Relating Tree Series Transducers and Weighted Tree Automata

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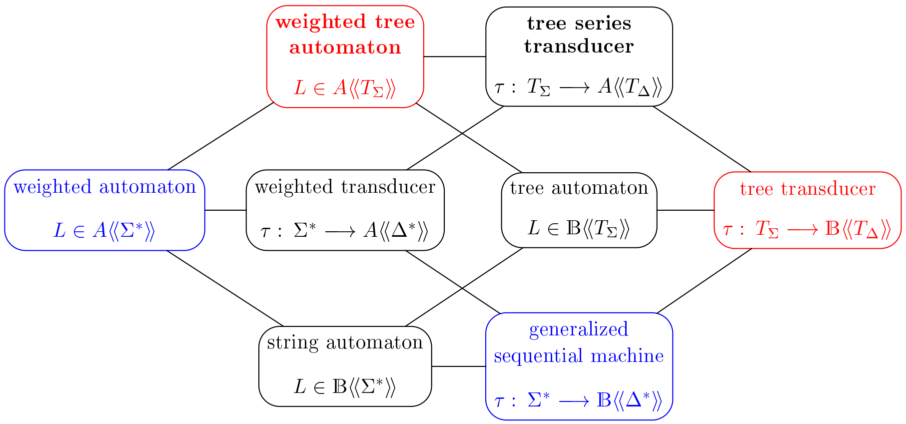
June 18, 2004

1. Semirings and DM-monoids
2. Bottom-Up DM-monoid Weighted Tree Automata
3. Establishing a Relationship

Generalization Hierarchy



Known Relations and Problems

- *String-based:*
  Theorem: For every generalized sequential machine M, there exists a weighted automaton N such that $\|M\| = \|N\|$.
  Required semiring: Start with the monoid $(\Delta^*, \circ, \varepsilon)$ and extend it to the semiring $(\mathbb{B}\langle\langle \Delta^* \rangle\rangle, \vee, \circ, \widetilde{0}, 1\,\varepsilon) \cong (\mathcal{P}(\Delta^*), \cup, \circ, \emptyset, \{\varepsilon\})$.

  Theorem: For every weighted transducer M, there exists a weighted automaton N such that $\|M\| = \|N\|$.
- *Tree-based:*
  Problem: For every tree transducer M, does there exist a weighted tree automaton N such that $\|M\| = \|N\|$?

  Problem: For every tree series transducer M, does there exist a weighted tree automaton N such that $\|M\| = \|N\|$?

Monoids

- A monoid is an algebraic structure $\mathcal{A} = (A, \oplus, \mathbf{0})$ such that
  (i) $\oplus$ is associative and
  (ii) $\mathbf{0}$ is the neutral element.
- $\mathcal{A}$ is complete, if

(C1) $\bigoplus_{i \in [n]} a_i = a_1 \oplus \cdots \oplus a_n$ for every $n \in \mathbb{N}$,

(C2) $\bigoplus_{j \in J} (\bigoplus_{i \in I_j} a_i) = \bigoplus_{i \in I} a_i$, if $I = \bigcup_{j \in J} I_j$ is a partition.

- $\mathcal{A}$ is naturally ordered, if the relation $\sqsubseteq \subseteq A^2$ defined by

$$a \sqsubseteq b \iff (\exists c \in A) : a \oplus c = b$$

  is a partial order.
- $\mathcal{A}$ is continuous, if $\mathcal{A}$ is *naturally ordered* and *complete* and

$$\bigoplus_{i \in I} a_i = \sup\{ \bigoplus_{i \in E} a_i \mid E \subseteq I, E \text{ finite} \}.$$

Semirings

- A semiring is an algebraic structure $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ such that
  - (i) $(A, \oplus, \mathbf{0})$ is a commutative monoid,
  - (ii) $(A, \odot, \mathbf{1})$ is a monoid,
  - (iii) $\mathbf{0}$ is the absorbing element with respect to $\odot$, and
  - (iv) $\odot$ (left and right) distributes over $\oplus$.
- $\mathcal{A}$ is complete, if $(A, \oplus, \mathbf{0})$ is complete and

(C3) $a \odot (\bigoplus_{i \in I} a_i) = \bigoplus_{i \in I} (a \odot a_i)$ and $(\bigoplus_{i \in I} a_i) \odot a = \bigoplus_{i \in I} (a_i \odot a)$.

- $\mathcal{A}$ is naturally ordered, if $(A, \oplus, \mathbf{0})$ is naturally ordered.
- $\mathcal{A}$ is continuous, if $\mathcal{A}$ is *naturally ordered* and *complete* and $(A, \oplus, \mathbf{0})$ is continuous.

Examples of Semirings

- $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ the semiring of natural numbers,
- $\mathbb{N}_\infty = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ the complete semiring of natural numbers,
- $\mathrm{Trop} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ the tropical semiring,
- $\mathbb{B} = (\{\bot, \top\}, \vee, \wedge, \bot, \top)$ the Boolean semiring,
- $\mathrm{Lang}_\Sigma = (\mathcal{P}(\Sigma^*), \cup, \circ, \emptyset, \{\varepsilon\})$ the formal language semiring,
- and generally any ring or field

Semiring	commutative	complete	naturally ordered	continuous
$\mathbb{N}$	yes	NO	yes	NO
$\mathbb{N}_\infty$	yes	yes	yes	yes
Trop	yes	yes	yes	yes
$\mathbb{B}$	yes	yes	yes	yes
Lang_Σ	NO	yes	yes	yes

Semimodules (I)

$\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ semiring and $\mathcal{D} = (D, +, 0)$ commutative monoid

- $\mathcal{D}$ is a semimodule of $\mathcal{A}$, if there is a mapping $\cdot : A \times D \longrightarrow D$ such that

(S1) $(a_1 \oplus a_2) \cdot d = a_1 \cdot d + a_2 \cdot d$ and $a \cdot (d_1 + d_2) = a \cdot d_1 + a \cdot d_2$,

(S2) $(a_1 \odot a_2) \cdot d = a_1 \cdot (a_2 \cdot d)$,

(S3) $\mathbf{0} \cdot d = 0 = a \cdot 0$, and $\mathbf{1} \cdot d = d$.

- $\mathcal{D}$ is a complete semimodule of $\mathcal{A}$, if $\mathcal{D}$ is a *semimodule* of $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{A}$ are *complete*, and

$$(\bigoplus_{i \in I} a_i) \cdot d = \sum_{i \in I} (a_i \cdot d) \qquad \text{and} \qquad a \cdot (\sum_{i \in I} d_i) = \sum_{i \in I} (a \cdot d_i).$$

Examples of semimodules (I)

Examples: $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ semiring, $\mathcal{A}_\oplus = (A, \oplus, \mathbf{0})$, and $\mathcal{D} = (D, +, 0)$ commutative monoid.

- $\mathcal{A}_\oplus$ is a semimodule of $\mathcal{A}$, and
- $\mathcal{A}_\oplus$ is a complete semimodule of $\mathcal{A}$, if $\mathcal{A}$ is complete.

	semimodule of $\mathbb{N}$	complete semimodule of $\mathbb{N}_\infty$	semimodule of $\mathbb{B}$	complete semimodule of $\mathbb{B}$
$\mathcal{D}$ is a ..., if $\mathcal{D}$ is ...	always	continuous	idempotent	idempotent, continuous

DM-monoids

$(D, +, 0)$ commutative monoid, Ω set, and $\mathrm{rk} : \Omega \longrightarrow \mathbb{N}$ mapping such that $\omega : D^k \longrightarrow D$ for every $\omega \in \Omega^{(k)}$.

- $\mathcal{D} = (D, +, 0, \Omega)$ is a distributive multi-operator monoid (DM-monoid), if
  (i) $\omega(d_1, \ldots, 0, \ldots, d_k) = 0$, and
  (ii) $\omega(d_1, \ldots, d + d_i, \ldots, d_k) = \omega(d_1, \ldots, d, \ldots, d_k) + \omega(d_1, \ldots, d_k)$.
- $\mathcal{D}$ is complete, if $(D, +, 0)$ is *complete* and

$$\omega(\sum_{i_1 \in I_1} d_{i_1}, \ldots, \sum_{i_k \in I_k} d_{i_k}) = \sum_{(\forall j \in [k]):\, i_j \in I_j} \omega(d_{i_1}, \ldots, d_{i_k}).$$

- $\mathcal{D}$ is naturally ordered, whenever $(D, +, 0)$ is *naturally ordered*.
- $\mathcal{D}$ is continuous, if $\mathcal{D}$ is *complete* and $(D, +, 0)$ is *continuous*.

DM-monoid Weighted Tree Automata — Syntax

Σ ranked alphabet, I, Ω non-empty sets.

- A tree representation over I, Σ, and Ω is $\mu = (\, \mu_k \mid k \in \mathbb{N} \,)$ such that

$$\mu_k : \Sigma^{(k)} \longrightarrow \Omega^{I \times I^k}.$$

- A (bottom-up) DM-monoid weighted tree automaton (DM-wta) $M = (I, \Sigma, \mathcal{D}, F, \mu)$ consists of
  - non-empty set I of states,
  - ranked alphabet Σ of input symbols,
  - a *complete* DM-monoid $(D, +, 0, \Omega)$,
  - a final weight map $F : I \longrightarrow \Omega^{(1)}$, and
  - a tree representation μ over I, Σ, and Ω such that $\mu_k : \Sigma^{(k)} \longrightarrow {\Omega^{(k)}}^{I \times I^k}$.

Excursion: Tree Series

$(A, \oplus, \mathbf{0})$ commutative, complete monoid, Σ ranked alphabet, and $X_k = \{x_1, \ldots, x_k\}$.

- A tree series ψ is a mapping $\psi : T_\Sigma(X_k) \longrightarrow A$.
- $A\langle\langle T_\Sigma(X_k)\rangle\rangle$ is the set of all tree series.
- The sum of $\psi_1, \psi_2 \in A\langle\langle T_\Sigma(X_k)\rangle\rangle$ is $(\psi_1 \oplus \psi_2, s) = (\psi_1, s) \oplus (\psi_2, s)$.
- $\widetilde{\mathbf{0}} \in A\langle\langle T_\Sigma(X_k)\rangle\rangle$ is such that $(\widetilde{\mathbf{0}}, s) = \mathbf{0}$ for all $s \in T_\Sigma(X_k)$.
- $(A\langle\langle T_\Sigma(X_k)\rangle\rangle, \oplus, \widetilde{\mathbf{0}})$ is a complete monoid.
- Tree series substiution of $\psi_1, \ldots, \psi_k \in A\langle\langle T_\Sigma\rangle\rangle$ into $\psi \in A\langle\langle T_\Sigma(X_k)\rangle\rangle$ is

$$\psi \longleftarrow (\psi_1, \ldots, \psi_k) = \sum_{\substack{s \in T_\Sigma(X_k), \\ (\forall i \in [k]): s_i \in T_\Sigma}} \Big((\psi, s) \odot \prod_{i \in [k]} (\psi_i, s_i) \Big)\, s[s_1, \ldots, s_k].$$

DM-monoids (II)

$(D, +, 0, \Omega)$ DM-monoid, $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ semiring.

- $\mathcal{D}$ is semimodule of $\mathcal{A}$, if $(D, +, 0)$ is a *semimodule* of $\mathcal{A}$ and

$$\omega(d_1, \ldots, a \cdot d_i, \ldots, d_k) = a \cdot \omega(d_1, \ldots, d_k).$$

- $\mathcal{D}$ is a complete semimodule of $\mathcal{A}$, if $\mathcal{D}$ is *complete* and $(D, +, 0)$ is a *complete semimodule* of $\mathcal{A}$.

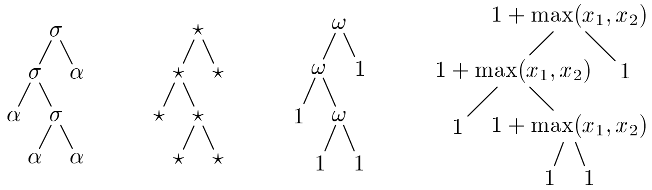
Examples:

- Let $\Omega^{(k)} = \{\, \underline{a}^{(k)} \mid a \in A \,\}$ with $\underline{a}^{(k)}(d_1, \ldots, d_k) = a \odot d_1 \odot \cdots \odot d_k$. Then $\mathcal{D}_\mathcal{A} = (A, \oplus, \mathbf{0}, \Omega)$ is a DM-monoid, which is complete (continuous), whenever $\mathcal{A}$ is so.
- Let $\Omega^{(k)} = \{\, \underline{\psi}^{(k)} \mid \psi \in A\langle\langle T_\Delta(X_k)\rangle\rangle \,\}$ with $\underline{\psi}^{(k)}(\psi_1, \ldots, \psi_k) = \psi \longleftarrow (\psi_1, \ldots, \psi_k)$. Then $\mathcal{D}_{A\langle\langle T_\Delta(X)\rangle\rangle} = (A\langle\langle T_\Delta\rangle\rangle, \oplus, \widetilde{\mathbf{0}}, \Omega)$ is a DM-monoid, which is complete (continuous), whenever $\mathcal{A}$ is so.

Example DM-wta

$\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, $\mathcal{N} = (\mathbb{N} \cup \{\infty\}, \min, \infty, \Omega)$ DM-monoid with $\Omega = \{\omega^{(2)}, \mathrm{id}^{(1)}, 1^{(0)}\}$ and $\omega(n_1, n_2) = 1 + \max(n_1, n_2)$

- DM-wta $M_E = (\{\star\}, \Sigma, \mathcal{N}, F, \mu)$ with $F_\star = \mathrm{id}$, $\mu_0(\alpha)_\star = 1$, and $\mu_2(\sigma)_{\star,(\star,\star)} = \omega$
- $(\|M_E\|, s) = \mathrm{height}(s)$



Constructing a Monoid (I)

$\mathcal{D} = (D, +, 0, \Omega)$ DM-monoid, $\Omega X = \{\, \omega(x_1, \ldots, x_k) \mid k \in \mathbb{N}, \omega \in \Omega^{(k)} \,\}$

Theorem: There exists monoid $(B, \leftarrow, \varepsilon)$ such that $D \cup \Omega X \subseteq B$ and for all $d_1, \ldots, d_k \in D$

$$\omega(d_1, \ldots, d_k) = \overline{\omega}(x_1, \ldots, x_k) \leftarrow d_1 \leftarrow \cdots \leftarrow d_k.$$

Proof sketch: Let $\Omega' = \Omega \cup D$.

- Define $h : T_{\Omega'}(X) \longrightarrow T_{\Omega'}(X)$ for every $v \in D \cup X$ by

$$h(v) = v$$

$$h(\omega(s_1, \ldots, s_k)) = \begin{cases} \omega(h(s_1), \ldots, h(s_k)) & \text{, if } h(s_1), \ldots, h(s_k) \in D \\ \overline{\omega}(h(s_1), \ldots, h(s_k)) & \text{, otherwise} \end{cases}$$

- $h(s) \in \widehat{T_{\Omega'}}(X_n)$, whenever $s \in \widehat{T_{\Omega'}}(X_n)$.

DM-monoid Weighted Tree Automata — Semantics

$\mathcal{D} = (D, +, 0, \Omega)$ complete DM-monoid, $M = (I, \Sigma, \mathcal{D}, F, \mu)$ DM-wta.

- Define $h_\mu : T_\Sigma \longrightarrow D^I$ by

$$h_\mu(\sigma(s_1, \ldots, s_k))_i = \sum_{i_1, \ldots, i_k \in I} \mu_k(\sigma)_{i,(i_1,\ldots,i_k)}(h_\mu(s_1)_{i_1}, \ldots, h_\mu(s_k)_{i_k})$$

  The tree series recognized by M is $(\|M\|, s) = \sum_{i \in I} F_i(h_\mu(s)_i)$.

- A run on $t \in T_\Sigma$ is a map $r : \mathrm{sub}(t) \longrightarrow I$ and $R(t)$ is the set of all runs on t. The weight of r is given by $\mathrm{wt}_r : \mathrm{sub}(t) \longrightarrow D$

$$\mathrm{wt}_r(\sigma(s_1, \ldots, s_k)) = \mu_k(\sigma)_{r(\sigma(s_1,\ldots,s_k)),(r(s_1),\ldots,r(s_k))}(\mathrm{wt}_r(s_1), \ldots, \mathrm{wt}_r(s_k)) \;.$$

  The run-based semantics of M is $(|M|, s) = \sum_{r \in R(s)} F_{r(s)}(\mathrm{wt}_r(s))$.

Theorem: $\|M\| = |M|$.

Weighted Tree Automata & Tree Series Transducers

$M = (I, \Sigma, \mathcal{D}, F, \mu)$ DM-wta, $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ semiring, and Δ ranked alphabet

- M is a weighted tree automaton (wta), if $\mathcal{D} = \mathcal{D}_\mathcal{A} = (A, \oplus, \mathbf{0}, \Omega)$ with $\Omega^{(k)} = \{\, \underline{a}^{(k)} \mid a \in A \,\}$ and $\underline{a}^{(k)}(d_1, \ldots, d_k) = a \odot d_1 \odot \cdots \odot d_k$.
- M is a tree series transducer (tst), if $\mathcal{D} = \mathcal{D}_{A\langle\langle T_\Delta(X)\rangle\rangle} = (A\langle\langle T_\Delta\rangle\rangle, \oplus, \widetilde{\mathbf{0}}, \Omega)$ with $\Omega^{(k)} = \{\, \underline{\psi}^{(k)} \mid \psi \in A\langle\langle T_\Delta(X_k)\rangle\rangle \,\}$ with $\underline{\psi}^{(k)}(\psi_1, \ldots, \psi_k) = \psi \longleftarrow (\psi_1, \ldots, \psi_k)$.

Theorem: Let M_1 be a wta and M_2 be a tst.

(i) There exists a DM-wta M such that $\|M\| = \|M_1\|$.

(ii) There exists a DM-wta M such that $\|M\| = \|M_2\|$.

From a Monoid to a Semiring (I)

$\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ semiring, DM-monoid $\mathcal{D} = (D, +, 0, \Omega)$ complete semimodule of $\mathcal{A}$, $\varphi_1, \ldots, \varphi_k \in A\langle\langle D\rangle\rangle$.

- Lift mapping $\leftarrow : B^2 \longrightarrow B$ to a mapping $\leftarrow : (A\langle\langle B\rangle\rangle)^2 \longrightarrow A\langle\langle B\rangle\rangle$ by

$$\psi_1 \leftarrow \psi_2 = \bigoplus_{b_1, b_2 \in B} \big((\psi_1, b_1) \odot (\psi_2, b_2)\big)\, (b_1 \leftarrow b_2).$$

- Define sum of a series $\varphi \in A\langle\langle D\rangle\rangle$ (summed in D) by $\sum : A\langle\langle D\rangle\rangle \longrightarrow D$

$$\sum \varphi = \sum_{d \in D} (\varphi, d) \cdot d.$$

- **Theorem:**
  (i) $\sum(\bigoplus_{i \in I} \varphi_i) = \sum_{i \in I} \sum \varphi_i$ for every family $(\, \varphi_i \mid i \in I\,)$ of series and
  (ii) $\omega(\sum \varphi_1, \ldots, \sum \varphi_k) = \sum\big(\overline{\omega}(x_1, \ldots, x_k) \leftarrow \varphi_1 \leftarrow \cdots \leftarrow \varphi_k\big)$.

Establishing a Relationship

- **Theorem:** For every tst M_1, there exists a wta M such that $\|M\| = \|M_1\|$.
- **Theorem:** For every deterministic tst M_2, there exists a deterministic wta M such that $\|M\| = \|M_2\|$.
- **Theorem:** For every tree transducer M_3, there exists a wta M such that $\|M\| = \|M_3\|$.
- **Theorem:** For every tst M_4 over an idempotent, continuous semiring, there exists a wta M such that $\|M\| = \|M_4\|$.

Constructing a Monoid (II)

- Let $s(\!|t|\!) = s[t, x_{k+1}, x_{k+2}, \ldots, x_{k+n-1}]$ for $s \in \widehat{T_\Sigma}(X_n)$ and $t \in \widehat{T_\Sigma}(X_k)$ (non-identifying tree substitution).
- $B = D^* \cup \bigcup_{n \in \mathbb{N}_+} D^* \cdot \widehat{T_{\Omega'}}(X_n)$.
- Define $\leftarrow : B^2 \longrightarrow B$ for every $a \in D^*$, $b \in B$, $s \in \widehat{T_{\Omega'}}(X_n)$, $t \in D \cup \widehat{T_{\Omega'}}(X_n)$ by

$$\begin{aligned} a \leftarrow b &= a{\cdot}b \\ a{\cdot}s \leftarrow \varepsilon &= a{\cdot}s \\ a{\cdot}s \leftarrow t{\cdot}b &= a{\cdot}(h(s(\!|t|\!))) \leftarrow b. \end{aligned}$$

- $(B, \leftarrow, \varepsilon)$ is a monoid.
- $\omega(d_1, \ldots, d_k) = \overline{\omega}(x_1, \ldots, x_k) \leftarrow d_1 \leftarrow \cdots \leftarrow d_k$.

From a Monoid to a Semiring (I)

$\mathcal{D} = (D, +, 0, \Omega)$ continuous DM-monoid. $M_1 = (I, \Sigma, \mathcal{D}, F_1, \mu_1)$ DM-wta.

- **Theorem:** There exists a semiring $(C, \oplus, \leftarrow, \widetilde{\mathbf{0}}, \varepsilon)$ such that $D \cup \Omega X \subseteq C$ and for all $d_1, \ldots, d_k \in D$
  (i) $\omega(d_1, \ldots, d_k) = \overline{\omega}(x_1, \ldots, x_k) \leftarrow d_1 \leftarrow \cdots \leftarrow d_k$,
  (ii) $\sum(\bigoplus_{i \in I} d_i) = \sum_{i \in I} d_i$.
  Proof sketch: Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ semiring such that $\mathcal{D}$ is a complete semimodule of $\mathcal{A}$. There exists a monoid $(B, \leftarrow, \varepsilon)$ such that (i) holds. Let $C = A\langle\langle B\rangle\rangle$ and $\leftarrow : C^2 \longrightarrow C$ be the extension of $\leftarrow$ on B.
- **Theorem:** There exists a wta $M = (I, \Sigma, \mathcal{B}, F, \mu)$ such that $\|M_1\| = \sum \|M\|$.

Conclusions

- the study of arbitrary weighted tree automata provides results for tree series transducers
- e.g., a pumping lemma for tree series transducers can be derived from a pumping lemma for weighted tree automata
- unfortunately, few results for weighted tree automata over non-commutative semirings exist

Thank You for Your Attention.

Pumping Lemma for DM-wta

$\mathcal{D} = (D, +, 0, \Omega)$ DM-monoid, $L \in \mathcal{L}^d_\Sigma(\mathcal{D})$, and $\Omega' = \Omega \cup D$.

Theorem: There exists $m \in \mathbb{N}$ such that for every $t \in \text{supp}(L)$ with $\text{height}(t) \geq m + 1$ there exist $C, C' \in \widehat{T_\Sigma}(X_1)$, $s \in T_\Sigma$, and $a, a' \in \widehat{T_{\Omega'}}(X_1)$, and $d \in D$ such that

- $t = C[C'[s]]$,
- $\text{height}(C[s]) \leq m + 1$ and $C \neq x_1$, and
- $(L, C'[C^n[s]]) = a' \leftarrow a^n \leftarrow d$ for every $n \in \mathbb{N}$.