Compositions of Tree Series Transformations

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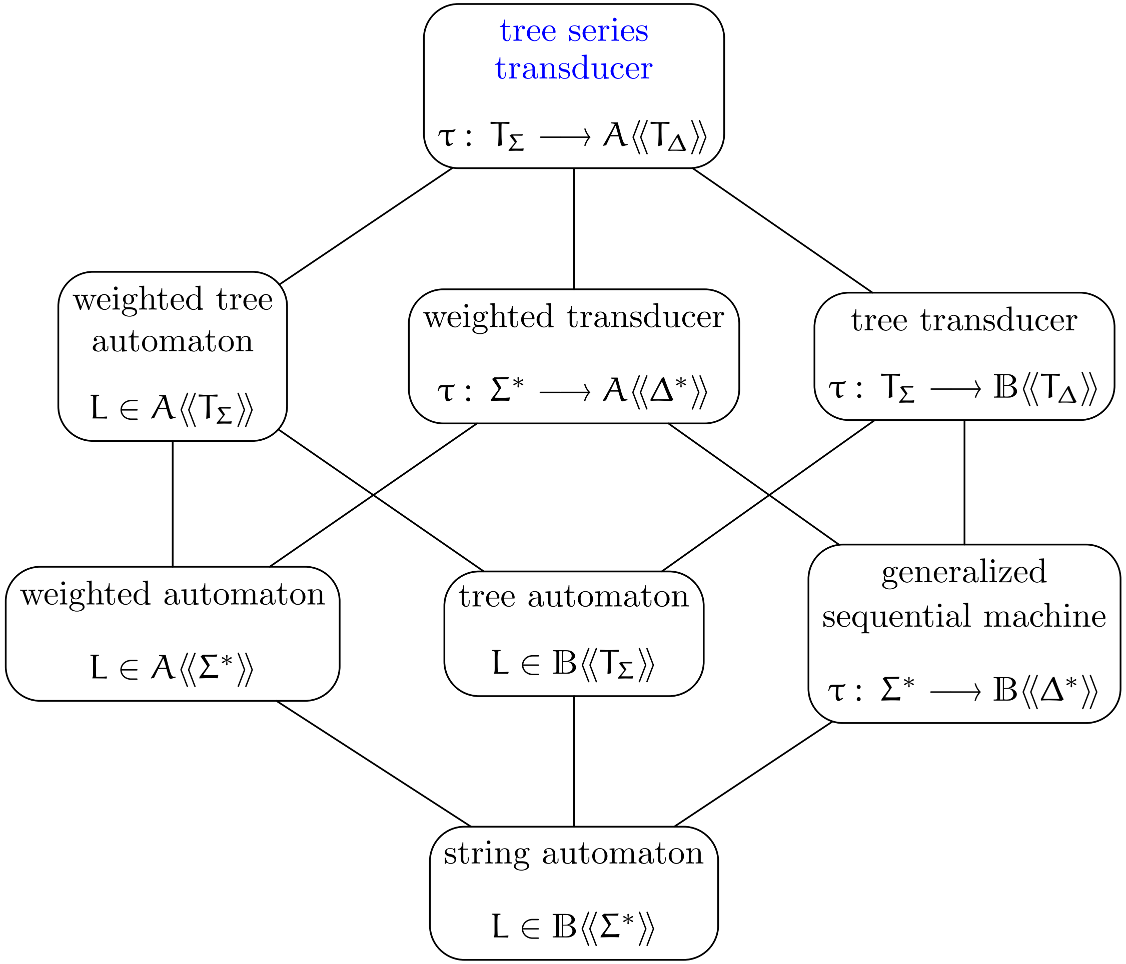
Motivation

- straightforward generalization of tree transducers and weighted tree automata
- can be used for code selection [Borchardt 04]
- potential uses in connection with tree banks

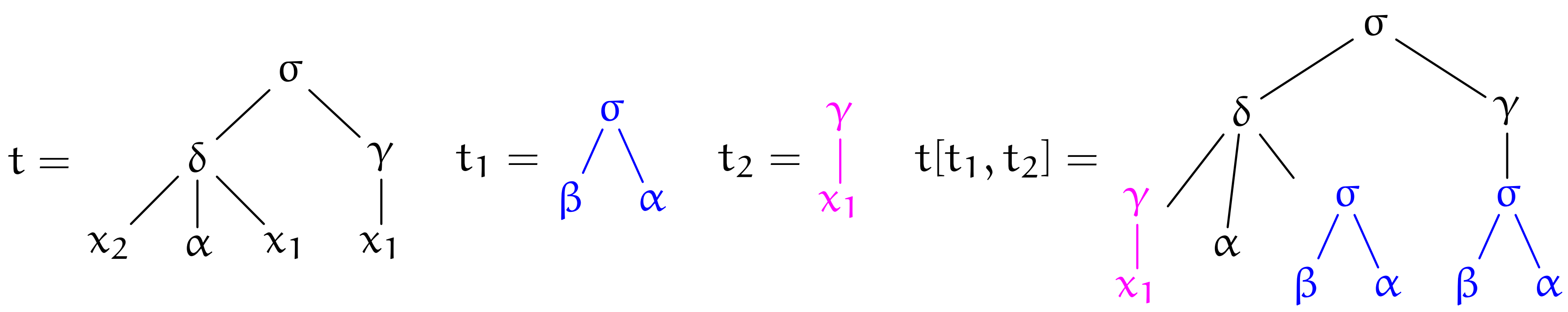
Generalization Hierarchy



Trees

Σ ranked alphabet, $\Sigma_k \subseteq \Sigma$ symbols of rank k, $X = \{ x_i \mid i \in \mathbb{N}_+ \}$

- $T_\Sigma(X)$ set of Σ-trees indexed by X,
- $T_\Sigma = T_\Sigma(\emptyset)$,
- $t \in T_\Sigma(X)$ is *linear* (resp., *non-deleting*) in $Y \subseteq X$, if every $y \in Y$ occurs at most (resp., at least) once in t,
- $t[t_1, \ldots, t_k]$ denotes the tree substitution of t_i for x_i in t



Semirings

A *semiring* is an algebraic structure $\mathcal{A} = (A, \oplus, \odot)$

- $(A, \oplus)$ is a commutative monoid with neutral element 0,
- $(A, \odot)$ is a monoid with neutral element 1,
- 0 is absorbing wrt. $\odot$, and
- $\odot$ distributes over $\oplus$.

Examples:

- semiring of non-negative integers $\mathbb{N}_\infty = (\mathbb{N} \cup \{\infty\}, +, \cdot)$
- Boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge)$
- tropical semiring $\mathbb{T} = (\mathbb{N} \cup \{\infty\}, \min, +)$
- any ring, field, etc.

Properties of Semirings

We say that $\mathcal{A}$ is

- *commutative*, if $\odot$ is commutative,
- *idempotent*, if $a \oplus a = a$,
- *complete*, if there is a operation $\bigoplus_I : A^I \longrightarrow A$ such that
  1. $\bigoplus_{i \in I} a_i = a_{i_1} \oplus \cdots \oplus a_{i_n}$, if $I = \{i_1, \ldots, i_n\}$, and
  2. $\bigoplus_{i \in I} a_i = \bigoplus_{j \in J} \bigoplus_{i \in I_j} a_i$, if $I = \bigcup_{j \in J} I_j$ is a partition of I, and
  3. $\bigoplus_{i \in I}(a \odot a_i) = a \odot \bigoplus_{i \in I} a_i$ and $\bigoplus_{i \in I}(a_i \odot a) = \left(\bigoplus_{i \in I} a_i\right) \odot a$,
- *completely idempotent*, if it is complete with $\bigoplus_{i \in I} a = a$ for every non-empty I.

Semiring	Commutative	Idempotent	Complete	Completely Idempotent
$\mathbb{N}_\infty$	YES	no	YES	no
$\mathbb{B}$	YES	YES	YES	YES
$\mathbb{T}$	YES	YES	YES	YES

Tree Series

$\mathcal{A} = (A, \oplus, \odot)$ semiring, Σ ranked alphabet

Mappings $\varphi : T_\Sigma(X) \longrightarrow A$ are also called *tree series*

- the set of all tree series is $A\langle\langle T_\Sigma(X)\rangle\rangle$,
- the *coefficient* of $t \in T_\Sigma(X)$ in φ, i.e., $\varphi(t)$, is denoted by (φ, t),
- the *sum* is defined pointwise $(\varphi_1 \oplus \varphi_2, t) = (\varphi_1, t) \oplus (\varphi_2, t)$,
- the *support* of φ is $\mathrm{supp}(\varphi) = \{t \in T_\Sigma(X) \mid (\varphi, t) \neq 0\}$,
- φ is *linear* (resp., *non-deleting* in $Y \subseteq X$), if $\mathrm{supp}(\varphi)$ is a set of trees, which are linear (resp., non-deleting in Y),
- the series φ with $\mathrm{supp}(\varphi) = \emptyset$ is denoted by $\widetilde{0}$.

Example: $\varphi = 1\,\alpha + 1\,\beta + 3\,\sigma(\alpha, \alpha) + \ldots + 3\,\sigma(\beta, \beta) + 5\,\sigma(\alpha, \sigma(\alpha, \alpha)) + \ldots$

Tree Series Substitution

$\mathcal{A} = (A, \oplus, \odot)$ complete semiring, $\varphi, \psi_1, \ldots, \psi_k \in A\langle\langle T_\Sigma(X)\rangle\rangle$

Pure substitution of $(\psi_1, \ldots, \psi_k)$ into φ:

$$\varphi \longleftarrow (\psi_1, \ldots, \psi_k) = \bigoplus_{\substack{t \in \text{supp}(\varphi), \\ (\forall i \in [k]):\, t_i \in \text{supp}(\psi_i)}} (\varphi, t) \odot (\psi_1, t_1) \odot \cdots \odot (\psi_k, t_k)\, t[t_1, \ldots, t_k]$$

o*-substitution* of $(\psi_1, \ldots, \psi_k)$ into φ:

$$\varphi \stackrel{o}{\longleftarrow} (\psi_1, \ldots, \psi_k) = \bigoplus_{\substack{t \in \text{supp}(\varphi), \\ (\forall i \in [k]):\, t_i \in \text{supp}(\psi_i)}} (\varphi, t) \odot (\psi_1, t_1)^{|t|_{x_1}} \odot \cdots \odot (\psi_k, t_k)^{|t|_{x_k}}\, t[t_1, \ldots, t_k]$$

Example: $5\,\sigma(x_1, x_1) \longleftarrow (2\,\alpha) = 10\,\sigma(\alpha, \alpha)$ and $5\,\sigma(x_1, x_1) \stackrel{o}{\longleftarrow} (2\,\alpha) = 20\,\sigma(\alpha, \alpha)$

Distributivity

$$\Big(\bigoplus_{i\in I}\varphi_i\Big) \stackrel{m}{\longleftarrow} \Big(\bigoplus_{i_1\in I_1}\psi_{1i_1},\ldots,\bigoplus_{i_k\in I_k}\psi_{ki_k}\Big) = \bigoplus_{\substack{i\in I,\\ (\forall j\in[k])\colon i_j\in I_j}} \varphi_i \stackrel{m}{\longleftarrow} (\psi_{1i_1},\ldots,\psi_{ki_k})$$

Substitution	Sufficient condition for distributivity
pure substitution	always
o-substitution	φ_i linear, $\mathcal{A}$ completely idempotent
OI-substitution	φ_i linear and non-deleting [Kuich 99]

Associativity

$$\left(\varphi \overset{m}{\longleftarrow} (\psi_1, \ldots, \psi_k)\right) \overset{m}{\longleftarrow} (\tau_1, \ldots, \tau_n) = \varphi \overset{m}{\longleftarrow} (\psi_1 \overset{m}{\longleftarrow} (\tau_1, \ldots, \tau_n), \ldots, \psi_k \overset{m}{\longleftarrow} (\tau_1, \ldots, \tau_n))$$

Substitution	Sufficient condition for associativity
pure substitution	special associativity law
o-substitution	$\varphi, \psi_1, \ldots, \psi_k$ linear, $\mathcal{A}$ zero-divisor free and completely idempotent
OI-substitution	φ_i linear and non-deleting [Kuich 99]

Special associativity law: $\mathrm{var}(\varphi) \subseteq J$, partition $(\, I_j \,)_{j \in J}$ of I with $\mathrm{var}(\psi_j) \subseteq X_{I_j}$ for every $j \in J$

$$\left(\varphi \longleftarrow (\, \psi_j \,)_{j \in J}\right) \longleftarrow (\, \tau_i \,)_{i \in I} = \varphi \longleftarrow \left(\psi_j \longleftarrow (\, \tau_i \,)_{i \in I_j}\right)_{j \in J}$$

Linearity

$$(a \odot \varphi) \overset{m}{\longleftarrow} (a_1 \odot \psi_1, \ldots, a_k \odot \psi_k) = a \odot a_1 \odot \cdots \odot a_k \odot \varphi \overset{m}{\longleftarrow} (\psi_1, \ldots, \psi_k)$$

Substitution	Sufficient condition for distributivity
pure substitution	always
o-substitution	$a_i \in \{0, 1\}$ or special linearity law
OI-substitution	φ_i linear and non-deleting [Kuich 99]

Special linearity law: tree $t \in T_\Sigma(X_k)$

$$(a\, t) \overset{o}{\longleftarrow} (a_1 \odot \psi_1, \ldots, a_k \odot \psi_k) = a \odot a_1^{|t|_1} \odot \cdots \odot a_k^{|t|_k} \odot \left(t \overset{o}{\longleftarrow} (\psi_1, \ldots, \psi_k)\right)$$

Tree Series Transducers

Definition: A *(bottom-up) tree series transducer* (tst) is a system $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$

- Q is a non-empty set of *states*,
- Σ and Δ are input and output ranked alphabets,
- $\mathcal{A} = (A, \oplus, \odot)$ is a complete semiring,
- $F \in A\langle\langle T_\Delta(X_1)\rangle\rangle^Q$ is a vector of *final outputs*,
- $\mu = (\mu_k)_{k \in \mathbb{N}}$ with $\mu_k : \Sigma_k \longrightarrow A\langle\langle T_\Delta(X_k)\rangle\rangle^{Q \times Q^k}$.

If Q is finite and $\mu_k(\sigma)_{q,\vec{q}}$ is polynomial, then M is called *finite*.

Semantics of Tree Series Transducers

$m \in \{\varepsilon, o\}$, $q \in Q$, $t \in T_\Sigma$, $\varphi \in A\langle\langle T_\Sigma \rangle\rangle$

Definition: The mapping $h_\mu^m : T_\Sigma \longrightarrow A\langle\langle T_\Delta \rangle\rangle^Q$ is defined as

$$h_\mu^m(\sigma(t_1, \ldots, t_k))_q = \bigoplus_{q_1, \ldots, q_k \in Q} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \overset{m}{\longleftarrow} (h_\mu^m(t_1)_{q_1}, \ldots, h_\mu^m(t_k)_{q_k})$$

and $h_\mu^m(\varphi)_q = \bigoplus_{t \in T_\Sigma} (\varphi, t) \cdot h_\mu^m(t)_q$.

- the m-*tree-to-tree-series transformation* $\|M\|^m : T_\Sigma \longrightarrow A\langle\langle T_\Delta \rangle\rangle$ computed by M is $(\|M\|^m, t) = \bigoplus_{q \in Q} F_q \overset{m}{\longleftarrow} (h_\mu^m(t)_q)$ and
- the m-*tree-series-to-tree-series transformation* $|M|^m : A\langle\langle T_\Sigma \rangle\rangle \longrightarrow A\langle\langle T_\Delta \rangle\rangle$ computed by M is $(|M|^m, \varphi) = \bigoplus_{t \in T_\Sigma} (\varphi, t) \odot (\|M\|^m, t)$.

Extension

$(Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ be a bottom-up tree series transducer, $m \in \{\varepsilon, o\}$, $\vec{q} \in Q^k$, $q \in Q$, $\varphi \in A\langle\langle T_\Sigma(X_k)\rangle\rangle$

Definition: We define $h^{\vec{q}}_{\mu,m} : T_\Sigma(X_k) \longrightarrow A\langle\langle T_\Delta(X_k)\rangle\rangle^Q$

$$h^{\vec{q}}_{\mu,m}(x_i)_q = \begin{cases} 1\, x_i & \text{, if } q = q_i \\ \widetilde{0} & \text{, otherwise} \end{cases}$$

$$h^{\vec{q}}_{\mu,m}(\sigma(t_1, \dots, t_k))_q = \bigoplus_{p_1,\dots,p_k \in Q} \mu_k(\sigma)_{q,p_1\dots p_k} \overset{m}{\longleftarrow} (h^{\vec{q}}_{\mu,m}(t_1)_{p_1}, \dots, h^{\vec{q}}_{\mu,m}(t_k)_{p_k})$$

We define $h^{\vec{q}}_{\mu,m} : A\langle\langle T_\Sigma(X_k)\rangle\rangle \longrightarrow A\langle\langle T_\Delta(X_k)\rangle\rangle^Q$ by

$$h^{\vec{q}}_{\mu,m}(\varphi)_q = \bigoplus_{t \in T_\Sigma(X_I)} (\varphi, t) \odot h^{\vec{q}}_{\mu,m}(t)_q$$

Composition Construction

$M_1 = (Q_1, \Sigma, \Delta, \mathcal{A}, F_1, \mu_1)$ and $M_2 = (I_2, \Delta, \Gamma, \mathcal{A}, F_2, \mu_2)$ tree series transducer

Definition: The m-*product of* M_1 and M_2, denoted by $M_1 \cdot_m M_2$, is the tree series transducer

$$M = (I_1 \times I_2, \Sigma, \Gamma, \mathcal{A}, F, \mu)$$

- $F_{pq} = \bigoplus_{i \in Q_2} (F_2)_i \overset{m}{\longleftarrow} h^{q}_{\mu_2, m}((F_1)_p)_i$
- $\mu_k(\sigma)_{pq,(p_1 q_1, \ldots, p_k q_k)} = h^{q_1 \ldots q_k}_{\mu_2, m}((\mu_1)_k(\sigma)_{p, p_1 \ldots p_k})_q$.

Main Theorem

$\mathcal{A}$ commutative semiring, M_1 and M_2 tree series transducer

Theorem: $|M_1 \cdot_m M_2|^m = |M_1|^m \circ |M_2|^m$, if

- $m = \varepsilon$ and M_1 is non-deleting and linear, or
- $m = o$ and M_1 is linear, M_2 is non-deleting and linear, and $\mathcal{A}$ is completely idempotent.

Corollary:

- $\text{nl-BOT}_{\text{ts-ts}}(\mathcal{A}) \circ \text{BOT}_{\text{ts-ts}}(\mathcal{A}) = \text{BOT}_{\text{ts-ts}}(\mathcal{A})$.
- $\text{l-BOT}^{o}_{\text{ts-ts}}(\mathcal{A}) \circ \text{nl-BOT}^{o}_{\text{ts-ts}}(\mathcal{A}) = \text{l-BOT}^{o}_{\text{ts-ts}}(\mathcal{A})$, provided that $\mathcal{A}$ is completely idempotent.

References

[Borchardt 04] B. Borchardt:

[Kuich 99] W. Kuich:

[Engelfriet et al 02] J. Engelfriet, Z. Fülöp, and H. Vogler:

[Fülöp et al 03] Z. Fülöp and H. Vogler: Tree Series Transformations that Respect Copying