, the semiring $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is supposed to be partially ordered, i.e., there is a partial order $\preceq \subseteq A \times A$, which is preserved by the two operations $\oplus$ and $\odot$, and $\mathbf{1} \preceq \mathbf{1} \oplus \mathbf{1}$. In particular, any naturally ordered semiring fulfils these conditions. Finally, in Section 5 the incomparability results outlined above are derived.

2 Preliminaries

In this section we present some basic notions and notations required in the sequel. The first subsection recalls partial orders (cf. [DP02]) and associated notions. Words (cf. [MS97]) and trees (cf. [GS84, GS97]) are considered in the second subsection, whereas the third subsection is dedicated to algebraic structures and, in particular, (partially ordered) semirings [Kui97, HW98, Gol99]. Finally, this section is concluded by the presentation of formal tree series (cf. [BR82, Kui97]) and two definitions of substitutions for formal tree series (cf. [EFV02, FV03]).

2.1 Partial Orders

The set $\{0, 1, 2, \dots\}$ of all non-negative integers is denoted by $\mathbb{N}$, and the set $\mathbb{N} \setminus \{0\}$ of all positive integers is denoted by $\mathbb{N}_+$. For every two integers $i, j \in \mathbb{N}$ the interval $[i, j]$ denotes the subset $\{ n \in \mathbb{N} \mid i \leq n \leq j \}$, and we use $[j]$ to abbreviate $[1, j]$. Recall that the *cardinality* of a finite set S, i.e., the number of elements of S, is denoted by $\mathrm{card}(S)$, and for every set S the set of all subsets of S, also called the *power set of* S, is denoted by $\mathcal{P}(S)$.

Given a non-empty set A, a binary relation $\preceq \subseteq A \times A$ is called *partial order (on A)*, if $\preceq$ is (i) *reflexive*, i.e., for every element $a \in A$ we have $a \preceq a$, (ii) *antisymmetric*, i.e., for every two elements $a_1, a_2 \in A$ the facts $a_1 \preceq a_2$ and $a_2 \preceq a_1$ imply $a_1 = a_2$, and (iii) *transitive*, i.e., for every three elements $a_1, a_2, a_3 \in A$ with $a_1 \preceq a_2$ and $a_2 \preceq a_3$ also $a_1 \preceq a_3$ holds.

The pair $(A, \preceq)$ is termed *partially ordered set*, and we represent the pair by A alone whenever $\preceq$ is understood from the context. Let $a_1, a_2 \in A$. The fact that neither $a_1 \preceq a_2$ nor $a_2 \preceq a_1$ (or equivalently: a_1 and a_2 are *incomparable*) is expressed as $a_1 \bowtie a_2$. In case there are no incomparable elements, the partial order $\preceq$ is said to be a *total order*. As usual, the *strict order* $\prec \subseteq A \times A$ corresponding to a partial order $\preceq$ is derived by defining $a_1 \prec a_2$, if and only if $a_1 \preceq a_2$ and $a_1 \neq a_2$.

Let $S \subseteq A$. An element $u \in A$ is called *upper bound of* S, if $s \preceq u$ for every element $s \in S$. The set of all upper bounds of S is denoted by $\uparrow S$, and the least element of $\uparrow S$, also called *supremum of* S, is denoted by $\sup S$, if it exists. If, for every non-empty and finite subset $S \subseteq A$, there is an upper bound (i.e., $\uparrow S \neq \emptyset$), then A is called *directed*. Moreover, should even the supremum $\sup S$ exist for every non-empty and finite subset $S \subseteq A$, then $(A, \preceq)$ is said to be a $\vee$*-semilattice* (read: supremum-semilattice).

Next we define mappings between partially ordered sets $(A, \preceq_A)$ and $(B, \preceq_B)$ which are compatible with the order. An *order-preserving* mapping $f : A \longrightarrow B$ is a mapping satisfying for every two elements $a_1, a_2 \in A$ with $a_1 \preceq_A a_2$ the condition $f(a_1) \preceq_B f(a_2)$. Finally, we recall the notion of majorizations. Let $f : D \longrightarrow A$ be a mapping and $g = (\, g_i \mid i \in I \,)$ be a family of mappings each of type $g_i : D \longrightarrow A$ for a set D and an index set I. The mapping f is said to be a *majorization of* g, if $f(d) \in \uparrow \{\, g_i(d) \mid i \in I \,\}$ for every element $d \in D$, and the majorization f is called *tight*, if $f(d) = \sup \{\, g_i(d) \mid i \in I \,\}$ for every element $d \in D$ (hence a majorization can only be tight, if these suprema exist). Note that, in particular, if $f(d) \in \{\, g_i(d) \mid i \in I \,\}$ for every element $d \in D$, then f is tight.

2.2 Words and trees

By a *word of length* $n \in \mathbb{N}$ we mean an element of the n-fold Cartesian product $S^n = S \times \cdots \times S$ of a set S. The set of all words over S is denoted by S^*, where the particular element $() \in S^0$, called the *empty word*, is displayed as ε. The *length of a word* $w \in S^*$ is denoted by $|w|$; thus $|\varepsilon| = 0$.

Every non-empty and finite set S is called *alphabet* of which elements are termed *symbols*. A *ranked alphabet* is defined to be a pair (Σ, rk), wherein Σ is an alphabet and the mapping $\mathrm{rk} : \Sigma \longrightarrow \mathbb{N}$ associates to every symbol of Σ its finite *rank*. For every $k \in \mathbb{N}$ we use $\Sigma^{(k)} \subseteq \Sigma$ to denote the set of symbols having rank k, i.e., $\Sigma^{(k)} = \{\, \sigma \in \Sigma \mid \mathrm{rk}(\sigma) = k \,\}$. In the following, we will usually assume that the rk-mapping is implicitly given. Hence we identify (Σ, rk) with Σ and specify the ranked alphabet by listing the elements of Σ with their ranks put in parentheses as superscripts as in $\{\sigma^{(2)}, \alpha^{(0)}\}$, for example. The *maximal rank* of the ranked alphabet Σ, denoted by mx_Σ, is defined to be the maximal integer $m \in \mathbb{N}$ such that $\Sigma^{(m)} \neq \emptyset$. If $\mathrm{mx}_\Sigma = 1$, then Σ is said to be a *unary ranked alphabet*.

Let Σ be a ranked alphabet and $X = \{\, x_i \mid i \in \mathbb{N}_+ \,\}$ be a countable set of (formal) variables. The set of (finite, labeled, and ordered) Σ*-trees indexed by* $V \subseteq X$, denoted by $T_\Sigma(V)$, is inductively defined to be the smallest set T such that (i) $V \cup \Sigma^{(0)} \subseteq T$ and (ii) for every $k \in \mathbb{N}_+$, symbol $\sigma \in \Sigma^{(k)}$, and k elements $t_1, \ldots, t_k \in T$ also $\sigma(t_1, \ldots, t_k) \in T$. The set T_Σ of *ground* Σ-trees is an abbreviation for $T_\Sigma(\emptyset)$.

Let $V \subseteq X$ be a subset of X, and let $t \in T_\Sigma(V)$ be a Σ-tree indexed by V. The number of occurrences of a given variable $v \in V$ in t is denoted by $|t|_v$. The tree t is called *linear in* V (*non-deleting in* V), if every variable $v \in V$ occurs at most once, i.e., $|t|_v \leq 1$, (at least once, i.e., $|t|_v \geq 1$) in t. Observe that in order for a tree to be non-deleting, the set V needs to be finite. Since we will often deal with finite subsets $V \subset X$, we introduce for every integer $n \in \mathbb{N}$ the denotation X_n to stand for the set $\{\, x_i \mid i \in [n] \,\}$ (note that $X_0 = \emptyset$).

We distinguish a subset $\widehat{T_\Sigma}(X_n) \subseteq T_\Sigma(X_n)$ as follows. Let a tree $t \in T_\Sigma(X_n)$ be in $\widehat{T_\Sigma}(X_n)$, if and only if for every index $i \in [n]$ the variable x_i occurs exactly once in t, i.e., $|t|_{x_i} = 1$, and reading the leaves of the tree t left-to-right, the variables occur in the order $x_1, \ldots, x_n$. Note that elements of $\widehat{T_\Sigma}(X_k)$ are linear and non-deleting in X_n and $\widehat{T_\Sigma}(X_0) = T_\Sigma$.

We recursively define the standard mapping $\mathrm{height} : T_\Sigma(V) \longrightarrow \mathbb{N}_+$ by the following equalities:

- for every tree $t \in V \cup \Sigma^{(0)}$: $\mathrm{height}(t) = 1$ and

- for every $k \in \mathbb{N}_+$, symbol $\sigma \in \Sigma^{(k)}$, and k trees $t_1, \ldots, t_k \in T_\Sigma(V)$:

$$\mathrm{height}(\sigma(t_1, \ldots, t_k)) = 1 + \max\{\, \mathrm{height}(t_i) \mid i \in [k] \,\}\,.$$

Given an integer $n \in \mathbb{N}$, a Σ-tree $t \in T_\Sigma(X_n)$ indexed by X_n, and n trees $s_1, \ldots, s_n \in T_\Sigma(V)$, the expression $t[s_1, \ldots, s_n]$ denotes the result of replacing (in parallel), for every index $i \in [n]$, every occurrence of x_i in the tree t with the tree s_i, i.e., $x_j[s_1, \ldots, s_n] = s_j$ for every index $j \in [n]$ and for every integer $k \in \mathbb{N}$, symbol $\sigma \in \Sigma^{(k)}$, and k trees $t_1, \ldots, t_k \in T_\Sigma(X_n)$

$$(\sigma(t_1, \ldots, t_k))[s_1, \ldots, s_n] = \sigma(t_1[s_1, \ldots, s_n], \ldots, t_k[s_1, \ldots, s_n]).$$

Let Σ be a ranked alphabet with just a single non-nullary symbol, i.e., $\bigcup_{k \in \mathbb{N}_+} \Sigma^{(k)} = \{\sigma\}$. The set of *fully balanced trees (over Σ)* is defined to be the smallest subset $T \subseteq T_\Sigma$ such that $\Sigma^{(0)} \subseteq T$ and, whenever $t \in T$, also $\sigma(t, \ldots, t) \in T$.

2.3 Monoids and (partially ordered) semirings

A *monoid* is defined to be an algebraic structure $\mathcal{A} = (A, \otimes, \mathbf{1})$ consisting of a *carrier set* A together with a binary operation $\otimes \colon A \times A \longrightarrow A$ and a constant element $\mathbf{1} \in A$, such that the operation $\otimes$ is *associative*, i.e., for every three elements $a_1, a_2, a_3 \in A$ the equality $a_1 \otimes (a_2 \otimes a_3) = (a_1 \otimes a_2) \otimes a_3$ is met, and the constant element $\mathbf{1}$ is the *unit element* with respect to operation $\otimes$, i.e., for every element $a \in A$ we demand $a \otimes \mathbf{1} = \mathbf{1} \otimes a = a$. Further, the monoid $\mathcal{A}$ is said to be *commutative*, if for every two elements $a_1, a_2 \in A$ the equality $a_1 \otimes a_2 = a_2 \otimes a_1$ is fulfilled.

By a *semiring (with one and absorbing zero)* we mean an algebraic structure $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ with the operations of *addition* $\oplus$ and *multiplication* $\odot$, of which $(A, \oplus, \mathbf{0})$ and $(A, \odot, \mathbf{1})$ are monoids, also called the *additive monoid* and the *multiplicative monoid*, respectively. Additionally, the former monoid is required to be commutative and the monoids are connected via the *distributivity laws*, i.e., for every three elements $a_1, a_2, a_3 \in A$ the equalities $a_1 \odot (a_2 \oplus a_3) = (a_1 \odot a_2) \oplus (a_1 \odot a_3)$ and $(a_1 \oplus a_2) \odot a_3 = (a_1 \odot a_3) \oplus (a_2 \odot a_3)$ hold, and the *absorption laws*, i.e., for every element $a \in A$ it holds that $a \odot \mathbf{0} = \mathbf{0} \odot a = \mathbf{0}$. By convention, we assume that multiplication has a higher (binding) priority than addition, e.g., we read $a_1 \oplus a_2 \odot a_3$ as $a_1 \oplus (a_2 \odot a_3)$.

As usual, for every element $a \in A$ and integer $n \in \mathbb{N}$ we denote by a^n the multiplication $a \odot \cdots \odot a$ containing n-times the factor a and set $a^0 = \mathbf{1}$. Moreover, given a family $(\, a_i \in A \mid i \in [n] \,)$ of semiring elements a_i, we also use the *sum notation* $\sum_{i \in [n]} a_i = a_1 \oplus \cdots \oplus a_n$ to denote the *sum*, i.e., the result of the addition, of all elements of the family and the *product notation* $\prod_{i \in [n]} a_i = a_1 \odot \cdots \odot a_n$ to abbreviate the *product*, i.e., the result of the multiplication, of all elements of the family, where the order is determined by the total order $1 < \ldots < n$ on the index set. Note that $\sum_{i \in [0]} a_i = \mathbf{0}$ and $\prod_{i \in [0]} a_i = \mathbf{1}$. Finally, we will also use the sum notation with arbitrary index sets, given that only finitely many summands are non-zero, exploiting the fact that $\oplus$ is commutative, i.e., the order in which elements are summed up is irrelevant.

Important semirings are, for example,

- the *semiring of the non-negative integers* $\mathbb{N}_\infty = (\mathbb{N} \cup \{+\infty\}, +, \cdot, 0, 1)$ with the common operations of addition and multiplication extended to $(+\infty)$ as follows: for every element $a \in \mathbb{N}_+ \cup \{+\infty\}$ both $a + (+\infty) = (+\infty) + a = (+\infty)$ and $a \cdot (+\infty) = (+\infty) \cdot a = (+\infty)$,

- the *tropical semiring* $\mathbb{T} = (\mathbb{N} \cup \{+\infty\}, \min, +, (+\infty), 0)$ with minimum and addition both extended to $(+\infty)$ such that $(+\infty)$ is the unit element with respect to min and $+$ is the addition of the semiring $\mathbb{N}_\infty$,

- the *arctic semiring* $\mathbb{A} = (\mathbb{N} \cup \{-\infty\}, \max, +, (-\infty), 0)$ with maximum and addition both extended to $(-\infty)$ such that $(-\infty)$ is the unit element with respect to max and $+$ is the addition of the semiring $\mathbb{N}_\infty$,

- the *boolean semiring* $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ with the usual operations of disjunction and conjunction as addition and multiplication, respectively,

- the semiring $\mathbb{Z}_4 = (\{0,1,2,3\}, +, \cdot, 0, 1)$ with the following operation tables,

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

- the min-max *semiring over the reals* $\mathbb{R}_{\min,\max} = (\mathbb{R} \cup \{+\infty, -\infty\}, \min, \max, (+\infty), (-\infty))$ with the common minimum and maximum operation, and
- the *language semiring* $\mathbb{L}_S = (\mathcal{P}(S^*), \cup, \circ, \emptyset, \{\varepsilon\})$ for some alphabet S with set union as addition and concatenation of words, lifted to sets of words, as multiplication.

Several more examples of semirings can be found, e.g., in [HW98, Gol99]. For the sake of simplicity, we assume $\mathbf{0} \neq \mathbf{1}$ for all semirings we consider, i.e., we ignore the trivial semiring with the singleton carrier set.

Semirings having a finite carrier set are called *finite* semirings, and a *commutative* semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is defined to be a semiring, in which the multiplicative monoid $(A, \odot, \mathbf{1})$ is commutative. Moreover, the semiring $\mathcal{A}$ is called *additively idempotent*, if the addition fulfils $\mathbf{1} \oplus \mathbf{1} = \mathbf{1}$. In additively idempotent semirings we immediately have $a \oplus a = a$ for every element $a \in A$ by distributivity.

Finally, the semiring $\mathcal{A}$ is called *multiplicatively periodic*, if for every element $a \in A$ there exist non-negative integers $i, j \in \mathbb{N}$ such that $i < j$ and $a^i = a^j$. Any difference $j - i$ for such integers i and j is called *a period of* the element a. The smallest period of a is usually called *the period of* a. Consequently, the semiring $\mathcal{A}$ is called *multiplicatively non-periodic*, if it is not multiplicatively periodic. Note that a multiplicatively non-periodic semiring is necessarily infinite, and a finite semiring must apparently be multiplicatively periodic.

Table 1 summarizes the properties of each of the above mentioned important semirings, where S is an arbitrary non-trivial, i.e., $\mathrm{card}(S) > 1$, alphabet. If S is a singleton, then the semiring $\mathbb{L}_S$ is commutative.

semiring	commutative	additively idempotent	finite	multiplicatively periodic
$\mathbb{N}_\infty$	yes	no	no	no
$\mathbb{A}$	yes	yes	no	no
$\mathbb{T}$	yes	yes	no	no
$\mathbb{B}$	yes	yes	yes	yes
$\mathbb{Z}_4$	yes	no	yes	yes
$\mathbb{R}_{\min,\max}$	yes	yes	no	yes
$\mathbb{L}_S$	no	yes	no	no

Table 1: Various semirings and their properties.

Now we consider semirings of which the carrier set is a partially ordered set. In principle, we could allow any partial order on the carrier set, but in our context it is more useful to consider a partial order such that the operations of the semiring are actually order-preserving. Thus, given a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ and a partial order $\preceq \subseteq A \times A$, we say that $\preceq$ *partially orders* $\mathcal{A}$ or, equivalently, the semiring $\mathcal{A}$ is *partially ordered (by $\preceq$)*, if the following two conditions are satisfied for every four elements $a_1, a_2, b_1, b_2 \in A$:

(OP$\oplus$) if $a_1 \preceq a_2$ and $b_1 \preceq b_2$, then the inequality $a_1 \oplus b_1 \preceq a_2 \oplus b_2$ holds and

(OP$\odot$) if $a_1 \preceq a_2$ and $b_1 \preceq b_2$, then we have $a_1 \odot b_1 \preceq a_2 \odot b_2$.

The literature contains several different notions of partially ordered semirings. For example, in [HW98, Gol99] order-preservation with respect to the multiplication (OP$\odot$) is only demanded for multiplication with *positive* elements $a \in A$, i.e., $\mathbf{0} \preceq a$. However, the positive elements of such a partially ordered semiring form a partially ordered sub-semiring according to our definition. In contrast, the definition of partially ordered semirings in [Kui97], additionally, requires every element to be positive.

In the sequel we will denote a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ partially ordered by $\preceq$ simply by $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ and call it *totally ordered*, if $\preceq$ is a total order. The set $P_{\mathcal{A}} = \{ a \in A \mid \mathbf{0} \preceq a \}$ is called the *(additive) positive cone* of the semiring $\mathcal{A}$ (via $\preceq$), whereas $N_{\mathcal{A}} = \{ a \in A \mid a \preceq \mathbf{0} \}$ is defined to be the *(additive) negative cone* of the semiring $\mathcal{A}$ (note $P_{\mathcal{A}} \cap N_{\mathcal{A}} = \{\mathbf{0}\}$). Moreover, we say that the partially ordered semiring $\mathcal{A}$ has the monotonicity property (MO$\oplus$), if

(MO$\oplus$) the inequality $\mathbf{1} \preceq \mathbf{1} \oplus \mathbf{1}$ holds.

Note that property (MO$\oplus$) implies that for every element $a \in A$ the inequality $a \preceq a \oplus a$ holds. Moreover, every additively idempotent and partially ordered semiring trivially satisfies property (MO$\oplus$). Finally, if every semiring element of a partially ordered semiring is comparable to its additive unit $\mathbf{0}$, then Observation 2.1 characterizes property (MO$\oplus$).

2.1 Observation (Property (MO$\oplus$) characterized)
Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ be a partially ordered semiring. If $A = P_{\mathcal{A}} \cup N_{\mathcal{A}}$, then the following statements are equivalent.

(i) The semiring $\mathcal{A}$ fulfils property (MO$\oplus$).

(ii) For every element $a \in N_{\mathcal{A}}$ the condition $a = a \oplus a$ holds.

(iii) The semiring $\mathcal{A}$ is additively idempotent or $A = P_{\mathcal{A}}$.

Proof. We prove the implications (i) implies (iii), (iii) implies (ii), and (ii) implies (i).

- (i) $\rightarrow$ (iii): By assumption $\mathbf{1} \preceq \mathbf{1} \oplus \mathbf{1}$. Since $\mathbf{1} \in (P_{\mathcal{A}} \cup N_{\mathcal{A}})$, either $\mathbf{1} \in P_{\mathcal{A}}$ or $\mathbf{1} \in N_{\mathcal{A}} \setminus \{\mathbf{0}\}$. In the former case $\mathbf{0} \preceq \mathbf{1}$ and, consequently, $\mathbf{0} \preceq a$ for every element $a \in A$ by (OP$\odot$). Thus $A = P_{\mathcal{A}}$. In the latter case $\mathbf{1} \prec \mathbf{0}$ and thereby also $\mathbf{1} \oplus \mathbf{1} \preceq \mathbf{1}$ by (OP$\oplus$). Together with the assumption $\mathbf{1} \preceq \mathbf{1} \oplus \mathbf{1}$ we have $\mathbf{1} \oplus \mathbf{1} = \mathbf{1}$ by antisymmetry. Hence $\mathcal{A}$ is additively idempotent.

- (iii) $\rightarrow$ (ii): Assume that $A = P_{\mathcal{A}}$. Then, since $P_{\mathcal{A}} \cap N_{\mathcal{A}} = \{\mathbf{0}\}$, it holds that $N_{\mathcal{A}} = \{\mathbf{0}\}$ and clearly $\mathbf{0} = \mathbf{0} \oplus \mathbf{0}$. On the other hand, if $\mathcal{A}$ is additively idempotent, then the statement is trivial.

- (ii) $\rightarrow$ (i): Since $A = P_{\mathcal{A}} \cup N_{\mathcal{A}}$, we either have $\mathbf{1} \in P_{\mathcal{A}}$ or $\mathbf{1} \in N_{\mathcal{A}} \setminus \{\mathbf{0}\}$. In the former case $\mathbf{0} \preceq \mathbf{1}$, thus $\mathbf{1} \preceq \mathbf{1} \oplus \mathbf{1}$ by (OP$\oplus$), and in the latter case $\mathbf{1} = \mathbf{1} \oplus \mathbf{1}$ by assumption. Hence the semiring fulfils property (MO$\oplus$). ■

We conclude that a totally ordered semiring possesses property (MO$\oplus$), if and only if the semiring is partially ordered according to the definition of [Kui97] or additively idempotent. The next observation groups together some simple statements concerning partially ordered semirings. Basically, the statements lift several conditions like (OP$\oplus$), (OP$\odot$), and (MO$\oplus$) from exactly two elements to several elements. The proofs are straightforward and therefore left to the reader.

2.2 Observation (Basic properties of partially ordered semirings)
Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ be a partially ordered semiring.

(i) Let $n \in \mathbb{N}$, and moreover, let $(\, a_i \in A \mid i \in [n] \,)$ and $(\, b_i \in A \mid i \in [n] \,)$ be two families of semiring elements such that for every index $i \in [n]$ the inequality $a_i \preceq b_i$ holds. Then $\sum_{i \in [n]} a_i \preceq \sum_{i \in [n]} b_i$ and $\prod_{i \in [n]} a_i \preceq \prod_{i \in [n]} b_i$.

(ii) Assume that the semiring $\mathcal{A}$ has property (MO$\oplus$), and let $a \in A$ be a semiring element. For every two integers $m, n \in \mathbb{N}_+$, if $m \leq n$, then we conclude $\sum_{i \in [m]} a \preceq \sum_{i \in [n]} a$. Note that $m = 0$ is excluded, because there may be an element $a \in A$ with $\mathbf{0} \not\preceq a$.

(iii) Let $b \in A$ be an element with $\mathbf{1} \preceq b$. For every two integers $m, n \in \mathbb{N}$ with $m \leq n$ the inequality $b^m \preceq b^n$ holds. □

Generalizing the usual total order $\leq$ on the non-negative integers, some semirings are partially ordered by a partial order defined in terms of the semiring addition. For instance, consider the relation $\sqsubseteq \subseteq A \times A$, defined for every two elements $a_1, a_2 \in A$ by $a_1 \sqsubseteq a_2$, if and only if there exists an element $a \in A$ such that $a_1 \oplus a = a_2$. The semiring $\mathcal{A}$ is said to be *naturally ordered*, if the relation $\sqsubseteq$ is a partial order (actually it suffices to show that $\sqsubseteq$ is antisymmetric). We will always write $\sqsubseteq$ for the natural order.

The next theorem of [Kui97] establishes that naturally ordered semirings are partially ordered by $\sqsubseteq$. In addition, they always fulfil property (MO$\oplus$) and have a directed carrier set.

2.3 Theorem (Theorem 2.1 of [Kui97])
Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a naturally ordered semiring.

(i) For every element $a \in A$ the condition $\mathbf{0} \sqsubseteq a$ holds.

(ii) Addition and multiplication are order-preserving (with respect to the natural order $\sqsubseteq$), i.e., (OP$\oplus$) and (OP$\odot$) are met. □

Thus, $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \sqsubseteq)$ is a partially ordered semiring with property (MO$\oplus$), which follows from Observation 2.1, and directed carrier set by Observation 2.4. Note that a similar statement can be derived for additively idempotent semirings. Namely, an additively idempotent semiring induces a $\vee$-semilattice partially ordering the semiring (e.g., [Wec92]).

2.4 Observation (Semirings and directedness)
Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ be a partially ordered semiring with $A = P_{\mathcal{A}} \cup N_{\mathcal{A}}$. Then the carrier set A is directed.

Proof. Let $S \subseteq A$ be a finite and non-empty subset. Further, let $u = \sum \{ a \mid a \in S \cap P_{\mathcal{A}} \}$, which is a finite sum. We prove $u \in \uparrow S$, which implies that the carrier set A is directed. Apparently, $\mathbf{0} \preceq u$ by Observation 2.2(i). Hence $a \preceq u$ for every negative element $a \in S \cap N_{\mathcal{A}}$. Moreover, $a \preceq u$ for every positive element $a \in S \cap P_{\mathcal{A}}$ by Observation 2.2(i). Consequently, by $A = P_{\mathcal{A}} \cup N_{\mathcal{A}}$ we conclude $a \preceq u$ for every $a \in S$; hence $u \in \uparrow S$. ■

Table 2 displays the order-related properties of the introduced semirings. Therein, column 3 identifies a partial order, which partially orders the semiring, and the properties to the right of column 3 are relative to that particular partial order.

semiring $\mathcal{A}$	naturally ordered	partially ordered	totally ordered	property (MO$\oplus$)	$P_{\mathcal{A}}$	$N_{\mathcal{A}}$
$\mathbb{N}_\infty$	yes	yes (by $\leq$)	yes	yes	$\mathbb{N} \cup \{+\infty\}$	$\{0\}$
$\mathbb{A}$	yes	yes (by $\leq$)	yes	yes	$\mathbb{N} \cup \{-\infty\}$	$\{-\infty\}$
$\mathbb{T}$	yes	yes (by $\sqsubseteq$)	yes	yes	$\mathbb{N} \cup \{+\infty\}$	$\{+\infty\}$
$\mathbb{T}$	yes	yes (by $\leq$)	yes	yes	$\{+\infty\}$	$\mathbb{N} \cup \{+\infty\}$
$\mathbb{B}$	yes	yes (by $\sqsubseteq$)	yes	yes	$\{0, 1\}$	$\{0\}$
$\mathbb{Z}_4$	no	no				
$\mathbb{R}_{\min,\max}$	yes	yes (by $\sqsubseteq$)	yes	yes	$\mathbb{R} \cup \{+\infty, -\infty\}$	$\{+\infty\}$
$\mathbb{L}_S$	yes	yes (by $\subseteq$)	no	yes	$\mathcal{P}(S^*)$	$\{\emptyset\}$

Table 2: Various semirings and their order-related properties.

Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ be a partially ordered semiring. In the sequel we will often consider partially ordered semirings with property (MO$\oplus$) and, in some cases, directed carrier set. Thus, Table 3 summarizes the relation between other properties (naturally ordered, totally ordered, and additively idempotent) and the aforementioned ones. For example, if the semiring $\mathcal{A}$ is totally ordered, then the carrier set A is directed, while, in general, it cannot be concluded that the semiring $\mathcal{A}$ has property (MO$\oplus$). The counterexamples which illustrate the two negative statements of Table 3 are as follows.

Property ...	implies property (MO$\oplus$)	implies directedness of the carrier set A
naturally ordered ($\preceq = \sqsubseteq$)	**yes** (cf. Thm. 2.3)	**yes** (cf. Thm. 2.3)
additively idempotent	**yes**	**no**
additively idempotent and $A = P_{\mathcal{A}} \cup N_{\mathcal{A}}$	**yes**	**yes** (cf. Obs. 2.4)
totally ordered	**no**	**yes**
totally ordered and $A = P_{\mathcal{A}}$	**yes** (cf. Obs. 2.1)	**yes**

Table 3: Relating various properties of a partially ordered semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$.

- The semiring $\mathbb{B}$ with the trivial partial order $=$ is additively idempotent, but the carrier set is not directed, because $\uparrow\{0, 1\} = \emptyset$.
- The semiring $(\{0, 1, 2\}, +, \cdot, 0, 1, \preceq)$ completely determined by $1 + 1 = 2 + 1 = 2 + 2 = 2$ and $2 \cdot 2 = 2$ (note that the remaining cases are such that 0 and 1 are unit elements with respect to addition and multiplication, respectively, and 0 is absorbing), is totally ordered by $2 \prec 1 \prec 0$, but $1 + 1 \prec 1$.

2.4 Formal tree series

Let Δ be a ranked alphabet and $V \subseteq X$ be a subset of variables. Every mapping $\varphi : T_\Delta(V) \longrightarrow A$ from Δ-trees indexed by V into a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is called *(formal) tree series (over Δ, V, and A)*. We use $A\langle\!\langle T_\Delta(V)\rangle\!\rangle$ to denote the set of all formal tree series over Δ, V, and A. Given a tree $t \in T_\Delta(V)$, we usually write (φ, t), termed the *coefficient* of t, instead of $\varphi(t)$ and $\sum_{t \in T_\Delta(V)} (\varphi, t)\, t$ instead of the tree series φ. For example,

$$\sum_{t \in T_\Delta(V)} \text{height}(t)\, t$$

is the tree series, which associates to every tree its height.

The *support* of a tree series $\varphi \in A\langle\!\langle T_\Delta(V)\rangle\!\rangle$ is the set $\text{supp}(\varphi) = \{\, t \in T_\Delta(V) \mid (\varphi, t) \neq \mathbf{0} \,\}$. Whenever $\text{supp}(\varphi)$ is a singleton, φ is said to be a *monomial*, and φ is said to be *polynomial*, if $\text{supp}(\varphi)$ is finite. The set of all polynomial formal tree series (over Δ, V, and A) is denoted by $A\langle T_\Delta(V)\rangle$. Moreover, if there is an element $a \in A$ such that for every tree $t \in T_\Delta(V)$ the coefficient $(\varphi, t) = a$ is constant, then the tree series φ is said to be *constant*, and we use $\widetilde{a}$ to abbreviate such a tree series φ.

Tree substitution can be generalized to tree languages [ES77, ES78] as well as tree series. Following the IO-substitution approach, the common definition of tree series substitution found, for example, in [EFV02] lets $n \in \mathbb{N}$ be an integer, $\varphi \in A\langle\!\langle T_\Delta(X_n)\rangle\!\rangle$ be a tree series, and $(\psi_1, \ldots, \psi_n) \in A\langle\!\langle T_\Delta(V)\rangle\!\rangle^n$ be an n-tuple of tree series. *(Pure) substitution of* the tuple $(\psi_1, \ldots, \psi_n)$ *into* the tree series φ, denoted by $\varphi \longleftarrow (\psi_1, \ldots, \psi_n)$, is then defined by

$$\varphi \longleftarrow (\psi_1, \ldots, \psi_n) = \sum_{\substack{t \in \text{supp}(\varphi),\\ (\forall i \in [n]) \colon t_i \in \text{supp}(\psi_i)}} \Big((\varphi, t) \odot \prod_{i \in [n]} (\psi_i, t_i) \Big)\, t[t_1, \ldots, t_n]. \tag{1}$$

Irrespective of the number of occurrences of a formal variable x_i for some $i \in [n]$, the coefficient (ψ_i, t_i) is taken into account exactly once, even if the variable does not appear at all in the tree t. This particularity led to the introduction of a different notion of substitution defined in [FV03] as follows.

$$\varphi \overset{o}{\longleftarrow} (\psi_1, \ldots, \psi_n) = \sum_{\substack{t \in \text{supp}(\varphi),\\ (\forall i \in [n]) \colon t_i \in \text{supp}(\psi_i)}} \Big((\varphi, t) \odot \prod_{i \in [n]} (\psi_i, t_i)^{|t|_{x_i}} \Big)\, t[t_1, \ldots, t_n]. \tag{2}$$

This notion of substitution, called *o-substitution*, takes the coefficient (ψ_i, t_i) into account as often as the corresponding formal variable x_i appears in the tree t. The next proposition lists some properties common to both types of substitution. In particular, consider the third property in case the modifier mod is o, i.e., o-substitution is used.

2.5 Proposition (Proposition 3.4 of [FV03])
Let $n \in \mathbb{N}$ be an integer and $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring. Moreover, let $\varphi \in A\langle\langle T_\Delta(X_n)\rangle\rangle$ and $\psi_1, \ldots, \psi_n \in A\langle\langle T_\Delta(V)\rangle\rangle$ for some $V \subseteq X$. Then for every modifier mod $\in \{\varepsilon, o\}$ and index $i \in [n]$ we have

(i) $\varphi \overset{\text{mod}}{\longleftarrow} () = \varphi$,

(ii) $\widetilde{\mathbf{0}} \overset{\text{mod}}{\longleftarrow} (\psi_1, \ldots, \psi_n) = \widetilde{\mathbf{0}}$, and

(iii) $\varphi \overset{\text{mod}}{\longleftarrow} (\psi_1, \ldots, \psi_{i-1}, \widetilde{\mathbf{0}}, \psi_{i+1}, \ldots, \psi_n) = \widetilde{\mathbf{0}}$. □

Finally, in [Kui99] a notion of substitution based on the OI-substitution approach [ES77, ES78] is introduced. There the number of occurrences of a certain formal variable is taken into account as well. However, in this paper we will exclusively deal with the IO-substitution approach.

3 Tree series transducers

In this section we recall the notions of bottom-up and top-down tree series transducers from [EFV02]. Figure 1 attempts to display the automata and transducer concepts subsumed by tree series transducers. Roughly speaking, moving upwards-left adds weights (costs or multiplicity) to the current model, moving upwards-right performs the generalization from strings to trees, and finally, moving left-to-right adds an output component to the current model.

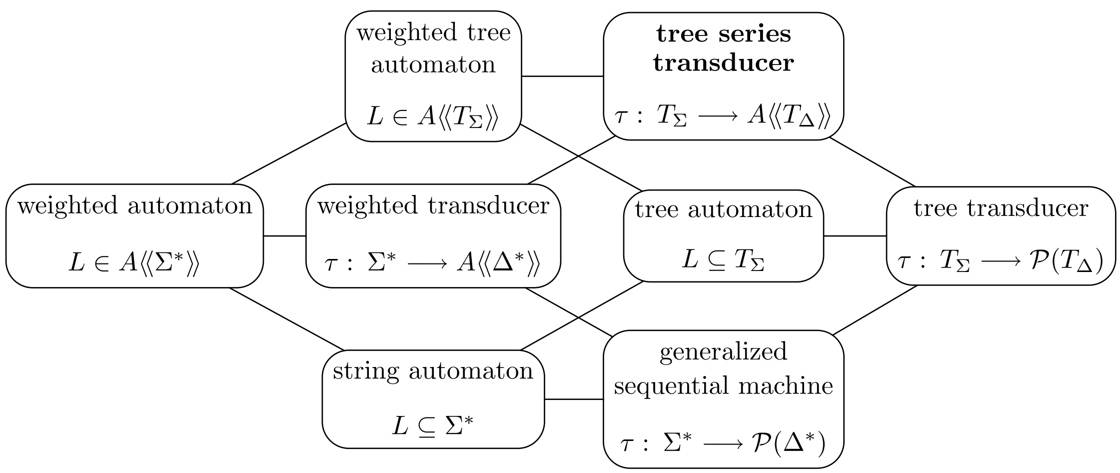

Figure 1: Generalization hierarchy

Before we proceed with the definition of tree series transducers, we recall some basic notions concerning matrices. Let I and J be countable *index sets* and let S be a set of *entries*. An $(I \times J)$-*matrix over* S is a mapping $M : I \times J \longrightarrow S$. The set of all matrices over S with indices of $I \times J$ is denoted by $S^{I \times J}$. The element $M(i, j)$ is called the (i, j)-*entry* of the matrix M and also written as $M_{i,j}$. If it is understood that the matrix M is a *row-vector* or *column-vector* (i.e., I or J is a singleton set, respectively), then we generally omit the element of the singleton set when indexing elements of the matrix M. Accordingly, we write, for example, M^I instead of $M^{I \times \{1\}}$, whenever we do not want to stress that the matrix M is a column-vector.

Next we define tree representations which encode the transitions and output trees of tree series transducers. Let Q be a finite set representing the state set of a tree series transducer. For every subset $V \subseteq X$ we abbreviate the set $\{\, q(v) \mid q \in Q, v \in V \,\}$ by $Q(V)$. Roughly speaking, the tree representation is a family of mappings, each of which maps an input symbol to a matrix indexed by states (more formally, by a state and an element of $Q(X)^*$). The entries of those matrices are tree series over Δ, X, and A, where Δ is the output ranked alphabet and A is the carrier set of some semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$.

3.1 Definition (Tree representation)
Given a finite set Q of states, input and output ranked alphabet Σ and Δ, respectively, and a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, we define a *tree representation (over Q, Σ, Δ, and A)* to be a family $(\,\mu_k \mid k \in \mathbb{N}\,)$ of mappings, where for every $k \in \mathbb{N}$ the mapping μ_k has type

$$\mu_k : \Sigma^{(k)} \longrightarrow A\langle\langle T_\Delta(X)\rangle\rangle^{Q \times Q(X_k)^*},$$

and for every $\sigma \in \Sigma^{(k)}$ we have $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$ for only finitely many indices $(q, w) \in Q \times Q(X_k)^*$. Moreover, for every index $(q, w) \in Q \times Q(X_k)^*$ we demand $\mu_k(\sigma)_{q,w} \in A\langle\langle T_\Delta(X_{|w|})\rangle\rangle^{Q \times Q(X_k)^{|w|}}$.

- The tree representation μ is called *bottom-up*, if for every integer $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and index $(q, w) \in Q \times Q(X_k)^*$ with $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$ we have $w = q_1(x_1) \ldots q_k(x_k)$ for some states $q_1, \ldots, q_k \in Q$.
- The tree representation μ is called *top-down*, if for every integer $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and index $(q, w) \in Q \times Q(X_k)^*$ we have $\operatorname{supp}(\mu_k(\sigma)_{q,w}) \subseteq \widehat{T_\Delta}(X_{|w|})$.

Finally, if for every $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and index $(q, w) \in Q \times Q(X_k)^*$ the tree series $\mu_k(\sigma)_{q,w}$ is polynomial, i.e., $\mu_k(\sigma)_{q,w} \in A\langle T_\Delta(X)\rangle$, then the tree representation μ is called *polynomial*. □

Note that polynomial tree representations are finitely representable, due to the finiteness of the input ranked alphabet Σ and the fact that for every $k \in \mathbb{N}$ almost all entries in the matrices in the range of the mappings μ_k are the constant zero tree series $\widetilde{\mathbf{0}}$. A tree series transducer is now basically just a tree representation together with supportive information about the state set Q, the input and output ranked alphabet, and the semiring. Additionally, we distinguish certain states, which will be called designated states. Depending on the mode of traversing the input, these might be initial or final states.

3.2 Definition (Tree series transducer)
A *tree series transducer (over Σ and Δ)* is defined as a six-tuple $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$, where

- Q and $D \subseteq Q$ are non-empty, finite sets of *states* and *designated states*, respectively,
- Σ and Δ are the *input* and *output ranked alphabet*, respectively; both disjoint to Q;
- $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring, and
- μ is a tree representation over Q, Σ, Δ, and A.

The tree series transducer M inherits the properties bottom-up, top-down, and polynomial from its tree representation, i.e., M is called *bottom-up* (*top-down*, *polynomial*), if the tree representation μ is bottom-up (top-down, polynomial). In case of a bottom-up (top-down) tree series transducer, the set D of designated states is also called set of *final states* (*initial states*). □

For the rest of the paper we only consider polynomial tree series transducers which are bottom-up or top-down. For an investigation of general tree series transducers, we refer the reader to [EFV02, FV03]. In order to have a concise notation, we drop the variables from the second index component in the tree representation of a bottom-up tree representation, e.g., we write $\mu_k(\sigma)_{q,(q_1,\ldots,q_k)}$ instead of $\mu_k(\sigma)_{q,(q_1(x_1),\ldots,q_k(x_k))}$, if μ is a bottom-up tree representation.

3.3 Definition (Subclasses of bottom-up tree series transducers)
Let $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a bottom-up tree series transducer. We say that the bottom-up tree series transducer M is

- *deterministic*, if for every $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and k states $q_1, \ldots, q_k \in Q$ there exists at most one state $q \in Q$ such that $\mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \neq \widetilde{\mathbf{0}}$, and for such a state $q \in Q$ the cardinality of the set $\mathrm{supp}(\mu_k(\sigma)_{q,(q_1,\ldots,q_k)})$ is at most one,

- *total*, if $F = Q$ and for every $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and k states $q_1, \ldots, q_k \in Q$ there exists at least one state $q \in Q$ such that $\mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \neq \widetilde{\mathbf{0}}$,

- and a *homomorphism*, if it is total and deterministic with a singleton set Q. □

Similarly these concepts (of determinism, totality, and homomorphism) can be defined for top-down tree series transducers.

3.4 Definition (Subclasses of top-down tree series transducers)
Let $M = (Q, \Sigma, \Delta, \mathcal{A}, I, \mu)$ be a top-down tree series transducer. We say that the top-down tree series transducer M is

- *deterministic*, if the set I of initial states is a singleton, for every $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and state $q \in Q$ there exists at most one word $w \in Q(X_k)^*$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$, and for such a word $w \in Q(X_k)^*$ the cardinality of the set $\mathrm{supp}(\mu_k(\sigma)_{q,w})$ is at most one,

- *total*, if for every $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and state $q \in Q$ there exists at least one word $w \in Q(X_k)^*$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$,

- and a *homomorphism*, if it is total and deterministic with a singleton set Q. □

Note that a deterministic tree series transducer is necessarily polynomial. Finally, we should assign a formal semantics to polynomial tree series transducers. In fact, we define two different semantics; namely, we define the tree-to-tree-series transformation and the o-tree-to-tree-series transformation computed by a polynomial tree series transducer. Both are defined in the very same manner; the only difference is the type of substitution used.

3.5 Definition (Semantics of polynomial tree series transducers)
Let $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ be a polynomial tree series transducer over semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$.

(i) For every modifier $\mathrm{mod} \in \{\varepsilon, o\}$, integer $k \in \mathbb{N}$, and input symbol $\sigma \in \Sigma^{(k)}$ the tree representation μ induces a mapping

$$\overline{\mu_k(\sigma)}^{\mathrm{mod}} : \left(A\langle\langle T_\Delta\rangle\rangle^Q\right)^k \longrightarrow A\langle\langle T_\Delta\rangle\rangle^Q$$

defined componentwise for every state $q \in Q$ and k vectors $R_1, \ldots, R_k \in A\langle\langle T_\Delta\rangle\rangle^Q$ by

$$\overline{\mu_k(\sigma)}^{\mathrm{mod}}(R_1, \ldots, R_k)_q = \sum_{\substack{w \in Q(X_k)^*,\\ w=(q_1(x_{i_1}),\ldots,q_l(x_{i_l}))}} \mu_k(\sigma)_{q,w} \overset{\mathrm{mod}}{\longleftarrow} \left((R_{i_1})_{q_1}, \ldots, (R_{i_l})_{q_l}\right).$$

Note that

$$\left(A\langle\langle T_\Delta\rangle\rangle^Q, \left(\overline{\mu_k(\sigma)}^{\mathrm{mod}} \;\middle|\; k \in \mathbb{N}, \sigma \in \Sigma^{(k)}\right)\right)$$

defines a Σ-algebra, and T_Σ is the free Σ-algebra. Thus there exists a unique homomorphism $h_\mu^{\mathrm{mod}} : T_\Sigma \longrightarrow A\langle\langle T_\Delta\rangle\rangle^Q$, which is defined for every $k \in \mathbb{N}$, input symbol $\sigma \in \Sigma^{(k)}$, and k trees $s_1, \ldots, s_k \in T_\Sigma$ by

$$h_\mu^{\mathrm{mod}}(\sigma(s_1, \ldots, s_k)) = \overline{\mu_k(\sigma)}^{\mathrm{mod}}(h_\mu^{\mathrm{mod}}(s_1), \ldots, h_\mu^{\mathrm{mod}}(s_k)).$$

(ii) The mod-*tree-to-tree-series transformation*, abbreviated mod-t-ts transformation, computed by the polynomial tree series transducer M is the mapping $\tau_M^{\mathrm{mod}} : T_\Sigma \longrightarrow A\langle\langle T_\Delta\rangle\rangle$ specified for every input tree $s \in T_\Sigma$ by $\tau_M^{\mathrm{mod}}(s) = \sum_{q \in D} h_\mu^{\mathrm{mod}}(s)_q$. □

3.6 Example (Height bottom-up tree series transducer)
The bottom-up tree series transducer $M = (\{*\}, \Sigma, \Delta, \mathbb{A}, \{*\}, \mu)$ over the arctic semiring with input ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, output ranked alphabet $\Delta = \{\alpha^{(0)}\}$, and tree representation μ defined by

$$\mu_2(\sigma)_{*,(*,*)} = \max(1\, x_1\;,\; 1\, x_2) \qquad \text{and} \qquad \mu_0(\alpha)_{*,\varepsilon} = 1\,\alpha$$

is total and polynomial, but not deterministic and, consequently, no homomorphism. For every input tree $s \in T_\Sigma$ the o-t-ts transformation computed by M is $\tau^o_M(s) = \mathrm{height}(s)\,\alpha$. To illustrate the previous definition, we prove this property by structural induction over the input tree $s \in T_\Sigma$.

Induction base: Let the input tree be $s = \alpha$. Then

$$\begin{aligned}
\tau^o_M(\alpha) &\overset{\text{Def. 3.5(ii)}}{=} \sum_{q\in\{*\}} h^o_\mu(\alpha)_q \;=\; h^o_\mu(\alpha)_* \overset{\text{Def. 3.5(i)}}{=} (\overline{\mu_0(\alpha)}^{\,o}())_* \\
&\overset{\text{Def. 3.5(i)}}{=} \mu_0(\alpha)_{*,\varepsilon} \overset{o}{\longleftarrow} () \overset{\text{Prop. 2.5(i)}}{=} \mu_0(\alpha)_{*,\varepsilon} \;=\; 1\,\alpha \;=\; \mathrm{height}(\alpha)\,\alpha.
\end{aligned}$$

Induction step: Let the input tree be $s = \sigma(s_1, s_2)$ for some input trees $s_1, s_2 \in T_\Sigma$. Note that $a^0 = 0$ in the arctic semiring. Further, recall that Equation (2) refers to the defining equation of o-substitution found in Subsection 2.4. We compute

$$\begin{aligned}
\tau^o_M(\sigma(s_1,s_2)) &\overset{\text{Def. 3.5(ii)}}{=} \sum_{q\in\{*\}} h^o_\mu(\sigma(s_1,s_2))_q \;=\; h^o_\mu(\sigma(s_1,s_2))_* \\
&\overset{\text{Def. 3.5(i)}}{=} (\overline{\mu_2(\sigma)}^{\,o}(h^o_\mu(s_1), h^o_\mu(s_2)))_* \\
&\overset{\text{Def. 3.5(i)}}{=} \sum_{(q_1,q_2)\in\{*\}^2} \mu_2(\sigma)_{*,(q_1,q_2)} \overset{o}{\longleftarrow} (h^o_\mu(s_1)_{q_1}, h^o_\mu(s_2)_{q_2}) \\
&= \mu_2(\sigma)_{*,(*,*)} \overset{o}{\longleftarrow} (h^o_\mu(s_1)_*, h^o_\mu(s_2)_*) \\
&= (\max(1\,x_1, 1\,x_2)) \overset{o}{\longleftarrow} (\sum_{q\in\{*\}} h^o_\mu(s_1)_q, \sum_{q\in\{*\}} h^o_\mu(s_2)_q) \\
&\overset{\text{Def. 3.5(ii)}}{=} (\max(1\,x_1, 1\,x_2)) \overset{o}{\longleftarrow} (\tau^o_M(s_1), \tau^o_M(s_2)) \\
&\overset{\text{I.H.}}{=} (\max(1\,x_1, 1\,x_2)) \overset{o}{\longleftarrow} (\mathrm{height}(s_1)\,\alpha\;,\; \mathrm{height}(s_2)\,\alpha) \\
&\overset{\text{Eq. (2)}}{=} \max(1 + \mathrm{height}(s_1)^1 + \mathrm{height}(s_2)^0, 1 + \mathrm{height}(s_1)^0 + \mathrm{height}(s_2)^1)\,\alpha \\
&= \max(1 + \mathrm{height}(s_1), 1 + \mathrm{height}(s_2))\,\alpha \\
&= (1 + \max(\mathrm{height}(s_1), \mathrm{height}(s_2)))\,\alpha \;=\; \mathrm{height}(\sigma(s_1,s_2))\,\alpha.
\end{aligned}$$

Note that the facts $\mathrm{supp}(\max(1\,x_1,\, 1\,x_2)) = \{x_1, x_2\}$ and $\mathrm{supp}(\tau^o_M(s_1)) = \mathrm{supp}(\tau^o_M(s_2)) = \{\alpha\}$ are used in the line, where Equation (2) was applied. □

In the sequel we will be interested in a comparison of the computational power of subclasses of bottom-up and top-down tree series transducers. More precisely, to every class of restricted bottom-up or top-down tree series transducers, where the restriction is characterized by some of the properties of Definition 3.3 and Definition 3.4, we associate the class of all mod-t-ts transformations computed by such tree series transducers. Then we compare such classes of mod-t-ts transformations by means of set inclusion. The next definition establishes shorthands for such classes of mod-t-ts transformations also taking the two different notions of substitution into account.

3.7 Definition (Classes of mod-t-ts transformations)
Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring and $\mathrm{mod} \in \{\varepsilon, o\}$. The class of all mappings

$$(\,\tau : T_\Sigma \longrightarrow A\langle\!\langle T_\Delta\rangle\!\rangle \mid \Sigma, \Delta \text{ ranked alphabets }\,)$$

of mod-t-ts transformations computed by polynomial bottom-up tree series transducers over the semiring $\mathcal{A}$ is denoted by p–BOT$^{\mathrm{mod}}(\mathcal{A})$. Moreover, the class d–BOT$^{\mathrm{mod}}(\mathcal{A})$ (t–BOT$^{\mathrm{mod}}(\mathcal{A})$, dt–BOT$^{\mathrm{mod}}(\mathcal{A})$, h–BOT$^{\mathrm{mod}}(\mathcal{A})$) is defined to be the class of all mod-t-ts transformations computed by deterministic (total, deterministic and total, homomorphism) bottom-up tree series transducers over $\mathcal{A}$.

Likewise, we use the classes p–TOP$^{\mathrm{mod}}(\mathcal{A})$, d–TOP$^{\mathrm{mod}}(\mathcal{A})$, t–TOP$^{\mathrm{mod}}(\mathcal{A})$, dt–TOP$^{\mathrm{mod}}(\mathcal{A})$, and h–TOP$^{\mathrm{mod}}(\mathcal{A})$ to stand for the corresponding classes of mod-t-ts transformations computed by top-down tree series transducers over $\mathcal{A}$. □

3.8 Theorem (Lemma 5.1 and Theorem 5.2 of [FV03])

(i) Let $M = (Q, \Sigma, \Delta, \mathcal{A}, I, \mu)$ be a polynomial top-down tree series transducer. Then for every input tree $s \in T_\Sigma$ and state $q \in Q$ we have $h_\mu(s)_q = h_\mu^o(s)_q$.

(ii) For every $x \in \{\mathrm{p}, \mathrm{d}, \mathrm{t}, \mathrm{dt}, \mathrm{h}\}$ the equality x–TOP$(\mathcal{A}) = x$–TOP$^o(\mathcal{A})$ holds. □

Next we recall a property of deterministic tree series transducers which are bottom-up or top-down. Roughly speaking, the addition of the underlying semiring is completely irrelevant concerning computations of a deterministic tree series transducer, i.e., all computations are performed in the multiplicative monoid of the semiring $\mathcal{A}$. Formally speaking, the proposition shows that the conditions imposed on the tree representation μ of a deterministic tree series transducer can be lifted to the level of the homomorphic extension h_μ^{mod} for any modifier mod $\in \{\varepsilon, o\}$.

3.9 Proposition (Proposition 3.12 of [EFV02])
Let $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ be a deterministic (bottom-up or top-down) tree series transducer and mod $\in \{\varepsilon, o\}$. Then for every input tree $s \in T_\Sigma$ there exists at most one state $q \in Q$ such that $h_\mu^{\mathrm{mod}}(s)_q \neq \widetilde{\mathbf{0}}$, and if $h_\mu^{\mathrm{mod}}(s)_q \neq \widetilde{\mathbf{0}}$, then it is a monomial. Hence $\tau_M^{\mathrm{mod}}(s)$ is either $\widetilde{\mathbf{0}}$ or a monomial.

Proof. The proof of the statement concerning mod $= \varepsilon$ can be found in Proposition 3.12 of [EFV02], and the proof of the statement with mod $= o$ uses exactly the same argumentation. ■

Before we proceed with the next section, we explicitly exclude certain non-interesting tree series transducers. We will call a tree series transducer $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ *non-trivial*, if $\Sigma^{(0)} \neq \emptyset$ and there exist an integer $k \in \mathbb{N}$, an input symbol $\sigma \in \Sigma^{(k)}$, and a state $q \in Q$ such that $\mu_k(\sigma)_{q,\varepsilon} \neq \widetilde{\mathbf{0}}$. Hence, in particular, for bottom-up tree series transducers, non-triviality implies that there exists a nullary input symbol $\alpha \in \Sigma^{(0)}$ satisfying the condition above. Moreover, the above definition implies $\Delta^{(0)} \neq \emptyset$ for every non-trivial tree series transducer M. Let M be a trivial tree series transducer. Then the tree series $\tau_M^{\mathrm{mod}}(s)$ is the constant zero tree series, i.e., $\tau_M^{\mathrm{mod}}(s) = \widetilde{\mathbf{0}}$, for every input tree $s \in T_\Sigma$ and modifier mod $\in \{\varepsilon, o\}$. Since this particular case is not interesting, we will assume that all tree series transducers, which are considered in the rest of the paper, are non-trivial.

4 Coefficient majorization

Throughout the rest of the paper $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ will be a partially ordered semiring with property (MO⊕). Thus, for example, the semirings $\mathbb{N}_\infty$, $\mathbb{A}$, $\mathbb{T}$, $\mathbb{B}$, $\mathbb{R}_{\min,\max}$, and $\mathbb{L}_S$ for an alphabet S are suitable semirings, whereas $\mathbb{Z}_4$ is excluded. Moreover, $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ will always be a non-trivial polynomial tree series transducer over the semiring $\mathcal{A}$, which is bottom-up in Subsection 4.2 and top-down in Subsection 4.3. Finally, let mod $\in \{\varepsilon, o\}$.

4.1 The general approach

In this section we will approximate the cost of an output tree which is in the support of a tree series in the range of the mod-t-ts transformation computed by M. More precisely, we derive mappings, called coefficient majorizations, $f : \mathbb{N}_+ \longrightarrow A$ such that for every $n \in \mathbb{N}_+$ the approximation

$f(n) \in \uparrow C_M^{\mathrm{mod}}(n)$ holds, where the set $C_M^{\mathrm{mod}}(n) \subseteq A$ of *coefficients generated by M on input trees of height n* is defined to be

$$C_M^{\mathrm{mod}}(n) = \left\{ (h_\mu^{\mathrm{mod}}(s)_q, t) \;\middle|\; q \in Q, s \in T_\Sigma, \mathrm{height}(s) = n, t \in \mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_q) \right\}.$$

This gives rise to a property of polynomial tree series transducers. We will exploit this property in the sequel to reprove some recent results and also provide some insight into the relation between the two different modes of traversing the input tree, i.e., bottom-up and top-down, and the two types of substitution, i.e., pure and o-substitution, established in the previous section.

We start by defining some constants associated with the polynomial tree series transducer M. Later on they will provide the abstraction from a concrete tree series transducer used in our majorizations.

4.1 Definition (Constants associated with a polynomial tree series transducer)
We define the following constants representing basic facts of $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$:

- the *maximal rank* $r_M \in \mathbb{N}$ of M is defined by $r_M = \mathrm{mx}_\Sigma$,
- the number $d_M \in \mathbb{N}_+$ of *follow-up states* (or: *successor states*), is defined by

$$d_M = \begin{cases} 1 & \text{, if } M \text{ is deterministic} \\ \mathrm{card}(Q) & \text{, if } M \text{ is bottom-up and not deterministic ,} \\ \mathrm{card}(Q) \cdot r_M & \text{, otherwise} \end{cases}$$

- the *maximal support cardinality* $e_M \in \mathbb{N}$ is defined by

$$e_M = \max\left\{ \mathrm{card}(\mathrm{supp}(\mu_k(\sigma)_{q,w})) \;\middle|\; k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q, w \in Q(X_k)^* \right\},$$

- the *maximal variable degree* $u_{M,\mathrm{mod}} \in \mathbb{N}$ is defined by

$$u_{M,\mathrm{mod}} = \begin{cases} r_M & \text{, if mod} = \varepsilon \\ \max\limits_{\substack{k\in\mathbb{N}, \sigma\in\Sigma^{(k)}, q\in Q, \\ w\in Q(X_k)^*, t\in\mathrm{supp}(\mu_k(\sigma)_{q,w})}} \sum_{x\in X_{|w|}} |t|_x & \text{, if mod} = o \end{cases},$$

- and the *maximal length* $v_M \in \mathbb{N}$ of the second index in any matrix in the range of μ_k is defined by

$$v_M = \max\left\{ |w| \;\middle|\; k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q, w \in Q(X_k)^*, \mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}} \right\}.$$

□

Let us discuss those constants in some more detail. The constant r_M represents the maximal number of direct subtrees of any tree. Of course, this number coincides with the maximal rank of the input symbols. Next we consider a state $q \in Q$ and a word $w \in Q(X_k)^*$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$. The constant d_M represents the number of possible combinations for a single symbol of the word w. For a deterministic tree series transducer d_M is apparently 1 by Proposition 3.9. Given that M is bottom-up, we have only $\mathrm{card}(Q)$ choices for the state, because the variable of X_k is uniquely determined by the position in the word w. Finally, for polynomial top-down tree series transducers we have $\mathrm{card}(Q) \cdot k$ choices, which is bound from above by $\mathrm{card}(Q) \cdot r_M$.

The intention of the constant e_M is obvious. Since M is polynomial, the constant e_M is well-defined. Lastly, the constants $u_{M,\mathrm{mod}}$ and v_M fulfil a similar purpose. They both limit the number of factors representing subtree weights in any multiplication occurring due to Equation (1) or Equation (2). The bottom-up case is handled by the constant $u_{M,\mathrm{mod}}$, where according to Equation (1) at most r_M such factors occur and according to Equation (2) there are at most as many such factors as there are variables in the tree selected out of the tree representation. In the top-down case there is no difference between pure substitution and o-substitution. Thus, there are at most as many factors as the length of the longest word $w \in Q(X_k)^*$ with $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$.

Note that $u_{M,\mathrm{mod}}$ and v_M are well-defined; for the former we need that the tree series transducer M is polynomial. Additionally, $v_M = r_M$, if M is bottom-up, and $v_M = u_{M,o}$, if M is top-down.

4.2 Definition (Upper bound of the coefficients of μ)
An element $c \in A$ with $\mathbf{1} \preceq c$ is called an *upper bound of the coefficients of* μ, if

$$c \in \uparrow \Big(\{\mathbf{1}\} \cup \Big\{ (\mu_k(\sigma)_{q,w}, t) \;\Big|\; k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q, w \in Q(X_k)^*, t \in \operatorname{supp}(\mu_k(\sigma)_{q,w}) \Big\} \Big) . \qquad \square$$

Note that such a $c \in A$ need not exist in general. However, the existence of such a c can be assured, for example, by demanding that the carrier set of the semiring is directed. In the following, we will often assume an upper bound c of the coefficients of μ, and apparently, to obtain the best results, c should be chosen as small as possible; hence c should be the supremum of the coefficients occurring in μ, if it exists.

Next we introduce particular mappings, namely cardinality majorizations, sum majorizations, and coefficient majorizations. The first of which, i.e., cardinality majorizations, given a positive integer $n \in \mathbb{N}_+$ are supposed to limit the cardinality of the support of the tree series $h_\mu^{\mathrm{mod}}(s)_q$ for every state $q \in Q$ and input tree $s \in T_\Sigma$ of height n. We will see later that thereby it also limits the cardinality of the support of the tree series $\tau_M^{\mathrm{mod}}(s)$.

The second type of majorizations shall provide an upper bound of the n-fold sum of a semiring element $a \in A$. This mapping represents internal knowledge of the operations of the concrete semiring and is provided externally. Using the semiring of non-negative integers $\mathbb{N}_\infty$, this mapping might, for example, be $g(n, a) = n \cdot a$ for every positive integer $n \in \mathbb{N}_+$ and semiring element $a \in \mathbb{N} \cup \{+\infty\}$. Moreover, this mapping also allows to omit unnecessary details, for the mapping is only required to approximate the sum; it need not return the precise sum.

The final type, i.e., the coefficient majorizations, given $n \in \mathbb{N}_+$ are supposed to limit all non-zero coefficients generated by M on input trees of height n, i.e., a coefficient majorization f must fulfil $f(n) \in \uparrow C_M^{\mathrm{mod}}(n)$.

4.3 Definition (Several majorizations)

- Any mapping $l : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ satisfying for every $n \in \mathbb{N}_+$, input tree $s \in T_\Sigma$ of height n, and state $q \in Q$ the condition $\operatorname{card}(\operatorname{supp}(h_\mu^{\mathrm{mod}}(s)_q)) \leq l(n)$ is called *cardinality majorization (with respect to M)*.
- Any mapping $g : \mathbb{N}_+ \times A \longrightarrow A$ such that for every $n \in \mathbb{N}_+$ and semiring element $a \in A$ we have $\sum_{i \in [n]} a \preceq g(n, a)$ is called *sum majorization (with respect to $\mathcal{A}$)*.
- Any mapping $f : \mathbb{N}_+ \longrightarrow A$ satisfying for every $n \in \mathbb{N}_+$, input tree $s \in T_\Sigma$ of height n, state $q \in Q$, and output tree $t \in \operatorname{supp}(h_\mu^{\mathrm{mod}}(s)_q)$ the property $(h_\mu^{\mathrm{mod}}(s)_q, t) \preceq f(n)$ is called *coefficient majorization (with respect to M)*.

Finally, a mapping $f : \mathbb{N}_+ \longrightarrow A$ is said to be a *coefficient majorization for a class of* mod*-t-ts transformations* $\mathfrak{T} = (\, \tau_i : T_\Sigma \longrightarrow A\langle\langle T_\Delta \rangle\rangle \mid i \in I \,)$, if for every $\tau_i \in \mathfrak{T}$, input tree $s \in T_\Sigma$ of height $n \in \mathbb{N}_+$, and output tree $t \in \operatorname{supp}(\tau_i(s))$ the condition $(\tau_i(s), t) \preceq f(n)$ holds. $\square$

The careful reader might have noticed that we used the term majorization rather informally. At this stage we will show that the defined notions really are majorizations in the strict sense, but will continue to use the more liberal version frequently in the following. Roughly speaking, we call a mapping majorization, if it limits another mapping or family of mappings from above "using less information".

We start by exhibiting the family of mappings for which a cardinality majorization l is a majorization. Let $B_{M,s,q,\mathrm{mod}} : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ be the mapping defined for every input tree $s \in T_\Sigma$, state $q \in Q$, and integer $n \in \mathbb{N}_+$ by the following case analysis.

$$B_{M,s,q,\mathrm{mod}}(n) = \begin{cases} \operatorname{card}(\operatorname{supp}(h_\mu^{\mathrm{mod}}(s)_q)) & \text{, if } n = \operatorname{height}(s) \\ 1 & \text{, otherwise} \end{cases} .$$

Now it is easily seen that l is a cardinality majorization, if and only if l is a majorization for the mappings $B_{M,s,q,\mathrm{mod}}$. Sum majorizations are apparently majorizations, and, finally, a coefficient majorization f is a majorization for the following family of mappings. Let $C_{M,s,q,t,\mathrm{mod}} : \mathbb{N}_+ \longrightarrow A$ be a mapping defined for every input tree $s \in T_\Sigma$, state $q \in Q$, output tree $t \in T_\Delta$, and integer $n \in \mathbb{N}_+$ by the following case analysis.

$$C_{M,s,q,t,\mathrm{mod}}(n) = \begin{cases} (h_\mu^{\mathrm{mod}}(s)_q, t) & \text{, if } n = \mathrm{height}(s), t \in \mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_q) \\ f(n) & \text{, otherwise} \end{cases} .$$

Lastly, we call a sum majorization order-preserving, if it is order-preserving with respect to the point-wise order on $\mathbb{N}_+ \times A$, which is defined for every two integers $n_1, n_2 \in \mathbb{N}_+$ and two semiring elements $a_1, a_2 \in A$ by $(n_1, a_1) \preceq (n_2, a_2)$, if and only if $n_1 \leq n_2$ and $a_1 \preceq a_2$. We continue by providing an example for each of the above defined majorizations using our running example bottom-up tree series transducer of Example 3.6. We highlight that every majorization of the example is order-preserving; a property which will be required often in the sequel.

4.4 Example (Continuing Example 3.6)
Let M be the (non-trivial) polynomial bottom-up tree series transducer of Example 3.6 and let $\mathrm{mod} = o$. Recall that the arctic semiring $\mathbb{A}$ (over which M is defined) fulfils the general conditions of this section.

- The mapping $l : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ defined by $l(n) = 1$ for every positive integer $n \in \mathbb{N}_+$ is a cardinality majorization by $\mathrm{card}(T_\Delta) = 1$. Moreover, l trivially preserves the natural order on $\mathbb{N}_+$, i.e., $a \leq b$ for positive integers $a, b \in \mathbb{N}_+$ implies $l(a) \leq l(b)$.
- The mapping $g : \mathbb{N}_+ \times (\mathbb{N} \cup \{-\infty\}) \longrightarrow \mathbb{N} \cup \{-\infty\}$ defined by $g(n, a) = a$ for every $n \in \mathbb{N}_+$ and semiring element $a \in \mathbb{N} \cup \{-\infty\}$ is a sum majorization due to the additive idempotency of $\mathbb{A}$. Note that g is also order-preserving.
- The mapping $f : \mathbb{N}_+ \longrightarrow \mathbb{N} \cup \{-\infty\}$ defined by $f(n) = n$ is a coefficient majorization, which is easily derived from Example 3.6. □

Now we discuss the general approach used to derive a first coefficient majorization. Let $s \in T_\Sigma$ be an input tree. Using an order-preserving cardinality majorization l and an order-preserving sum majorization g, we can introduce the ample coefficient majorization associated with l, g, and c. The different modifiers, i.e., $\mathrm{mod} = \varepsilon$ or $\mathrm{mod} = o$, are taken care of by the maximal variable degree $u_{M,\mathrm{mod}}$ (cf. Definition 4.1) in case M is bottom-up, while the mod-t-ts transformations computed by top-down tree series transducers using on the one hand $\mathrm{mod} = \varepsilon$ and on the other hand $\mathrm{mod} = o$ do not differ, i.e., $\tau_M = \tau_M^o$ (cf. Lemma 5.1 of [FV03]).

Roughly speaking, if the input tree s has height 1, then every support element of $h_\mu^{\mathrm{mod}}(s)_q$ has cost at most c, where c is an upper bound of the coefficients of μ (cf. Definition 4.2). Given an input tree s of height $n + 1$, we first compute an upper bound of all subtrees of height at most n. Since those weights are multiplied in Equation (1) and Equation (2), we take the result of the recursive call to the $u_{M,\mathrm{mod}}$-th power, if M is bottom-up, and to the v_M-th power, if M is top-down. Recall that $u_{M,\mathrm{mod}}$ and v_M are defined such that they hold the maximal number of multiplications in any product generated by a substitution. The final factor is provided by the tree representation, and thus c again provides a suitable upper bound of this factor.

Finally, by substitution, equal trees might arise such that the costs of those are going to be summed up. The cardinality majorization l with the help of the sum majorization g is going to provide an upper bound of this sum as we will see in Theorem 4.7 and Theorem 4.15.

In the subsections to follow we distinguish the two modes of traversing the input tree, namely bottom-up and top-down. In particular, in the top-down subsection we will casually refer to the bottom-up subsection, because the derived majorizations generally have the same structure such that properties only depending on the structure neatly carry over to the top-down case.

4.2 The bottom-up case

Recall that in this subsection M is always a (non-trivial) polynomial bottom-up tree series transducer. Moreover, let l be an order-preserving cardinality majorization and g be an order-preserving sum majorization. Lastly, let c be an upper bound of the coefficients of μ (cf. Definition 4.2). According to the outline just presented, we obtain the following coefficient majorization.

4.5 Definition (Ample coefficient majorization)
For every $n \in \mathbb{N}_+$ we recursively define the *ample coefficient majorization* $f^{\text{bot}}_{M,\text{mod},g,l,c} : \mathbb{N}_+ \longrightarrow A$ *associated with* l, g, *and* c to be

$$f^{\text{bot}}_{M,\text{mod},g,l,c}(n) = \begin{cases} c & \text{, if } n = 1 \\ g((d_M)^{r_M} \cdot e_M \cdot l(n-1)^{r_M}, c \odot f^{\text{bot}}_{M,\text{mod},g,l,c}(n-1)^{u_{M,\text{mod}}}) & \text{, if } n > 1 \end{cases} .$$

□

Thus the ample coefficient majorization depends on a cardinality majorization l and a sum majorization g, both of which are order-preserving, the modifier mod, the upper bound c, and the polynomial bottom-up tree series transducer M (or more specifically: a few characteristics of M). Next we prove that the ample coefficient majorization associated with l, g, and c is order-preserving with respect to the total order on $\mathbb{N}_+$ and the partial order on the semiring.

4.6 Lemma (The ample coefficient majorization is order-preserving)
The ample coefficient majorization $f^{\text{bot}}_{M,\text{mod},g,l,c}$ associated with l, g, and c is order-preserving, and for every $n \in \mathbb{N}_+$ the condition $\mathbf{1} \preceq f^{\text{bot}}_{M,\text{mod},g,l,c}(n)$ holds.

Proof. We first analyze two different cases.

Case (i): Let $u_{M,\text{mod}} = 0$, then $f^{\text{bot}}_{M,\text{mod},g,l,c}(n)^{u_{M,\text{mod}}} = \mathbf{1}$. For every $n \in \mathbb{N}_+$ we get

$$f^{\text{bot}}_{M,\text{mod},g,l,c}(n) = \begin{cases} c & \text{, if } n = 1 \\ g((d_M)^{r_M} \cdot e_M \cdot l(n-1)^{r_M}, c) & \text{, if } n > 1 \end{cases} ,$$

which shows that $f^{\text{bot}}_{M,\text{mod},g,l,c}$ is order-preserving in this case, because l and g are order-preserving. Moreover, $\mathbf{1} \preceq f^{\text{bot}}_{M,\text{mod},g,l,c}(n)$ due to the facts $\mathbf{1} \preceq c$ and $a \preceq \sum_{i \in [k]} a \preceq g(k, a)$ for every $k \in \mathbb{N}_+$ and $a \in A$ by Observation 2.2(ii).

Case (ii): Let $u_{M,\text{mod}} \neq 0$. First we prove that $\mathbf{1} \preceq f^{\text{bot}}_{M,\text{mod},g,l,c}(n)$ for every $n \in \mathbb{N}_+$ using induction on the integer n.

Induction base: Let $n = 1$. Then $f^{\text{bot}}_{M,\text{mod},g,l,c}(n) = c$ and thus $\mathbf{1} \preceq f^{\text{bot}}_{M,\text{mod},g,l,c}(n)$ by Definition 4.2.

Induction step: Let $n > 1$. By induction hypothesis $\mathbf{1} \preceq f^{\text{bot}}_{M,\text{mod},g,l,c}(n-1)$ and, due to Observation 2.2(iii), also $\mathbf{1} \preceq f^{\text{bot}}_{M,\text{mod},g,l,c}(n-1)^{u_{M,\text{mod}}}$. Since $\mathbf{1} \preceq c$, we can also conclude $f^{\text{bot}}_{M,\text{mod},g,l,c}(n-1)^{u_{M,\text{mod}}} \preceq c \odot f^{\text{bot}}_{M,\text{mod},g,l,c}(n-1)^{u_{M,\text{mod}}}$, due to property (OP$\odot$). Finally, $((d_M)^{r_M} \cdot e_M \cdot l(n-1)^{r_M}) \in \mathbb{N}_+$ and, consequently, by Observation 2.2(ii)

$$\begin{aligned} \mathbf{1} &\preceq f^{\text{bot}}_{M,\text{mod},g,l,c}(n-1)^{u_{M,\text{mod}}} \preceq c \odot f^{\text{bot}}_{M,\text{mod},g,l,c}(n-1)^{u_{M,\text{mod}}} \\ &\preceq \sum_{i \in [(d_M)^{r_M} \cdot e_M \cdot l(n-1)^{r_M}]} c \odot f^{\text{bot}}_{M,\text{mod},g,l,c}(n-1)^{u_{M,\text{mod}}} \\ &\preceq g((d_M)^{r_M} \cdot e_M \cdot l(n-1)^{r_M}, c \odot f^{\text{bot}}_{M,\text{mod},g,l,c}(n-1)^{u_{M,\text{mod}}}) = f^{\text{bot}}_{M,\text{mod},g,l,c}(n), \end{aligned}$$

which proves the statement.

Next we prove order-preservation. Assume an arbitrary $n \in \mathbb{N}_+$. By Observation 2.2(iii) the property $f^{\text{bot}}_{M,\text{mod},g,l,c}(n) \preceq f^{\text{bot}}_{M,\text{mod},g,l,c}(n)^{u_{M,\text{mod}}}$ holds due to $\mathbf{1} \preceq f^{\text{bot}}_{M,\text{mod},g,l,c}(n)$. Using the argumentation found in the previous induction step we obtain

$$\begin{aligned}
f^{\text{bot}}_{M,\text{mod},g,l,c}(n) \;&\preceq\; f^{\text{bot}}_{M,\text{mod},g,l,c}(n)^{u_{M,\text{mod}}} \;\preceq\; c \odot f^{\text{bot}}_{M,\text{mod},g,l,c}(n)^{u_{M,\text{mod}}} \\
&\preceq\; \sum_{i \in [(d_M)^{r_M} \cdot e_M \cdot l(n)^{r_M}]} c \odot f^{\text{bot}}_{M,\text{mod},g,l,c}(n)^{u_{M,\text{mod}}} \\
&\preceq\; g((d_M)^{r_M} \cdot e_M \cdot l(n)^{r_M}, c \odot f^{\text{bot}}_{M,\text{mod},g,l,c}(n)^{u_{M,\text{mod}}}) \;=\; f^{\text{bot}}_{M,\text{mod},g,l,c}(n+1),
\end{aligned}$$

which completes the proof. ■

Finally, we can now provide the property that every ample coefficient majorization associated with an order-preserving cardinality majorization l, an order-preserving sum majorization g, and an upper bound c of the coefficients of μ really provides a coefficient majorization.

4.7 Theorem (Coefficient majorization)
The ample coefficient majorization $f^{\text{bot}}_{M,\text{mod},g,l,c}$ is indeed a coefficient majorization, i.e., for every integer $n \in \mathbb{N}_+$, state $q \in Q$, input tree $s \in T_\Sigma$ of height n, and output tree $t \in \text{supp}(h^{\text{mod}}_\mu(s)_q)$ we have

$$(h^{\text{mod}}_\mu(s)_q, t) \preceq f^{\text{bot}}_{M,\text{mod},g,l,c}(n)$$

and thus $f^{\text{bot}}_{M,\text{mod},g,l,c}(n) \in \uparrow C_M(n)$. Moreover, $(\tau^{\text{mod}}_M(s), t'') \preceq g(\text{card}(Q), f^{\text{bot}}_{M,\text{mod},g,l,c}(n))$ for every $t'' \in \text{supp}(\tau^{\text{mod}}_M(s))$.

Proof. Recall the constants r_M, d_M, e_M, and $u_{M,\text{mod}}$ of Definition 4.1. We now prove the first statement by structural induction over the input tree $s \in T_\Sigma$.

Induction base: Let $\alpha \in \Sigma^{(0)}$ be an input symbol and $s = \alpha$ be the input tree of height 1. Since $t \in \text{supp}(h^{\text{mod}}_\mu(\alpha)_q)$, we have

$$(h^{\text{mod}}_\mu(\alpha)_q, t) \overset{\text{Def. 3.5(i)}}{=} (\mu_0(\alpha)_{q,\varepsilon}, t) \;\preceq\; c \;=\; f^{\text{bot}}_{M,\text{mod},g,l,c}(1).$$

Induction step: Let $k \in \mathbb{N}_+$ be an integer, $\sigma \in \Sigma^{(k)}$ be an input symbol, $s_1, \ldots, s_k \in T_\Sigma$ be trees, and $s = \sigma(s_1, \ldots, s_k)$ be the input tree of height n. Note that throughout the proof we will use the statements of Observation 2.2 without explicit reference.

$$\begin{aligned}
&(h^{\text{mod}}_\mu(\sigma(s_1, \ldots, s_k))_q, t) \\
&\overset{\text{Def. 3.5(i)}}{=} \left(\sum_{(q_1,\ldots,q_k) \in Q^k} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \overset{\text{mod}}{\longleftarrow} (h^{\text{mod}}_\mu(s_1)_{q_1}, \ldots, h^{\text{mod}}_\mu(s_k)_{q_k}), t \right) \\
&\overset{\text{Eq. (1), (2)}}{=} \sum_{\substack{w=(q_1,\ldots,q_k)\in Q^k, \\ t=t'[t_1,\ldots,t_k],\ t'\in\text{supp}(\mu_k(\sigma)_{q,w}), \\ (\forall i\in[k]):\ t_i\in\text{supp}(h^{\text{mod}}_\mu(s_i)_{q_i})}} (\mu_k(\sigma)_{q,w}, t') \odot \prod_{i\in[k]} (h^{\text{mod}}_\mu(s_i)_{q_i}, t_i)^{m_i} \\
&\qquad \text{where for every index } i \in [k]:\ m_i = \begin{cases} |t'|_{x_i} & \text{, if mod} = o \\ 1 & \text{, if mod} = \varepsilon \end{cases} \\
&\overset{\text{I.H. \& Lem. 4.6}}{\preceq} \sum_{\substack{w=(q_1,\ldots,q_k)\in Q^k, \\ t=t'[t_1,\ldots,t_k],\ t'\in\text{supp}(\mu_k(\sigma)_{q,w}), \\ (\forall i\in[k]):\ t_i\in\text{supp}(h^{\text{mod}}_\mu(s_i)_{q_i})}} c \odot \prod_{i\in[k]} f^{\text{bot}}_{M,\text{mod},g,l,c}(\text{height}(s_i))^{m_i} \\
&\overset{\text{Lem. 4.6}}{\preceq} \sum_{\substack{w=(q_1,\ldots,q_k)\in Q^k, \\ t=t'[t_1,\ldots,t_k],\ t'\in\text{supp}(\mu_k(\sigma)_{q,w}), \\ (\forall i\in[k]):\ t_i\in\text{supp}(h^{\text{mod}}_\mu(s_i)_{q_i})}} c \odot f^{\text{bot}}_{M,\text{mod},g,l,c}(n-1)^{m_1+\cdots+m_k}
\end{aligned}$$

$$
\begin{aligned}
&\overset{\sum_{i\in[k]} m_i \le u_{M,\mathrm{mod}}}{\preceq} \sum_{\substack{w=(q_1,\dots,q_k)\in Q^k,\\ t=t'[t_1,\dots,t_k],\ t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall i\in[k]):\ t_i\in\mathrm{supp}(h_\mu^{\mathrm{mod}}(s_i)_{q_i})}} c\odot f^{\mathrm{bot}}_{M,\mathrm{mod},g,l,c}(n-1)^{u_{M,\mathrm{mod}}} \\
&\overset{\dagger}{\preceq} \sum_{\substack{w=(q_1,\dots,q_k)\in Q^k,\\ t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall i\in[k]):\ t_i\in\mathrm{supp}(h_\mu^{\mathrm{mod}}(s_i)_{q_i})}} c\odot f^{\mathrm{bot}}_{M,\mathrm{mod},g,l,c}(n-1)^{u_{M,\mathrm{mod}}} \\
&\preceq \sum_{j\in[(d_M)^k\cdot e_M\cdot\prod_{i\in[k]} l(\mathrm{height}(s_i))]} c\odot f^{\mathrm{bot}}_{\mathrm{mod},M,g,l,c}(n-1)^{u_{M,\mathrm{mod}}} \\
&\overset{l\ \text{order-pres.}}{\preceq} \sum_{j\in[(d_M)^k\cdot e_M\cdot l(n-1)^k]} c\odot f^{\mathrm{bot}}_{M,\mathrm{mod},g,l,c}(n-1)^{u_{M,\mathrm{mod}}} \\
&\preceq \sum_{j\in[(d_M)^{r_M}\cdot e_M\cdot l(n-1)^{r_M}]} c\odot f^{\mathrm{bot}}_{M,\mathrm{mod},g,l,c}(n-1)^{u_{M,\mathrm{mod}}} \\
&\preceq g((d_M)^{r_M}\cdot e_M\cdot l(n-1)^{r_M}, c\odot f^{\mathrm{bot}}_{M,\mathrm{mod},g,l,c}(n-1)^{u_{M,\mathrm{mod}}}) \\
&= f^{\mathrm{bot}}_{M,\mathrm{mod},g,l,c}(n).
\end{aligned}
$$

The step at † is governed by $t \in \mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_q)$, which implies that there exists at least one non-zero summand of the sum. This concludes the proof of the first statement which easily allows us to derive the latter property.

$$
\begin{aligned}
(\tau_M^{\mathrm{mod}}(s), t'') &\overset{\text{Def. 3.5(ii)}}{=} \sum_{q\in D} (h_\mu^{\mathrm{mod}}(s)_q, t'') = \sum_{q\in\{\, p\in D \mid t''\in\mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_p)\,\}} (h_\mu^{\mathrm{mod}}(s)_q, t'') \\
&\overset{\text{Obs. 2.2(i)}}{\preceq} \sum_{q\in\{\, p\in D \mid t''\in\mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_p)\,\}} f^{\mathrm{bot}}_{M,\mathrm{mod},g,l,c}(n) \overset{\text{Obs. 2.2(ii)}}{\preceq} \sum_{q\in Q} f^{\mathrm{bot}}_{M,\mathrm{mod},g,l,c}(n) \\
&\preceq g(\mathrm{card}(Q), f^{\mathrm{bot}}_{M,\mathrm{mod},g,l,c}(n))
\end{aligned}
$$

The step labeled Obs. 2.2(ii) is possible, because $\{\, p \in D \mid t'' \in \mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_p)\,\} \neq \emptyset$. Hence, we have proved the latter statement. ■

Continuing with the running example, we derive the ample coefficient majorization associated with the cardinality majorization and sum majorization given in Example 4.4 and $c = 1$ for the polynomial bottom-up tree series transducer M presented in Example 3.6 with $\mathrm{mod} = o$.

4.8 Example (Continuing Example 3.6)
Let $M = (\{*\}, \Sigma, \Delta, \mathbb{A}, \{*\}, \mu)$ be the polynomial bottom-up tree series transducer of Example 3.6. First we recall that the arctic semiring is partially ordered by $\le$ and fulfils property (MO⊕); thus the ample coefficient majorization according to Definition 4.5 is defined.

The constants of Definition 4.1 can be instantiated to

$$
r_M = 2, \quad d_M = 1, \quad e_M = 2, \quad u_{M,o} = 1,
$$

and we let l be the order-preserving cardinality majorization and g be the order-preserving sum majorization both presented in Example 4.4, i.e., for every $n \in \mathbb{N}_+$ and semiring element $a \in \mathbb{N} \cup \{-\infty\}$ we have $l(n) = 1$ and $g(n, a) = a$. Finally, we let $c = 1$, which is an upper bound of the coefficients of μ according to Definition 4.2. Hence, we obtain the following ample coefficient majorization $f^{\text{bot}}_{M,o,g,l,1}$ associated with l, g, and 1. For every integer $n \in \mathbb{N}_+$

$$\begin{aligned} f^{\text{bot}}_{M,o,g,l,1}(n) &= \begin{cases} 1 & \text{, if } n = 1 \\ g(1^2 \cdot 2 \cdot l(n-1)^2, 1 + f^{\text{bot}}_{M,o,g,l,1}(n-1)^1) & \text{, if } n > 1 \end{cases} \\ &= \begin{cases} 1 & \text{, if } n = 1 \\ 1 + f^{\text{bot}}_{M,o,g,l,1}(n-1) & \text{, if } n > 1 \end{cases} \\ &= n. \end{aligned}$$

Theorem 4.7 applied to this example yields that for every integer $n \in \mathbb{N}_+$, input tree $s \in T_\Sigma$ of height n, and output tree $t \in \text{supp}(h^o_\mu(s)_*)$ the majorization $(h^o_\mu(s)_*, t) \leq n$ holds, which is consistent with the fact $(\tau^o_M(s), \alpha) = \text{height}(s)$ proved in Example 3.6. Note, furthermore, that $f^{\text{bot}}_{M,o,g,l,1}$ coincides with the coefficient majorization presented in Example 4.4. □

Up to now, we have derived a mapping $f^{\text{bot}}_{M,\text{mod},g,l,c}$ which limits the costs generated by M on an output subtree. By definition, $f^{\text{bot}}_{M,o,g,l,c}$ depends on two more mappings, i.e., an order-preserving cardinality majorization $l : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ and an order-preserving sum majorization $g : \mathbb{N}_+ \times A \longrightarrow A$. The sum majorization g is highly semiring specific and needs to be provided from outside, i.e., it cannot be deduced from properties of M. Later we will see, how restrictions on the semiring allow an easy definition of this mapping. The cardinality majorization l, which is required to be order-preserving in Definition 4.5, limits the support cardinality of the tree series computed. This mapping was also supplied from outside, but now we will directly associate to M an easy cardinality majorization l^{bot}_M, which can be used instead of the cardinality majorization l coming from outside. Next we show how to achieve this.

Given a polynomial bottom-up tree series transducer $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$, a modifier $\text{mod} \in \{\varepsilon, o\}$, and an integer $n \in \mathbb{N}_+$, we will limit the cardinality of the support of the tree series $h^{\text{mod}}_\mu(s)_q$ for every input tree $s \in T_\Sigma$ of height n and state $q \in Q$. The idea is to pessimistically assume that given an integer $k \in \mathbb{N}$, pairs of different trees $(t, t') \in T_\Delta(X_k)^2$, and $(t_1, t'_1), \ldots, (t_k, t'_k) \in \left(T_\Delta\right)^2$, the trees $t[t_1, \ldots, t_k]$ and $t'[t'_1, \ldots, t'_k]$ are different. This is – of course – not true in general, but it is appropriate for our cardinality majorization, because the number of different trees in the support might only be overestimated.

4.9 Definition (Ample cardinality majorization)
The *ample cardinality majorization associated with* M is the mapping $l^{\text{bot}}_M : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ defined for every $n \in \mathbb{N}_+$ by

$$l^{\text{bot}}_M(n) = (d_M)^{\sum_{i \in [1,n-1]} r^i_M} \cdot (e_M)^{\sum_{i \in [0,n-1]} r^i_M} = \begin{cases} e_M & \text{, if } n = 1 \\ (d_M)^{r_M} \cdot e_M \cdot l^{\text{bot}}_M(n-1)^{r_M} & \text{, if } n > 1 \end{cases}.$$ □

Recall that for the definition of the ample coefficient majorization associated with l^{bot}_M, some order-preserving sum majorization g, and c in Definition 4.5, l^{bot}_M is required to be order-preserving. We show this fact in the following observation.

4.10 Observation (The ample cardinality majorization is order-preserving)
The ample cardinality majorization l^{bot}_M defined in Definition 4.9 is order-preserving.

Proof. We need to show that the mapping $l^{\text{bot}}_M : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ given by

$$l^{\text{bot}}_M(n) = d_M^{(\sum_{i \in [1,n-1]} r^i_M)} e_M^{(\sum_{i \in [0,n-1]} r^i_M)}$$

is order-preserving. This, however, is immediate by $d_M \neq 0$ and $e_M \neq 0$. ■

4.11 Lemma (Ample cardinality majorization)
The ample cardinality majorization associated with M is a cardinality majorization, i.e., for every integer $n \in \mathbb{N}_+$, state $q \in Q$, and input tree $s \in T_\Sigma$ of height n the statement

$$\mathrm{card}(\mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_q)) \leq l_M^{\mathrm{bot}}(n)$$

holds.

Proof. We prove the statement by structural induction over the input tree $s \in T_\Sigma$.

Induction base: Let $\alpha \in \Sigma^{(0)}$ be an input symbol, and let $s = \alpha$ be the input tree of height 1. Then

$$\mathrm{card}(\mathrm{supp}(h_\mu^{\mathrm{mod}}(\alpha)_q)) \overset{\text{Def. 3.5(i)}}{=} \mathrm{card}(\mathrm{supp}(\mu_0(\alpha)_{q,\varepsilon})) \leq e_M = l_M^{\mathrm{bot}}(1).$$

Induction step: Let $k \in \mathbb{N}_+$ be an integer, $\sigma \in \Sigma^{(k)}$ be an input symbol, $s_1, \ldots, s_k \in T_\Sigma$ be input subtrees, and $s = \sigma(s_1, \ldots, s_k)$ be the input tree of height n.

$$\begin{aligned}
&\mathrm{card}(\mathrm{supp}(h_\mu^{\mathrm{mod}}(\sigma(s_1, \ldots, s_k))_q)) \\
\overset{\text{Def. 3.5(i)}}{=}\quad & \mathrm{card}\Big(\mathrm{supp}\Big(\sum_{w=(q_1,\ldots,q_k)\in Q^k} \mu_k(\sigma)_{q,w} \overset{\mathrm{mod}}{\longleftarrow} (h_\mu^{\mathrm{mod}}(s_1)_{q_1}, \ldots, h_\mu^{\mathrm{mod}}(s_k)_{q_k})\Big)\Big) \\
\overset{\text{Eq. (1), (2)}}{=}\quad & \mathrm{card}\Big(\mathrm{supp}\Big(\sum_{\substack{w=(q_1,\ldots,q_k)\in Q^k,\\ t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall i\in[k]):\, t_i\in\mathrm{supp}(h_\mu^{\mathrm{mod}}(s_i)_{q_i})}} \big((\mu_k(\sigma)_{q,w}, t') \odot \prod_{i\in[k]} (h_\mu^{\mathrm{mod}}(s_i)_{q_i}, t_i)^{m_i}\big)\, t'[t_1, \ldots, t_k]\Big)\Big) \\
& \text{where for every index } i \in [k] :\ m_i = \begin{cases} |t'|_{x_i} & \text{, if mod} = o \\ 1 & \text{, if mod} = \varepsilon \end{cases} \\
\overset{\text{I.H. \& Prop. 3.9}}{\leq}\quad & (d_M)^k \cdot e_M \cdot l_M^{\mathrm{bot}}(\mathrm{height}(s_1)) \cdot \ldots \cdot l_M^{\mathrm{bot}}(\mathrm{height}(s_k)) \\
\overset{\text{Obs. 4.10}}{\leq}\quad & (d_M)^k \cdot e_M \cdot l_M^{\mathrm{bot}}(n-1)^k \\
\leq\quad & (d_M)^{r_M} \cdot e_M \cdot l_M^{\mathrm{bot}}(n-1)^{r_M} \\
=\quad & l_M^{\mathrm{bot}}(n)
\end{aligned}$$

■

Finally, we use the ample cardinality majorization l_M^{bot} as particular parameter in the ample coefficient majorization and obtain the following coefficient majorization $f_{M,\mathrm{mod},g,c}^{\mathrm{bot}} = f_{M,\mathrm{mod},g,l_M^{\mathrm{bot}},c}^{\mathrm{bot}}$, which now only depends on the constants, an order-preserving sum majorization, and c.

4.12 Corollary ($f_{M,\mathrm{mod},g,l_M^{\mathrm{bot}},c}^{\mathrm{bot}}$ is a coefficient majorization)
The *ample coefficient majorization* $f_{M,\mathrm{mod},g,c}^{\mathrm{bot}} : \mathbb{N}_+ \longrightarrow A$ *associated with* g and c defined for every integer $n \in \mathbb{N}_+$ by

$$f_{M,\mathrm{mod},g,c}^{\mathrm{bot}}(n) = \begin{cases} c & \text{, if } n = 1 \\ g(l_M^{\mathrm{bot}}(n), c \odot f_{M,\mathrm{mod},g,c}^{\mathrm{bot}}(n-1)^{u_{M,\mathrm{mod}}}) & \text{, if } n > 1 \end{cases}$$

is a coefficient majorization, i.e., for every integer $n \in \mathbb{N}_+$, state $q \in Q$, input tree $s \in T_\Sigma$ of height n, and output tree $t \in \mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_q)$ we have

$$(h_\mu^{\mathrm{mod}}(s)_q, t) \preceq f_{M,\mathrm{mod},g,c}^{\mathrm{bot}}(n)$$

and thus $f_{M,\mathrm{mod},g,c}^{\mathrm{bot}}(n) \in \uparrow C_M(n)$. Moreover, $(\tau_M^{\mathrm{mod}}(s), t'') \preceq g(\mathrm{card}(Q), f_{M,\mathrm{mod},g,c}^{\mathrm{bot}}(n))$ for every $t'' \in \mathrm{supp}(\tau_M^{\mathrm{mod}}(s))$.

Proof. We just supply the ample cardinality majorization l_M^{bot} associated with M as the order-preserving cardinality majorization to the ample coefficient majorization defined in Definition 4.5. By Theorem 4.7 we obtain the stated. ■

4.3 The top-down case

In this subsection we consider polynomial top-down tree series transducers and derive similar majorizations for them. Thus, M always denotes a (non-trivial) polynomial top-down tree series transducer in this subsection. Moreover, we again let l be an order-preserving cardinality majorization, g be an order-preserving sum majorization, and c be an upper bound of the coefficients of μ.

4.13 Definition (Ample coefficient majorization)
For every $n \in \mathbb{N}_+$ we define the *ample coefficient majorization* $f^{\text{top}}_{M,g,l,c} : \mathbb{N}_+ \longrightarrow A$ *associated with* l, g, *and* c to be

$$f^{\text{top}}_{M,g,l,c}(n) = \begin{cases} c & \text{, if } n = 1 \\ g((d_M)^{1+v_M} \cdot e_M \cdot l(n-1)^{v_M}, c \odot f^{\text{top}}_{M,g,l,c}(n-1)^{v_M}) & \text{, if } n > 1 \end{cases} \qquad \square$$

Note the structural similarity of $f^{\text{top}}_{M,g,l,c}$ and the ample coefficient majorization associated with l, g, and c of a polynomial bottom-up tree series transducer. Also note that $f^{\text{top}}_{M,g,l,c}$ does not depend on mod.

4.14 Lemma (Ample coefficient majorization is order-preserving)
The ample coefficient majorization $f^{\text{top}}_{M,g,l,c}$ associated with l, g, and c is order-preserving and for every $n \in \mathbb{N}_+$ the condition $\mathbf{1} \preceq f^{\text{top}}_{M,g,l,c}(n)$ holds.

Proof. The proof is analogous to the one of the bottom-up case found in Lemma 4.6 making a case analysis for v_M rather than $u_{M,\text{mod}}$. ■

Even the main theorem stating that the ample coefficient majorization is a coefficient majorization and its proof can be translated in a straightforward manner to the top-down case. The general approach remains the same, though there are some notational changes, so we resupply the proof. Due to Theorem 3.8(i) we can drop the modifier mod $\in \{\varepsilon, o\}$.

4.15 Theorem (Coefficient majorization)
The ample coefficient majorization $f^{\text{top}}_{M,g,l,c}$ is a coefficient majorization, i.e., for every integer $n \in \mathbb{N}_+$, state $q \in Q$, input tree $s \in T_\Sigma$ of height n, and output tree $t \in \text{supp}(h_\mu(s)_q)$ we have

$$(h_\mu(s)_q, t) \preceq f^{\text{top}}_{M,g,l,c}(n)$$

and thus $f^{\text{top}}_{M,g,l,c}(n) \in \uparrow C_M(n)$. Moreover, $(\tau_M(s), t'') \preceq g(\text{card}(Q), f^{\text{top}}_{M,g,l,c}(n))$ for every $t'' \in \text{supp}(\tau_M(s))$.

Proof. The proof of the latter statement is identical to the proof of the corresponding statement of Theorem 4.7. Thus, we continue with the former statement. Recall the constants d_M, e_M, and v_M of Definition 4.1. We prove the statement by structural induction over the input tree $s \in T_\Sigma$.

Induction base: Let $\alpha \in \Sigma^{(0)}$ be an input symbol and $s = \alpha$ be the input tree of height 1. Since $t \in \text{supp}(h_\mu(\alpha)_q)$,

$$(h_\mu(\alpha)_q, t) \overset{\text{Def. 3.5(i)}}{=} (\mu_0(\alpha)_{q,\varepsilon}, t) \preceq c = f^{\text{top}}_{M,g,l,c}(1).$$

Induction step: Let $k \in \mathbb{N}_+$ be an integer, $\sigma \in \Sigma^{(k)}$ be an input symbol, $s_1, \ldots, s_k \in T_\Sigma$ be trees, and $s = \sigma(s_1, \ldots, s_k)$ be the input tree of height n.

$$
\begin{aligned}
&(h_\mu(\sigma(s_1,\ldots,s_k))_q, t)\\
&\overset{\text{Def. 3.5(i)}}{=} \left(\sum_{k'\in\mathbb{N}, w=(q_1(x_{i_1}),\ldots,q_{k'}(x_{i_{k'}}))\in(Q(X_k))^{k'}} \mu_k(\sigma)_{q,w} \longleftarrow (h_\mu(s_{i_1})_{q_1},\ldots,h_\mu(s_{i_{k'}})_{q_{k'}}), t \right)\\
&\overset{\text{Eq. (1)}}{=} \sum_{\substack{k'\in\mathbb{N}, w=(q_1(x_{i_1}),\ldots,q_{k'}(x_{i_{k'}}))\in(Q(X_k))^{k'},\\ t=t'[t_1,\ldots,t_{k'}], t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall j\in[k'])\colon t_j\in\mathrm{supp}(h_\mu(s_{i_j})_{q_j})}} (\mu_k(\sigma)_{q,w}, t') \odot \prod_{j\in[k']} (h_\mu(s_{i_j})_{q_j}, t_j)\\
&\overset{\text{I.H. \& Lem. 4.14}}{\preceq} \sum_{\substack{k'\in\mathbb{N}, w=(q_1(x_{i_1}),\ldots,q_{k'}(x_{i_{k'}}))\in(Q(X_k))^{k'},\\ t=t'[t_1,\ldots,t_{k'}], t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall j\in[k'])\colon t_j\in\mathrm{supp}(h_\mu(s_{i_j})_{q_j})}} c \odot \prod_{j\in[k']} f^{\mathrm{top}}_{M,g,l,c}(\mathrm{height}(s_{i_j}))\\
&\overset{\text{Lem. 4.14}}{\preceq} \sum_{\substack{k'\in\mathbb{N}, w=(q_1(x_{i_1}),\ldots,q_{k'}(x_{i_{k'}}))\in(Q(X_k))^{k'},\\ t=t'[t_1,\ldots,t_{k'}], t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall j\in[k'])\colon t_j\in\mathrm{supp}(h_\mu(s_{i_j})_{q_j})}} c \odot f^{\mathrm{top}}_{M,g,l,c}(n-1)^{k'}\\
&\overset{k'\le v_M}{\preceq} \sum_{\substack{k'\in\mathbb{N}, w=(q_1(x_{i_1}),\ldots,q_{k'}(x_{i_{k'}}))\in(Q(X_k))^{k'},\\ t=t'[t_1,\ldots,t_{k'}], t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w})\\ (\forall j\in[k'])\colon t_j\in\mathrm{supp}(h_\mu(s_{i_j})_{q_j})}} c \odot f^{\mathrm{top}}_{M,g,l,c}(n-1)^{v_M}\\
&\overset{\dagger}{\preceq} \sum_{\substack{k'\in\mathbb{N}, w=(q_1(x_{i_1}),\ldots,q_{k'}(x_{i_{k'}}))\in(Q(X_k))^{k'},\\ t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}), (\forall j\in[k'])\colon t_j\in\mathrm{supp}(h_\mu(s_{i_j})_{q_j})}} c \odot f^{\mathrm{top}}_{M,g,l,c}(n-1)^{v_M}\\
&\preceq \sum_{j'\in\left[\left(\sum_{j\in[0,v_M]}(d_M)^j\right)\cdot e_M\cdot\prod_{j\in[v_M]} l(\mathrm{height}(s_{i_j}))\right]} c \odot f^{\mathrm{top}}_{M,g,l,c}(n-1)^{v_M}\\
&\preceq \sum_{j'\in[(d_M)^{1+v_M}\cdot e_M\cdot l(n-1)^{v_M}]} c \odot f^{\mathrm{top}}_{M,g,l,c}(n-1)^{v_M}\\
&\preceq g((d_M)^{1+v_M}\cdot e_M\cdot l(n-1)^{v_M}, c\odot f^{\mathrm{top}}_{M,g,l,c}(n-1)^{v_M})\\
&= f^{\mathrm{top}}_{M,g,l,c}(n)
\end{aligned}
$$

The step at † is governed by the fact $t \in \mathrm{supp}(h_\mu(s)_q)$, because thereby there exists at least one summand of the sum. ■

Finally we also derive an ample cardinality majorization for polynomial top-down tree series transducers.

4.16 Definition (Ample cardinality majorization)
The *ample cardinality majorization associated with* M is the mapping $l^{\mathrm{top}}_M : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ defined for every $n \in \mathbb{N}_+$ by

$$
l^{\mathrm{top}}_M(n) = \begin{cases} e_M & \text{, if } n = 1\\ (d_M)^{1+v_M}\cdot e_M\cdot l^{\mathrm{top}}_M(n-1)^{v_M} & \text{, if } n > 1 \end{cases}.
$$
□

4.17 Observation (The ample cardinality majorization is order-preserving)
The ample cardinality majorization l^{top}_M defined in Definition 4.16 is order-preserving. □

4.18 Lemma (Ample cardinality majorization)
The ample cardinality majorization associated with M is a cardinality majorization, i.e., for every integer $n \in \mathbb{N}_+$, state $q \in Q$, and input tree $s \in T_\Sigma$ of height n the statement

$$\mathrm{card}(\mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_q)) \leq l_M^{\mathrm{top}}(n)$$

holds.

Proof. The proof proceeds along the lines of the corresponding one concerning the bottom-up case (cf. Lemma 4.11) with just minor changes, most of which were already outlined in the proof of Theorem 4.15. Hence we leave the actual proof to the reader. ∎

4.19 Corollary ($f^{\mathrm{top}}_{M,g,l_M^{\mathrm{top}}}$ is a coefficient majorization)
The *ample coefficient majorization* $f^{\mathrm{top}}_{M,g,c} : \mathbb{N}_+ \longrightarrow A$ *associated with* g and c defined for every integer $n \in \mathbb{N}_+$ by

$$f^{\mathrm{top}}_{M,g,c}(n) = \begin{cases} c & \text{, if } n = 1 \\ g(l_M^{\mathrm{top}}(n), c \odot f^{\mathrm{top}}_{M,g,c}(n-1)^{v_M}) & \text{, if } n > 1 \end{cases}$$

is a coefficient majorization. Moreover, $(\tau_M(s), t'') \preceq g(\mathrm{card}(Q), f^{\mathrm{top}}_{M,g,c}(n))$ for every tree $t'' \in \mathrm{supp}(\tau_M(s))$.

Proof. We just supply the ample cardinality majorization l_M^{top} associated with M as the order-preserving cardinality majorization to the ample coefficient majorization defined in Definition 4.13. By Theorem 4.15 we obtain the statements. ∎

Again note the similarity of the coefficient majorizations $f^{\mathrm{top}}_{M,g,c}$ and $f^{\mathrm{bot}}_{M,\mathrm{mod},g,c}$ for a top-down and a bottom-up tree series transducer. To be specific, only the constants $u_{M,\mathrm{mod}}$ and v_M exchange their place and, moreover, we already observed that $v_M = u_{M,o}$, thus the majorizations even become equal in case mod $= o$.

5 Incomparability results

In the first part of this section we will reprove two recent results from [FV03] concerning growth properties of polynomial bottom-up tree series transducers using our coefficient majorization approach (i.e., using Theorem 4.7). The second part then focuses on some simplified coefficient majorization, which allows us to derive incomparability results for classes of mod-t-ts transformations computed by polynomial bottom-up as well as top-down tree series transducers.

5.1 Lemma (Lemma 5.14 of [FV03])
Let $N = (Q, \Sigma, \Delta, \mathbb{N}_\infty, F, \nu)$ be a polynomial bottom-up tree series transducer with input ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, output ranked alphabet $\Delta = \{\alpha^{(0)}\}$, and $+\infty$ does not occur as coefficient in any tree series of the tree representation ν. There exists a constant $b \in \mathbb{N}$ such that for every input tree $s \in T_\Sigma$ the approximation $(\tau_N^o(s), \alpha) \leq b^{\mathrm{height}(s)}$ holds.

Proof. We can instantiate Theorem 4.7, because the semiring of non-negative integers $\mathbb{N}_\infty$ is totally ordered by $\leq$. Also, since every element $n \in \mathbb{N} \cup \{+\infty\}$ is in the positive cone (i.e., $0 \leq n$), the semiring has property (MO⊕) by Observation 2.1. The constants of Definition 4.1 are instantiated as follows: $r_N = \mathrm{mx}_\Sigma = 2$, $e_N \leq \mathrm{card}(T_\Delta(X_2)) = 3$, and $u_{N,o} \leq 1$. Moreover, since the set $\mathbb{N}$ is directed, an upper bound $c \in \mathbb{N}$ of the coefficients of ν exists.

We choose the sum majorization $g : \mathbb{N}_+ \times (\mathbb{N} \cup \{+\infty\}) \longrightarrow \mathbb{N} \cup \{+\infty\}$ to be $g(n, a) = n \cdot a$ for every integer $n \in \mathbb{N}_+$ and semiring element $a \in \mathbb{N} \cup \{+\infty\}$. We note that g is order-preserving. Finally, we select the cardinality majorization $l : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ defined by $l(n) = 1$ for every integer $n \in \mathbb{N}_+$, which is trivially order-preserving and a cardinality majorization due to the fact $\mathrm{card}(\mathrm{supp}(h^o_\nu(s)_q)) \leq \mathrm{card}(T_\Delta) = 1$ for every input tree $s \in T_\Sigma$ and state $q \in Q$. Hence

$$\begin{aligned} f^{\mathrm{bot}}_{N,o,g,l,c}(n) &= \begin{cases} c & \text{, if } n = 1 \\ (d_N)^2 \cdot e_N \cdot c \cdot f^{\mathrm{bot}}_{N,o,g,l,c}(n-1)^{u_{N,o}} & \text{, if } n > 1 \end{cases} \\ &\leq (3 \cdot (d_N)^2 \cdot c)^{n-1} \cdot c \end{aligned}$$

and by setting $b = 3 \cdot \mathrm{card}(Q) \cdot (d_N)^2 \cdot c \neq +\infty$, because $c \neq +\infty$, we obtain by Theorem 4.7 and $0 \leq n$ for every $n \in \mathbb{N} \cup \{+\infty\}$ (note that Theorem 4.7 only holds for output trees in the support of the output tree series, hence the condition $0 \leq n$ is necessary): $(\tau^o_N(s), \alpha) \leq b^{\mathrm{height}(s)}$. ■

5.2 Lemma (Lemma 5.16 of [FV03])
Let $N = (Q, \Sigma, \Delta, \mathbb{N}_\infty, F, \nu)$ be a polynomial bottom-up tree series transducer with unary input ranked alphabet Σ and $+\infty$ does not occur as coefficient in any tree series of the tree representation ν. Then there is a constant $b \in \mathbb{N}$ such that for every input tree $s \in T_\Sigma$ and output tree $t \in T_\Delta$ the approximation $(\tau_N(s), t) \leq b^{\mathrm{height}(s)^2}$ holds.

Proof. Corollary 4.12 is applicable, because $\mathbb{N}_\infty$ fulfils the general restrictions imposed on the semiring. Apparently, the constants of Definition 4.1 are instantiated as follows: $r_N = \mathrm{mx}_\Sigma = 1$ and $u_{N,\varepsilon} = 1$. Again, since $\mathbb{N}$ is directed, there exists an upper bound $c \in \mathbb{N}$ of the coefficients of ν. Using the order-preserving sum majorization $g : \mathbb{N}_+ \times (\mathbb{N} \cup \{+\infty\}) \longrightarrow \mathbb{N} \cup \{+\infty\}$ defined for every integer $n \in \mathbb{N}_+$ and semiring element $a \in \mathbb{N} \cup \{+\infty\}$ by $g(n, a) = n \cdot a$ and the order-preserving ample cardinality majorization $l^{\mathrm{bot}}_N : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ associated with N of Definition 4.9, which is an order-preserving cardinality majorization (due to Observation 4.10 and Lemma 4.11), we obtain

$$\begin{aligned} f^{\mathrm{bot}}_{N,\varepsilon,g,c}(n) &= \begin{cases} c & \text{, if } n = 1 \\ (d_N)^{n-1} \cdot (e_N)^n \cdot c \cdot f^{\mathrm{bot}}_{N,\varepsilon,g,c}(n-1) & \text{, if } n > 1 \end{cases} \\ &= (d_N)^{\sum_{i \in [1,n-1]} i} \cdot (e_N)^{\sum_{i \in [2,n]} i} \cdot c^n \\ &\leq (d_N \cdot e_N \cdot c)^{\frac{(n+2)\cdot(n+1)}{2}}, \end{aligned}$$

which implies the bound shown in the lemma by setting $b = \mathrm{card}(Q) \cdot (d_N \cdot e_N \cdot c_N)^3$ as follows. Since $0 \leq n$ for every element $n \in \mathbb{N} \cup \{+\infty\}$ and $(d_N \cdot e_N \cdot c)^{\frac{(n+2)\cdot(n+1)}{2}} \leq b^{\mathrm{height}(s)^2}$, we obtain $(\tau_N(s), t) \leq b^{\mathrm{height}(s)^2}$ by Theorem 4.7. ■

In [FV03] the following corollary is proved using essentially Lemma 5.1 and Lemma 5.2 together with some examples required to show incomparability.

5.3 Corollary (Corollary 5.18 of [FV03])

$$\text{p–BOT}(\mathbb{N}_\infty) \bowtie \text{p–BOT}^o(\mathbb{N}_\infty)$$ □

Using the same approach one can also reprove Lemma 5.19 and Lemma 5.21 of [FV03]. Those two lemmata are used to prove Corollary 5.23 of [FV03], which essentially states the above for the tropical semiring.

As already pointed out, given certain additional restrictions the ample coefficient majorization can be simplified further. For example, if the tree series transducer is deterministic, then the addition of the semiring has no influence on the cost computation (cf. Proposition 3.9), because the costs are computed solely in the multiplicative monoid. Hence we can drop any restrictions concerning the addition and set the order-preserving sum majorization $g : \mathbb{N}_+ \times A \longrightarrow A$ required in Definition 4.5 and Definition 4.13 simply to $g(n, a) = a$ for every integer $n \in \mathbb{N}_+$ and semiring element $a \in A$. Likewise we can perform this simplification, if the underlying semiring is additively idempotent (and the polynomial tree series transducer need not be deterministic). Summing up, we arrive at the following corollary, which shows that, under these conditions a very simple mapping, called coefficient approximation, is a coefficient majorization. Recall that $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ is a (non-trivial) polynomial tree series transducer, and c is an upper bound of the coefficients of μ.

5.4 Definition (Coefficient approximation)
For every semiring element $a \in A$ and integer $z \in \mathbb{N}$ the mapping $f_{a,z} : \mathbb{N}_+ \longrightarrow A$ defined for every integer $n \in \mathbb{N}_+$ by $f_{a,z}(n) = a^{\sum_{i \in [0,n-1]} z^i}$ is called *coefficient approximation*. □

5.5 Corollary (Coefficient approximation)
Let mod $\in \{\varepsilon, o\}$. If (i) $\mathcal{A}$ is additively idempotent or (ii) M is deterministic, then the coefficient approximation $f_{c,z} : \mathbb{N}_+ \longrightarrow A$, where

$$z = \begin{cases} u_{M,\text{mod}} & \text{, if } M \text{ is bottom-up} \\ v_M & \text{, if } M \text{ is top-down} \end{cases}$$

is a coefficient majorization (with respect to M). Moreover $(\tau_M^{\text{mod}}(s), t) \preceq f_{c,z}(n)$ for every integer $n \in \mathbb{N}_+$, input tree $s \in T_\Sigma$ of height n, and output tree $t \in \text{supp}(\tau_M^{\text{mod}}(s))$.

Proof. To show that $f_{c,z}$ is a coefficient majorization, we show that $f_{c,z}$ is equal to the ample coefficient majorization $f^{\text{bot}}_{M,\text{mod},g,l,c}$ or $f^{\text{top}}_{M,g,l,c}$ (depending on whether M is bottom-up or top-down) for a particular cardinality majorization l and a particular sum majorization g. Let us first consider case (ii), i.e., M is deterministic. Here we set $g(n, a) = \sum_{i \in [n]} a$ for every integer $n \in \mathbb{N}_+$ and semiring element $a \in A$. Apparently, this is an order-preserving sum majorization. Moreover, we let $l(n) = 1$ for every integer $n \in \mathbb{N}_+$, which is an order-preserving cardinality majorization due to Proposition 3.9. Finally, we note that by determinism $d_M = 1$ and $e_M = 1$, thus the first argument of the sum majorization g will always be 1.

In case (i) we let l be an arbitrary order-preserving cardinality majorization, e.g., we could set $l = l_M^{\text{bot}}$ if M is bottom-up, and $l = l_M^{\text{top}}$ if M is top-down, and define the sum majorization $g : \mathbb{N}_+ \times A \longrightarrow A$ by $g(n, a) = a$ for every integer $n \in \mathbb{N}_+$ and semiring element $a \in A$. Note that g is an order-preserving sum majorization, because the semiring is assumed to be additively idempotent.

We continue in both cases by showing that $f_{c,z}(n) = h(n)$, where

$$h = \begin{cases} f^{\text{bot}}_{M,\text{mod},g,l,c} & \text{, if } M \text{ is bottom-up} \\ f^{\text{top}}_{M,g,l,c} & \text{, if } M \text{ is top-down} \end{cases}.$$

We prove this by induction on n as follows. Note that the induction base is handled in the first case of the case analysis, while the second case constitutes the induction step.

$$\begin{aligned} h(n) \quad &= \quad \begin{cases} c & \text{, if } n = 1 \\ g((d_M)^x \cdot e_M \cdot l(n-1)^y, c \odot h(n-1)^z) & \text{, if } n > 1 \end{cases} \\ &\qquad \text{where } x = y = r_M \text{ and } z = u_{M,\text{mod}}, \text{ if } M \text{ is bottom-up,} \\ &\qquad \text{otherwise } x = 1 + v_M \text{ and } y = z = v_M \\ &= \quad \begin{cases} c & \text{, if } n = 1 \\ c \odot h(n-1)^z & \text{, if } n > 1 \end{cases} \\ &\overset{\text{I.H.}}{=} \quad \begin{cases} c^{\sum_{i \in [0,1-1]} z^i} & \text{, if } n = 1 \\ c \odot \left(c^{\sum_{i \in [0,n-2]} z^i}\right)^z & \text{, if } n > 1 \end{cases} \\ &= \quad \begin{cases} c^{\sum_{i \in [0,1-1]} z^i} & \text{, if } n = 1 \\ c^{\sum_{i \in [0,n-1]} z^i} & \text{, if } n > 1 \end{cases} \\ &= \quad c^{\sum_{i \in [0,n-1]} z^i} \\ &= \quad f_{c,z}(n). \end{aligned}$$

Thus $h = f_{c,z}$ and by Theorem 4.7 and Theorem 4.15 it follows that $f_{c,z}$ is a coefficient majorization. It remains to show the latter statement of the corollary. In case (i) we have

$$(\tau_M^{\text{mod}}(s), t) \overset{\text{Def. 3.5(ii)}}{=} \sum_{q \in D} (h_\mu^{\text{mod}}(s)_q, t) \preceq \sum_{q \in D} f_{c,z}(\text{height}(s)) \overset{\dagger}{=} f_{c,z}(\text{height}(s)),$$

where at † we used that $\mathcal{A}$ is idempotent. In case (ii) we conclude that for some state $p \in Q$

$$(\tau_M^{\text{mod}}(s), t) \overset{\text{Def. 3.5(ii)}}{=} \sum_{q \in D} (h_\mu^{\text{mod}}(s)_q, t) \overset{\text{Prop. 3.9}}{=} (h_\mu^{\text{mod}}(s)_p, t) \preceq f_{c,z}(\text{height}(s)).$$

■

Let us have a look on classes of mod-t-ts transformations. Therefore, we distinguish two cases. Firstly, let $\mathcal{A}$ be additively idempotent. Then $f_{c,z}$ is a coefficient majorization for the class of mod-t-ts transformations computed by polynomial bottom-up tree series transducers N, which have c as an upper bound of their tree representation and $z = u_{N,\text{mod}}$. Moreover, $f_{c,z}$ is a coefficient majorization for the class of mod-t-ts transformations computed by polynomial top-down tree series transducers N, which have c as an upper bound of their tree representation and $z = v_N$. Secondly, in case the semiring $\mathcal{A}$ is not additively idempotent, $f_{c,z}$ is a majorization for the corresponding classes of mod-t-ts transformations computed by deterministic tree series transducers.

The following lemma provides the order-preservation of the coefficient approximation defined in Definition 5.4 with respect to each subscript and argument. This allows us to provide upper bounds of the subscripts and argument in order to obtain an upper bound of the cost of an output tree.

5.6 Lemma (Coefficient approximations are order-preserving in every argument)
Let $a, a' \in A$ be semiring elements with $\mathbf{1} \preceq a' \preceq a$ and let $z, z', n, n' \in \mathbb{N}$ be non-negative integers with $z' \leq z$ and $n' \leq n$. Then $f_{a',z'}(n') \preceq f_{a,z}(n)$.

Proof. We prove

$$f_{a',z'}(n') = (a')^{\sum_{i \in [0,n'-1]} (z')^i} \preceq a^{\sum_{i \in [0,n-1]} z^i} = f_{a,z}(n).$$

By $z' \leq z$ also $(z')^j \leq z^j$ for every $j \in \mathbb{N}$. Moreover $\sum_{i \in [0,n'-1]} (z')^i \leq \sum_{i \in [0,n-1]} z^i$ by $n' \leq n$, Observation 2.2(ii), and the previous argument. Finally, we apply $a' \preceq a$ to obtain

$$(a')^{\sum_{i \in [0,n'-1]} (z')^i} \preceq a^{\sum_{i \in [0,n'-1]} z^i} \preceq a^{\sum_{i \in [0,n-1]} z^i}$$

by Observation 2.2(iii) using $\mathbf{1} \preceq a' \preceq a$. ■

Next we establish that the coefficient approximation $f_{a,z}$ for deterministic tree series transducers as well as for polynomial tree series transducers using additively idempotent semirings, i.e., in cases where $f_{a,z}$ is actually a coefficient majorization according to Corollary 5.5, is tight (considered as a coefficient majorization for the aforementioned classes of mod-t-ts transformations, cf. note following Corollary 5.5). This result will be used in our main incomparability result to follow (cf. Lemma 5.9).

5.7 Lemma (Coefficient approximations are tight)
For every non-negative integer $u \in \mathbb{N}$ and semiring element $c \in A$ with $\mathbf{1} \preceq c$,

(i) there exists a homomorphism bottom-up tree series transducer M over $\Sigma = \{\sigma^{(u)}, \alpha^{(0)}\}$ and output ranked alphabet Δ (with $\Delta^{(0)} \neq \emptyset$) such that c is an upper bound of the coefficients of μ, $u_{M,\varepsilon} = u$, and for every integer $n \in \mathbb{N}_+$, if there exists a Σ-tree of height n, then there also exist an input tree $s \in T_\Sigma$ of height n and an output tree $t \in \text{supp}(\tau_M(s))$ such that

$$(\tau_M(s), t) = f_{c,u_{M,\varepsilon}}(n),$$

(ii) there exists a homomorphism bottom-up tree series transducer M over an input ranked alphabet Σ (with $\Sigma^{(0)} \neq \emptyset$ and, if $u > 0$, also $\bigcup_{k \in \mathbb{N}_+} \Sigma^{(k)} \neq \emptyset$) and $\Delta = \{\delta^{(u)}, \alpha^{(0)}\}$ such that c is an upper bound of the coefficient of μ, $u_{M,o} = u$, and for every integer $n \in \mathbb{N}_+$, if there exists a Σ-tree of height n, then there also exist an input tree $s \in T_\Sigma$ of height n and an output tree $t \in \mathrm{supp}(\tau^o_M(s))$ such that

$$(\tau^o_M(s), t) = f_{c,u_{M,o}}(n),$$

(iii) there exists a homomorphism top-down tree series transducer M over an input ranked alphabet Σ (with $\Sigma^{(0)} \neq \emptyset$ and, if $u > 0$, also $\bigcup_{k \in \mathbb{N}_+} \Sigma^{(k)} \neq \emptyset$) and $\Delta = \{\delta^{(u)}, \alpha^{(0)}\}$ such that c is an upper bound of the coefficients of μ, $v_M = u$, and for every integer $n \in \mathbb{N}_+$, if there exists any Σ-tree of height n, then there also exist an input tree $s \in T_\Sigma$ of height n and an output tree $t \in \mathrm{supp}(\tau_M(s))$ such that

$$(\tau_M(s), t) = f_{c,v_M}(n).$$

Consequently, the coefficient majorization $f_{c,u}$ is tight in the corresponding class of mod-t-ts transformations.

Proof. We prove the statements individually.

(i) Let mod $= \varepsilon$ and $\alpha \in \Delta^{(0)}$. We construct the homomorphism bottom-up tree series transducer $M = (\{*\}, \Sigma, \Delta, \mathcal{A}, \{*\}, \mu)$ with input ranked alphabet $\Sigma = \{\sigma^{(u)}, \alpha^{(0)}\}$ and tree representation μ specified by

$$\mu_0(\alpha)_{*,\varepsilon} = c\,\alpha \quad \text{and} \quad \mu_u(\sigma)_{*,(*,\dots,*)} = c\,\alpha.$$

Note $u_{M,\varepsilon} = u$. Moreover, let $s \in T_\Sigma$ be the fully balanced tree of height $n \in \mathbb{N}_+$. A straightforward structural induction shows that $(\tau_M(s), \alpha) = f_{c,u_{M,\varepsilon}}(n)$ as follows. The induction base is $(\tau_M(\alpha), \alpha) = c = f_{c,u_{M,\varepsilon}}(1)$. In the induction step we have for every input tree $s = \sigma(s', \dots, s')$ with subtree $s' \in T_\Sigma$ being a fully balanced tree of height $n-1$ the equality

$$\begin{aligned}
(\tau_M(\sigma(s', \dots, s')), \alpha) &\overset{\text{Def. 3.5(ii)}}{=} (h_\mu(\sigma(s', \dots, s'))_*, \alpha) \\
&\overset{\text{Eq. (1)}}{=} (\mu_u(\sigma)_{*,(*,\dots,*)}, \alpha) \odot (h_\mu(s')_*, \alpha) \odot \cdots \odot (h_\mu(s')_*, \alpha) \\
&\overset{\text{Def. 3.5(ii)}}{=} (\mu_u(\sigma)_{*,(*,\dots,*)}, \alpha) \odot (\tau_M(s'), \alpha)^u \\
&\overset{\text{I.H.}}{=} c \odot f_{c,u_{M,\varepsilon}}(\mathrm{height}(s'))^u \\
&= c \odot f_{c,u_{M,\varepsilon}}(n-1)^{u_{M,\varepsilon}} \\
&\overset{\text{Def. 5.4}}{=} f_{c,u_{M,\varepsilon}}(n).
\end{aligned}$$

(ii) Let mod $= o$ and $\alpha \in \Sigma^{(0)}$. We construct the homomorphism bottom-up tree series transducer $M = (\{*\}, \Sigma, \Delta, \mathcal{A}, \{*\}, \mu)$ with the output ranked alphabet $\Delta = \{\delta^{(u)}, \alpha^{(0)}\}$ and tree representation μ specified by

$$\mu_0(\alpha)_{*,\varepsilon} = c\,\alpha \quad \text{and, if } u > 0, \text{ then} \quad \mu_k(\sigma)_{*,(*,\dots,*)} = c\,\delta(x_1, \dots, x_1),$$

where for some $k \in \mathbb{N}_+$ we have $\sigma \in \Sigma^{(k)}$. We note that $u_{M,o} = u$. Moreover, one can easily show that for the fully balanced trees $s \in T_\Sigma$ of height $n \in \mathbb{N}_+$ we have $(\tau^o_M(s), s) = f_{c,u_{M,o}}(n)$ by a similar induction as in item (i).

(iii) Note that the homomorphism top-down tree series transducer required to show this statement is almost identical to the bottom-up tree series transducer presented in item (ii). Therefore, we just present the tree representation μ and assume that the remaining components are specified as in item (ii).

$$\mu_0(\alpha)_{*,\varepsilon} = c\,\alpha \quad \text{and, if } u > 0, \text{ then} \quad \mu_k(\sigma)_{*,(*(x_1),\ldots,*(x_1))} = c\,\delta(x_1,\ldots,x_u).$$

Again $v_M = u$. The proof of statement (iii) is analogous to the ones for the previous items and therefore it is omitted. ∎

The main theorem states the incomparability of the classes of mod-t-ts transformations computed by restricted tree series transducers using on the one hand pure substitution, i.e., mod $= \varepsilon$, and on the other hand o-substitution, i.e., mod $= o$, over the semiring $\mathcal{A}$ with two additional properties.

The first additional property will be a particular multiplicative non-periodicity of the semiring, namely we require that there is a semiring element $a \in A$ such that a has no period and $\mathbf{1} \preceq a$. Thus, we demand the existence of an element $a \in A$ such that $a^i \prec a^j$, if and only if $i < j$ for every two non-negative integers $i, j \in \mathbb{N}$. Secondly, we require for every $n \in \mathbb{N}$ and the previous element a, if $a^n = a_1 \odot b \odot a_2 \oplus d$ for some $a_1, a_2 \in A \setminus \{\mathbf{0}\}$ and some $b, d \in A$, then there also exists a non-negative integer $m \in \mathbb{N}$ such that $b \preceq a^m$. Roughly speaking, this property requires that any element which might occur in a decomposition of the element a^n can be bounded from above by a power of the element a. Semirings having those two properties are called weak a-growth semirings.

5.8 Definition (Weak a-growth semirings)
Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ be a partially ordered semiring and $a \in A$ be a semiring element. The semiring $\mathcal{A}$ is called *(multiplicative) weak a-growth* semiring, if a has no period in $\mathcal{A}$ and for every integer $n \in \mathbb{N}$ and every two non-zero elements $a_1, a_2 \in A \setminus \{\mathbf{0}\}$ and two elements $b, d \in A$ such that $a^n = a_1 \odot b \odot a_2 \oplus d$, there exists an integer $m \in \mathbb{N}$ such that $b \preceq a^m$. □

This property is trivially fulfilled, if there exists an element $a \in A$, which has no period and $\mathbf{1} \preceq a$, and for every $a_1, a_2 \in A \setminus \{\mathbf{0}\}$ the conditions $a_1 \preceq a_1 \odot a_2$, $a_2 \preceq a_1 \odot a_2$, and $\mathbf{0} \preceq \mathbf{1}$ hold. Among the important semirings

- the semiring of the non-negative integers $\mathbb{N}_\infty$ partially ordered by $\leq$ is a weak 2-growth semiring,
- the tropical semiring $\mathbb{T}$ partially ordered by $\leq$ is a weak 1-growth semiring,
- the arctic semiring $\mathbb{A}$ partially ordered by $\leq$ is a weak 1-growth semiring,
- and the language semiring $\mathbb{L}_S$ partially ordered by $\subseteq$ for some alphabet S is a weak s-growth semiring for any $s \in S$.

On the other hand, for every semiring element a the boolean semiring $\mathbb{B}$ and the min-max semiring $\mathbb{R}_{\min,\max}$ are no weak a-growth semirings, because they are multiplicatively periodic.

Next we are going to show that given an additively idempotent weak a-growth semiring, the classes of mod-t-ts transformations computed by polynomial bottom-up tree series transducers using on the one hand pure substitution (i.e., mod $= \varepsilon$) and on the other hand o-substitution (i.e., mod $= o$) are incomparable. Moreover, we also obtain the incomparability of the class of t-ts transformations computed by polynomial bottom-up tree series transducers and the class of t-ts transformations computed by polynomial top-down tree series transducers.

Before stating the incomparability theorem, we want to provide a sketch of the proof of it. Informally speaking, we show both directions by constructing a specific polynomial tree series transducer N using the particular coefficient $a \in A$, which has no period. Then the approximation mapping can be applied to every polynomial tree series transducer M, which is supposed to compute the same mod-t-ts transformation. By a careful choice of the input and output ranked alphabets we limit the constants $u_{M,\mathrm{mod}}$ and v_M. We then proceed by showing that the stated tree series transducer N has a higher growth rate than the polynomial tree series transducer M. Since the element $a \in A$ has no period the growth argument yields the desired contradiction.

5.9 Lemma (Incomparability results)
Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ be a partially ordered semiring with property (MO$\oplus$). Moreover, let $\mathcal{A}$ be a weak a-growth semiring for some element $a \in A$ such that $\mathbf{1} \preceq a$ and let

$$x = \begin{cases} \mathrm{p} & \text{, if } \mathcal{A} \text{ is additively idempotent} \\ \mathrm{d} & \text{, otherwise} \end{cases} .$$

We conclude

- h–BOT($\mathcal{A}$) $\not\subseteq$ x–BOT^o($\mathcal{A}$) and h–BOT^o($\mathcal{A}$) $\not\subseteq$ x–BOT($\mathcal{A}$),
- h–BOT($\mathcal{A}$) $\not\subseteq$ x–TOP($\mathcal{A}$) and h–TOP($\mathcal{A}$) $\not\subseteq$ x–BOT($\mathcal{A}$).

Proof. First we simultaneously prove h–BOT($\mathcal{A}$) $\not\subseteq$ x–BOT^o($\mathcal{A}$) and h–BOT($\mathcal{A}$) $\not\subseteq$ x–TOP($\mathcal{A}$). We consider the ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and the ranked alphabet $\Delta = \{\alpha^{(0)}\}$. Then, by Lemma 5.7(i) there is a homomorphism bottom-up tree series transducer M over Σ and Δ such that a is an upper bound of the coefficients of the tree representation of M, $u_{M,\varepsilon} = r_M = 2$, and for every positive integer $n \in \mathbb{N}_+$ there exist an input tree $s \in T_\Sigma$ of height n and an output tree $t \in \mathrm{supp}(\tau_M(s))$ such that $(\tau_M(s), t) = f_{a,u_{M,\varepsilon}}(n) = a^{2^n-1}$.

Now assume that there exists a polynomial tree series transducer $N = (Q, \Sigma, \Delta, \mathcal{A}, F, \nu)$, which is bottom-up or top-down and in case the semiring is not additively idempotent is also deterministic, with $\tau_N^o = \tau_M$. Since N is polynomial, there are only finitely many non-zero coefficients $c_1, \ldots, c_k \in A$ for some non-negative integer $k \in \mathbb{N}$ occurring in the tree series of the tree representation ν. Apparently, we can assume that for every coefficient c_j with $j \in [k]$ of the tree representation ν there exist elements $a_j, \bar{a}_j \in A \setminus \{\mathbf{0}\}$, $b_j \in A$, and $m_j \in \mathbb{N}$ such that the equality $a^{m_j} = a_j \odot c_j \odot \bar{a}_j \oplus b_j$ holds. If there is a coefficient c_j not obeying this property, then it cannot influence τ_N^o, because $\tau_N^o = \tau_M$ and every coefficient appearing in a tree series in the range of τ_M is a power of a. Thus, such coefficients c_j can be set to $\mathbf{1}$.

By the weak a-growth property, there is an $e_j \in \mathbb{N}$ such that $c_j \preceq a^{e_j}$. Consequently, $\max_{i\in[k]} a^{e_i} = a^{\max_{i\in[k]} e_i}$ is greater than every coefficient c_j and, thus, it is an upper bound of the coefficients. Hence $c' = a^m$ with $m = \max_{i\in[k]} e_i$ is an upper bound of the coefficients of ν. By Corollary 5.5 and Lemma 5.6 for every input tree $s \in T_\Sigma$ the cost for every output tree $t \in \mathrm{supp}(\tau_N^o(s))$ is approximated by

$$(\tau_N^o(s), t) \preceq f_{c',1}(\mathrm{height}(s)) = (c')^{\mathrm{height}(s)} = (a^m)^{\mathrm{height}(s)},$$

because $u_{N,o} \leq 1$ and $v_N \leq 1$ due to the specific form of Δ. However, it is known that there is no non-negative integer $m \in \mathbb{N}$ such that $m \cdot n \geq 2^n - 1$ for every positive integer $n \in \mathbb{N}_+$. Hence there exists a positive integer $n' \in \mathbb{N}_+$ such that $m \cdot n' < 2^{n'} - 1$. With this height n' there also exist an input tree $s' \in T_\Sigma$ and an output tree $t' \in \mathrm{supp}(\tau_M(s'))$ such that $(\tau_M(s'), t') = f_{a,2}(n') = a^{2^{n'}-1}$, whereas $(\tau_N^o(s'), t') \preceq a^{m\cdot n'}$ and $a^{m\cdot n'} \prec a^{2^{n'}-1}$, which yields a contradiction to the assumption $\tau_N^o = \tau_M$. Consequently, τ_M is neither in x–BOT^o($\mathcal{A}$) nor in x–TOP($\mathcal{A}$).

The remaining statements, i.e., h–BOT^o($\mathcal{A}$) $\not\subseteq$ x–BOT($\mathcal{A}$) and h–TOP($\mathcal{A}$) $\not\subseteq$ x–BOT($\mathcal{A}$), are established using the input ranked alphabet $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and output ranked alphabet $\Delta = \{\sigma^{(2)}, \alpha^{(0)}\}$. By Lemma 5.7(ii) there is a homomorphism bottom-up tree series transducer N such that a is an upper bound of the coefficients of the tree representation of N and $u_{N,o} = 2$ and by Lemma 5.7(iii) there is a homomorphism top-down tree series transducer N' with c being an upper bound of the coefficients of the tree representation of N' and $v_{N'} = 2$. Moreover, for every positive integer $n \in \mathbb{N}_+$ there exist input trees $s, s' \in T_\Sigma$ of height n and output trees $t \in \mathrm{supp}(\tau_N^o(s))$ and $t' \in \mathrm{supp}(\tau_{N'}(s'))$ such that

$$(\tau_N^o(s), t) = f_{a,u_{N,o}}(n) = a^{2^n-1} \quad \text{and} \quad (\tau_{N'}(s'), t') = (\tau_{N'}^o(s'), t') = f_{a,v_{N'}}(n) = a^{2^n-1}.$$

Let M be a polynomial bottom-up tree series transducer $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$, which additionally is restricted to be deterministic, if the semiring $\mathcal{A}$ is not additively idempotent, with $\tau_M = \tau_N^o$. An argumentation, which is analogous to the one in the first part of the proof, shows that for every input tree $s \in T_\Sigma$ and output tree $t \in \text{supp}(\tau_M(s))$ the condition $(\tau_M(s), t) \preceq (c')^{\text{height}(s)}$ holds, where $c' = a^m$ for some non-negative integer $m \in \mathbb{N}$. This again yields the desired contradiction, which finally establishes the lemma. ■

Using the previous lemma, we are now ready to state the promised incomparability results in our main theorem.

5.10 Theorem (Incomparability results)
Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ be a partially ordered semiring with property (MO⊕). Moreover, let $\mathcal{A}$ be a weak a-growth semiring for some element $a \in A$ such that $\mathbf{1} \preceq a$. For every

$$x, y \in \begin{cases} \{\mathrm{p}, \mathrm{d}, \mathrm{t}, \mathrm{dt}, \mathrm{h}\} & \text{, if } \mathcal{A} \text{ is additively idempotent} \\ \{\mathrm{d}, \mathrm{dt}, \mathrm{h}\} & \text{, otherwise} \end{cases}$$

we have

$$x\text{–BOT}(\mathcal{A}) \bowtie y\text{–BOT}^o(\mathcal{A}) \quad \text{and} \quad x\text{–BOT}(\mathcal{A}) \bowtie y\text{–TOP}(\mathcal{A}).$$

Proof. The theorem is an immediate consequence of Lemma 5.9. ■

Using the previous theorem we can now derive some incomparability results for some specific semirings. Consider the additively idempotent semirings introduced in the preliminaries (i.e., $\mathbb{T}$, $\mathbb{A}$, $\mathbb{B}$, $\mathbb{R}_{\text{min,max}}$, and $\mathbb{L}_S$). Two of them, namely the boolean semiring $\mathbb{B}$ and the min-max semiring $\mathbb{R}_{\text{min,max}}$, are not applicable, because they are multiplicatively periodic and, thus, cannot be weak a-growth semirings. For the remaining ones we derive the following statements.

5.11 Corollary (Instantiating Theorem 5.10)

(i) Let S be an alphabet. For every semiring $\mathcal{A} \in \{\mathbb{T}, \mathbb{A}, \mathbb{L}_S\}$ we obtain for every $x, y \in \{\mathrm{p}, \mathrm{d}, \mathrm{t}, \mathrm{dt}, \mathrm{h}\}$ the incomparabilities

$$x\text{–BOT}(\mathcal{A}) \bowtie y\text{–BOT}^o(\mathcal{A}) \quad \text{and} \quad x\text{–BOT}(\mathcal{A}) \bowtie y\text{–TOP}(\mathcal{A}),$$

(ii) and for the semiring $\mathbb{N}_\infty$ we obtain for every $x, y \in \{\mathrm{d}, \mathrm{dt}, \mathrm{h}\}$ the incomparabilities

$$x\text{–BOT}(\mathbb{N}_\infty) \bowtie y\text{–BOT}^o(\mathbb{N}_\infty) \quad \text{and} \quad x\text{–BOT}(\mathbb{N}_\infty) \bowtie y\text{–TOP}(\mathbb{N}_\infty).$$

Proof. Both results are immediate consequences of Theorem 5.10. ■

In fact, for $x = y = p$, the first part of Corollary 5.11(ii) is slightly weaker than Corollary 5.18 of [FV03], because in the latter result classes of polynomial t-ts transformations are compared (and not only deterministic ones). Also note that, for $\mathcal{A} = \mathbb{T}$ the first part of Corollary 5.11(i) restates Corollary 5.23 of [FV03].

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