Tree Series Transducers 
and Weighted Tree Automata

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Generalization Hierarchy

- **Weighted Automaton**
  \[ L \in A \langle T_\Sigma \rangle \]
  \[ \tau : \Sigma^* \rightarrow A \langle \Delta^* \rangle \]

- **Weighted Tree Automaton**
  \[ L \in A \langle T_\Sigma \rangle \]

- **Tree Series Transducer**
  \[ \tau : T_\Sigma \rightarrow A \langle T_\Delta \rangle \]

- **Tree Automaton**
  \[ L \subseteq T_\Sigma \]

- **Tree Transducer**
  \[ \tau : T_\Sigma \rightarrow \mathcal{P}(T_\Delta) \]

- **String Automaton**
  \[ L \subseteq \Sigma^* \]

- **Generalized Sequential Machine**
  \[ \tau : \Sigma^* \rightarrow \mathcal{P}(\Delta^*) \]

**Basic Definitions 2 December 2, 2003**
Bottom-Up Tree Series Transducers

\[ M = (Q, \Sigma, \Delta, A, F, \mu) \]

- **input and output ranked alphabet** \( \Sigma = \Delta = \{ \sigma(2), \alpha(0), \beta(0) \} \),
- **states and final states** \( Q = F = \{ p, q \} \),
- **semiring** \( A = \mathbb{P} = (\mathcal{P}(\mathbb{N}^*_1), \cup, \circ, \emptyset, \{\varepsilon\}) \) with \( P_1 \circ P_2 = \{ ab \mid a \in P_1, b \in P_2 \} \), and
- **tree representation** \( \mu \)

\[ \begin{align*}
\alpha & \rightarrow q \\
\beta & \rightarrow q \\
\sigma & \rightarrow \{\varepsilon\} \sigma \\
\sigma & \rightarrow \{ i \} x_i
\end{align*} \]
• a tree series $\varphi$ is a mapping of type $T_\Delta(V) \rightarrow A$; $(\varphi, t)$ is used to denote $\varphi(t)$

• the class of all tree series is denoted $A \langle \langle T_\Delta(V) \rangle \rangle$

• the support of a tree series $\varphi$ is defined to be $\text{supp}(\varphi) = \{ t \in T_\Delta(V) \mid (\varphi, t) \neq 0 \}$

• $\varphi$ is polynomial iff its support is finite; the corresponding class is $A(T_\Delta(V))$

• Let $\varphi \in A \langle \langle T_\Delta(X_k) \rangle \rangle$, $(\psi_1, \ldots, \psi_k) \in A \langle \langle T_\Delta(V) \rangle \rangle^k$. Substitution of $(\psi_1, \ldots, \psi_k)$ into $\varphi$ is

$$
\varphi \leftarrow (\psi_1, \ldots, \psi_k) = \sum_{\substack{t \in \text{supp}(\varphi) \\
(\forall i \in [k]): t_i \in \text{supp}(\psi_i) \}} \left( (\varphi, t) \circ (\psi_1, t_1) \circ \cdots \circ (\psi_k, t_k) \right) t[t_1, \ldots, t_k].
$$
Bottom-up Tree Series Transducers

\[ M = (Q, \Sigma, \Delta, A, F, \mu), \] where

- \( Q \) and \( F \subseteq Q \) are \textit{finite} sets of states and final states, resp.,
- \( \Sigma \) and \( \Delta \) are the input and output ranked alphabets, resp.,
- \( A = (A, \oplus, \odot, 0, 1) \) is a semiring
- \( \mu \) is a family of mappings \( (\mu_k)_{k \in \mathbb{N}} \) of type
  \[ \mu_k : \Sigma^{(k)} \rightarrow A^{\llbracket T_\Delta(X_k) \rrbracket Q \times Q^k}. \]
Semantics of Bottom-up Tree Series Transducers

\[ \mu_k(\sigma) : (A\langle\langle T_\Delta \rangle\rangle^Q)^k \rightarrow A\langle\langle T_\Delta \rangle\rangle^Q \]

\[ \mu_k(\sigma)(R_1, \ldots, R_k)_q = \sum_{(q_1, \ldots, q_k) \in Q^k} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \leftarrow ((R_1)_{q_1}, \ldots, (R_k)_{q_k}). \]

Initial homomorphism: \( h_\mu : T_\Sigma \rightarrow A\langle\langle T_\Delta \rangle\rangle^Q \)

\[ h_\mu(\sigma(s_1, \ldots, s_k)) = \mu_k(\sigma)(h_\mu(s_1), \ldots, h_\mu(s_k)) \]

**tree-to-tree-series transformation** computed by \( M \) is \( \tau_M : T_\Sigma \rightarrow A\langle\langle T_\Delta \rangle\rangle \)

\[ \tau_M(s) = \sum_{q \in F} h_\mu(s)_q \]
Bottom-up Weighted Tree Automata

\[ M = (Q, \Sigma, A, F, \mu), \]
where

- \( Q \) and \( F \subseteq Q \) are finite sets of states and final states, respectively,
- \( \Sigma \) is the input ranked alphabet, respectively,
- \( A = (A, \oplus, \odot, 0, 1) \) is a semiring,
- \( \mu \) is a family of mappings \( (\mu_k)_{k \in \mathbb{N}} \) of type \( \mu_k : \Sigma^{(k)} \rightarrow A^{Q \times Q^k} \).

Semantics is similarly defined as it is for bottom-up tree series transducers.
Let \( A = (A, \oplus, \odot, 0, 1) \) be a semiring. We define the following algebraic structure.

\[
B = A\langle T \rangle^* \odot (\{\varepsilon\} \cup \{(n, \varphi) \mid n \in \mathbb{N}_+, \varphi \in A\langle T \Delta(X_n) \rangle\}) \quad S = \mathbb{N}^B
\]

\( S = (S, \cup, \circ, \emptyset, \{\varepsilon\}) \) with addition being defined for every element \( b \in B \) and every two semiring elements \( S_1, S_2 \in S \) by

\[
(S_1 \cup S_2)(b) = S_1(b) + S_2(b).
\] (1)

This addition is trivially associative, commutative, and has unit element \( \emptyset : B \rightarrow \mathbb{N} \) which is defined for every \( b \in B \) to be \( \emptyset(b) = 0 \).

The multiplication is defined for every element \( b \in B \) and every two semiring elements \( S_1, S_2 \in S \) by

\[
(S_1 \circ S_2)(b) = \sum_{b_1, b_2 \in B, b = b_1 \leftrightarrow b_2} S_1(b_1) \cdot S_2(b_2).
\] (2)
Wrapping Substitution

On $B$ we define the following operation $\leftarrow : B^2 \rightarrow B$:

$$a \leftarrow b = a.b$$

if $a \in A\langle T_\Delta \rangle^*$ or $b = \varepsilon$,

$$a.(1, \phi) \leftarrow \psi.b = a.(\phi \leftarrow_0 \psi).b,$n

$$a.(n, \phi) \leftarrow \psi.b = a.(n-1, \phi \leftarrow_0 \psi) \leftarrow b$$

, if $n > 1$,

$$a.(n, \phi) \leftarrow (m, \psi) = a.(n-1+m, \phi \leftarrow_m \psi).$$

The substitutions $(\leftarrow_k : A\langle T_\Delta (X) \rangle \times A\langle T_\Delta (X_k) \rangle \rightarrow A\langle T_\Delta (X) \rangle \mid k \in \mathbb{N})$ are defined as follows.

$$a \leftarrow_k b = a[x_i/x_{i+k-1} \mid i > 1] \leftarrow (b)$$

**Lemma:** $(a \leftarrow b) \leftarrow c = a \leftarrow (b \leftarrow c)$.

**Lemma:** $(\phi \leftarrow_m \psi) \leftarrow_k \tau = \phi \leftarrow_{m-1+k} (\psi \leftarrow_k \tau)$ with $m \neq 0$. 
Remaining Questions and Literature

- Deterministic tree series transducers?
- Tree transducers, i.e., polynomial tree series transducers over \( \mathbb{B} \)
- Top-down tree series transducers?
- \( \sigma \)-substitution?

Some References:

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