Incomparability results for polynomial bottom-up tree series transducers

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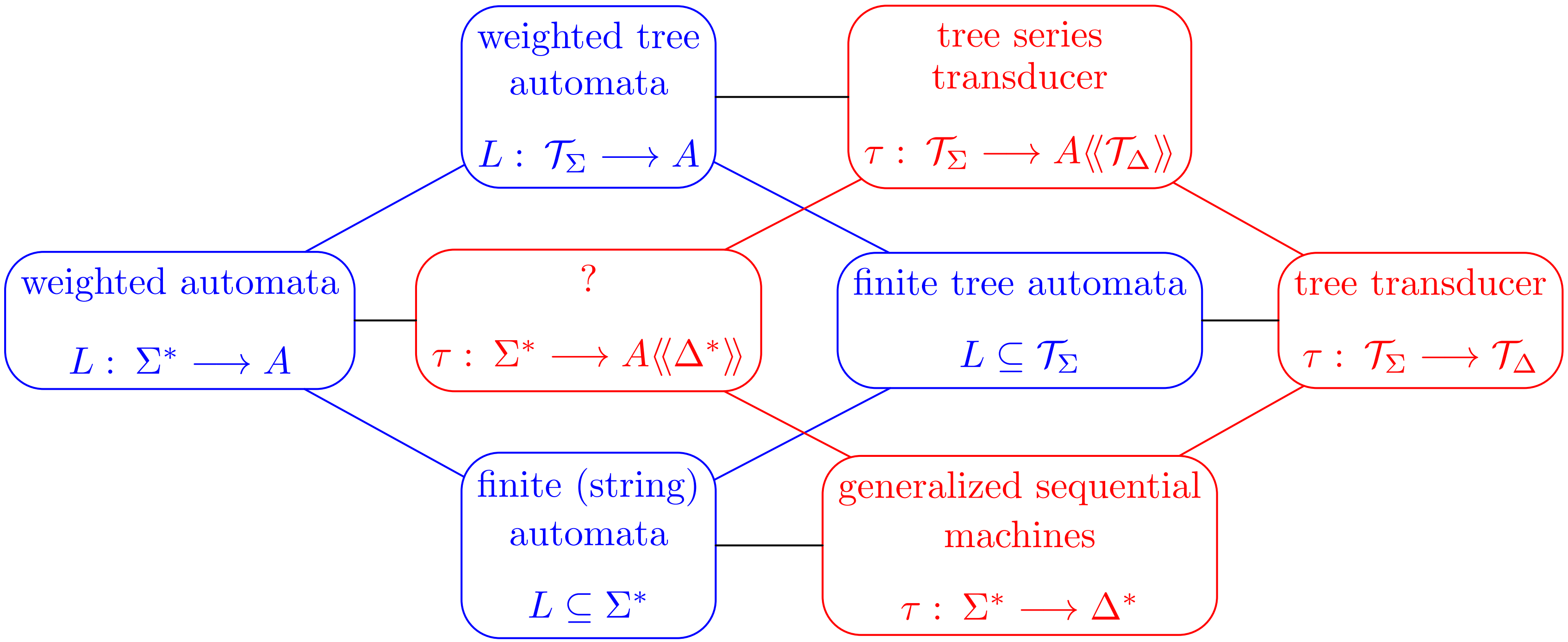
Generalization hierarchy



Semirings and Orders

- *semiring* $\mathfrak{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ comprises of a commutative monoid $(A, \oplus, \mathbf{0})$ and monoid $(A, \odot, \mathbf{1})$; $\odot$ distributes over $\oplus$ and $\mathbf{0}$ is absorbing
- *partial order* $\preceq \subseteq A \times A$ is reflexive, antisymmetric and transitive
- partial order $\preceq$ is *consistent*, iff every finite subset $S \subseteq A$ has an upper bound
- semiring $\mathfrak{A}$ is *partially ordered* by the partial order $\preceq$, iff $a_1 \preceq a_2$ implies
  1. $a_1 \oplus a \preceq a_2 \oplus a$,
  2. $a_1 \odot a \preceq a_2 \odot a$ and $a \odot a_1 \preceq a \odot a_2$
- semiring $\mathfrak{A}$ is *naturally ordered*, iff $\sqsubseteq \subseteq A \times A$ is a partial order, where

$$a \sqsubseteq b \qquad \text{iff} \qquad (\exists c \in A) : \; b = a \oplus c$$

- naturally ordered semirings are partially ordered by the consistent partial order $\sqsubseteq$

Examples

1. the *natural numbers* $\mathbb{N}_\infty = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ are naturally ordered by $\sqsubseteq = \leq$
2. the *Boolean semiring* $\mathbb{B} = (\{\bot, \top\}, \vee, \wedge, \bot, \top)$ is naturally ordered by $\bot < \top$
3. the *arctical semiring* $\mathbb{A} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ is naturally ordered by $\sqsubseteq = \leq$
4. the *tropical semiring* $\mathbb{T} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ is naturally ordered by $\sqsubseteq = \geq$ and (consistently) partially ordered by $\leq$

A semiring is *additively idempotent*, iff $a \oplus a = a$. The latter three presented semirings are additively idempotent.

A semiring preserves $\preceq$ under *continuation*, iff $a \preceq a \oplus a$. All additively idempotent as well as naturally ordered semirings preserve the order under continuation.

Tree Series

- a *tree series* φ is a mapping of type $\mathcal{T}_\Delta(V) \longrightarrow A$; (φ, t) is used to denote $\varphi(t)$
- the *class of all tree series* is denoted $A\langle\langle \mathcal{T}_\Delta(V) \rangle\rangle$
- the *support* of a tree series φ is defined to be $\mathrm{supp}(\varphi) = \{\, t \in \mathcal{T}_\Delta(V) \mid (\varphi, t) \neq \mathbf{0} \,\}$
- φ is *polynomial* iff its support is finite; the corresponding class is $A\langle \mathcal{T}_\Delta(V) \rangle$
- Let $\varphi \in A\langle\langle \mathcal{T}_\Delta(X_k) \rangle\rangle$, $(\psi_1, \ldots, \psi_k) \in A\langle\langle \mathcal{T}_\Delta(V) \rangle\rangle^k$. *Substitution* of $(\psi_1, \ldots, \psi_k)$ into φ is

$$\varphi \longleftarrow (\psi_1, \ldots, \psi_k) = \sum_{\substack{t \in \mathrm{supp}(\varphi) \\ (\forall i \in [k]):\, t_i \in \mathrm{supp}(\psi_i)}} \big((\varphi, t) \odot (\psi_1, t_1) \odot \cdots \odot (\psi_k, t_k)\big) t[t_1, \ldots, t_k].$$

- whereas o-*substitution* of $(\psi_1, \ldots, \psi_k)$ into φ is

$$\varphi \overset{o}{\longleftarrow} (\psi_1, \ldots, \psi_k) = \sum_{\substack{t \in \mathrm{supp}(\varphi) \\ (\forall i \in [k]):\, t_i \in \mathrm{supp}(\psi_i)}} \big((\varphi, t) \odot (\psi_1, t_1)^{|t|_{x_1}} \odot \cdots \odot (\psi_k, t_k)^{|t|_{x_k}}\big) t[t_1, \ldots, t_k].$$

Tree Series Transducers

$M = (Q, \Sigma, \Delta, \mathfrak{A}, Q_d, \mu)$, where

- Q and $Q_d \subseteq Q$ are *finite* sets of states and final states, resp.
- Σ and Δ are the input and output ranked alphabets, resp.
- $\mathfrak{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring
- μ is a family of mappings $(\mu_k)_{k \in \mathbb{N}}$ of type

$$\mu_k : \Sigma^{(k)} \longrightarrow A\langle\langle \mathcal{T}_\Delta(X_k) \rangle\rangle^{Q \times Q^k},$$

M is

1. *polynomial*, if every $\mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \in A\langle \mathcal{T}_\Delta(X_k) \rangle$
2. *deterministic*, if for every k-ary σ and $(q_1, \ldots, q_k) \in Q^k$ and q

$$\#(\mathrm{supp}(\mu_k(\sigma)_{q,(q_1,\ldots,q_k)})) \leq 1$$

   and for at most one q equality holds.

Semantics of tree series transducers

Let $\mathrm{mod} \in \{\varepsilon, o\}$.

$$\overline{\mu_k(\sigma)}^{\mathrm{mod}} : \left(A\langle\langle \mathcal{T}_\Delta \rangle\rangle^{Q \times \{1\}}\right)^k \longrightarrow A\langle\langle \mathcal{T}_\Delta \rangle\rangle^{Q \times \{1\}}$$

$$\overline{\mu_k(\sigma)}^{\mathrm{mod}}(R_1, \ldots, R_k)_q = \sum_{(q_1,\ldots,q_k) \in Q^k} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \stackrel{\mathrm{mod}}{\longleftarrow} \big((R_1)_{q_1}, \ldots, (R_k)_{q_k}\big).$$

Initial homomorphism: $h_\mu^{\mathrm{mod}} : \mathcal{T}_\Sigma \longrightarrow A\langle\langle \mathcal{T}_\Delta \rangle\rangle^{Q \times \{1\}}$

$$h_\mu^{\mathrm{mod}}(\sigma(s_1, \ldots, s_k)) = \overline{\mu_k(\sigma)}^{\mathrm{mod}}(h_\mu^{\mathrm{mod}}(s_1), \ldots, h_\mu^{\mathrm{mod}}(s_k))$$

mod-*semantics* of M is $\tau_M^{\mathrm{mod}} : \mathcal{T}_\Sigma \longrightarrow A\langle\langle \mathcal{T}_\Delta \rangle\rangle$

$$\tau_M^{\mathrm{mod}}(s) = \sum_{q \in Q_d} h_\mu^{\mathrm{mod}}(s)_q$$

Example

Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and $\Delta = \{\alpha^{(0)}\}$. The (bottom-up) tree series transducer

$$M = (\{*\}, \Sigma, \Delta, \mathfrak{A}, \{*\}, \mu),$$

where $\mu_0(\alpha)_{*,\varepsilon} = \mu_2(\sigma)_{*,(*,*)} = c\,\alpha$ for some $c \in A$, is *deterministic* and gives rise to the following *translations*:

- $\tau_M(s) = c^{\text{size}(s)}\ \alpha$ since

$$h_\mu(s)_* = \begin{cases} \Big(c \odot \underbrace{(h_\mu(s_1)_*, \alpha)}_{c^{\text{size}(s_1)}} \odot \underbrace{(h_\mu(s_2)_*, \alpha)}_{c^{\text{size}(s_2)}}\Big)\alpha & \text{, if } s = \sigma(s_1, s_2) \\ c\,\alpha & \text{, if } s = \alpha \end{cases}$$

- $\tau_M^o(s) = c\,\alpha$ since

$$h_\mu^o(s)_* = c\,\alpha$$

Requirements and constants

- *partially ordered semiring* $\mathfrak{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ via a *consistent* partial order $\preceq$
- $\preceq$ preserves order under *continuation*

or

- $\mathfrak{A}$ is *naturally ordered* and $\preceq = \sqsubseteq$

$M = (Q, \Sigma, \Delta, \mathfrak{A}, Q_d, \mu)$ is a *polynomial* bottom-up tree series transducer

1. the *maximal rank* $r = \max \left\{ k \in \mathbb{N} \;\middle|\; \Sigma^{(k)} \neq \emptyset \right\}$ of a symbol of Σ
2. the number of *follow-up states* $d = \begin{cases} 1 & \text{, if } M \text{ is deterministic} \\ \#(Q) & \text{, otherwise} \end{cases}$
3. the *maximal support cardinality* e of any tree series of μ

$$e = \max \left\{ \#(\mathrm{supp}(\mu_k(\sigma)_{q,w})) \;\middle|\; k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q, w \in Q^k \right\}.$$

Cardinality approximations

Lemma:

(a) $\#(\mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_q)) \leq \#(\mathcal{T}_\Delta)$ and

(b) $\#(\mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_q)) \leq d^{(\sum_{i=1}^{\mathrm{height}(s)-1} r^i)} e^{(\sum_{i=0}^{\mathrm{height}(s)-1} r^i)}$ hold.

Proof: (a) trivial, (b) straightforward induction over s

Observation: Both approximations are *monotonic* in s (or $\mathrm{height}(s)$).

Observation: For *deterministic* transducers:

$$\#(\mathrm{supp}(h_\mu^{\mathrm{mod}}(s)_q)) \leq 1$$

Constants again

1. the *maximal rank* $r = \max\left\{ k \in \mathbb{N} \;\middle|\; \Sigma^{(k)} \neq \emptyset \right\}$ of a symbol of Σ
2. the number of *follow-up states* $d = \begin{cases} 1 & \text{, if } M \text{ is deterministic} \\ \#(Q) & \text{, otherwise} \end{cases}$
3. the *maximal support cardinality* e of any tree series of μ
$$e = \max\left\{ \#(\mathrm{supp}(\mu_k(\sigma)_{q,w})) \;\middle|\; k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q, w \in Q^k \right\}.$$
4. an *upper bound* c of all coefficients appearing in the tree representation μ
$$c \in \left(\{\mathbf{1}\} \cup \left\{ (\mu_k(\sigma)_{q,w}, t) \;\middle|\; k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q, w \in Q^k, t \in \mathrm{supp}(\mu_k(\sigma)_{q,w}) \right\}\right)^u,$$
5. the *maximal number of variables* u in the support of any tree series of μ
$$u = \begin{cases} r & \text{, if } \mathrm{mod} = \varepsilon \\ \max_{\substack{k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q, \\ w \in Q^k, t \in \mathrm{supp}(\mu_k(\sigma)_{q,w})}} \sum_{x \in X_k} |t|_x & \text{, if } \mathrm{mod} = o \end{cases}$$

Approximations

- order-preserving cardinality approximation mapping $l : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$

approximation mapping $f^{\text{mod}}_{M,l} : \mathbb{N}_+ \longrightarrow A$

$$f^{\text{mod}}_{M,l}(n) = \begin{cases} c & \text{, if } n = 1 \\ \displaystyle\sum_{i=1}^{d^r e\, l(n-1)^r} c \odot f^{\text{mod}}_{M,l}(n-1)^u & \text{, if } n > 1 \end{cases}$$

In case M is *deterministic* or $\mathfrak{A}$ is *additively idempotent*

$$f^{\text{mod}}_{M}(n) = \begin{cases} c & \text{, if } n = 1 \\ c \odot f^{\text{mod}}_{M}(n-1)^u & \text{, if } n > 1 \end{cases}$$

Approximations (cont'd)

Lemma: For every $s \in \mathcal{T}_\Sigma$ and $t \in \operatorname{supp}(h_\mu^{\mathrm{mod}}(s)_q)$ with $q \in Q_d$ we have

$$(h_\mu^{\mathrm{mod}}(s)_q, t) \preceq f_{M,l}^{\mathrm{mod}}(\mathrm{height}(s)).$$

Example: $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, $\Delta = \{\alpha^{(0)}\}$ and $\mu_0(\alpha)_* = \mu_2(\sigma)_{*,(*,*)} = c\,\alpha$

- regular substitution: $r = u = 2$

$$(\tau_M(s), \alpha) \leq c^{2^{\mathrm{height}(s)}-1}$$

- o-substitution: $u = 0$

$$(\tau_M^o(s), \alpha) \leq c$$

Sharpness of the approximation

If $\mathfrak{A}$ is *additively idempotent* or M is *deterministic*, then the given approximation is *sharp*, i.e. given suitable c, u, there exists a M such that

$$(\forall n \in \mathbb{N})(\exists s \in \mathcal{T}_\Sigma)\, \text{height}(s) = n\, (\exists t \in \mathcal{T}_\Delta) : \quad (\tau_M^{\text{mod}}(s), t) = f_M^{\text{mod}}(n)$$

Proof: by Construction of a suitable M

Let p-BOT$(\mathfrak{A}) = \{\, \tau_M \mid M$ is a polynomial bottom-up transducer over $\mathfrak{A}\,\}$ and similarly also p-BOT$^o(\mathfrak{A})$, d-BOT$(\mathfrak{A})$, d-BOT$^o(\mathfrak{A})$.

Incomparability results

Lemma:

- partially ordered semiring $\mathfrak{A}$ ordered via a consistent order $\preceq$
- semiring $\mathfrak{A}$ *additively idempotent*
- exists $c \in A$ such that $(\forall i, j \in \mathbb{N})$ with $i < j$ the condition $c^i \prec c^j$ holds

$$p\text{-BOT}(\mathfrak{A}) \bowtie p\text{-BOT}^o(\mathfrak{A})$$

Proof: by contradiction using M and $N = (\{*\}, \Sigma, \Delta, \mathfrak{A}, \{*\}, \nu)$, where

$$\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\} \qquad \Delta = \{\sigma^{(2)}, \alpha^{(0)}\}$$

$\nu_0(\alpha)_{*,\varepsilon} = c\,\alpha$ and $\nu_1(\gamma)_{*,*} = c\,\sigma(x_1, x_1)$.

Incomparability results (cont'd)

Lemma:

- exists $c \in A$ such that $(\forall i, j \in \mathbb{N})$ with $i \neq j$ the condition $c^i \neq c^j$ holds

$$d\text{-BOT}(\mathfrak{A}) \bowtie d\text{-BOT}^o(\mathfrak{A})$$

Lemma:

- A finite or $\mathfrak{A}$ commutative
- $x \in \{n, l\}$ (non-deleting or linear)
- $(\forall a \in A)(\exists i, j \in \mathbb{N})$ with $i < j$ such that $a^i = a^j$

either $dx\text{-BOT}(\mathfrak{A}) \subseteq dx\text{-BOT}^o(\mathfrak{A})$ or $dx\text{-BOT}(\mathfrak{A}) \supseteq dx\text{-BOT}^o(\mathfrak{A})$