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Relating Tree Series Transducers and Weighted Tree Automata

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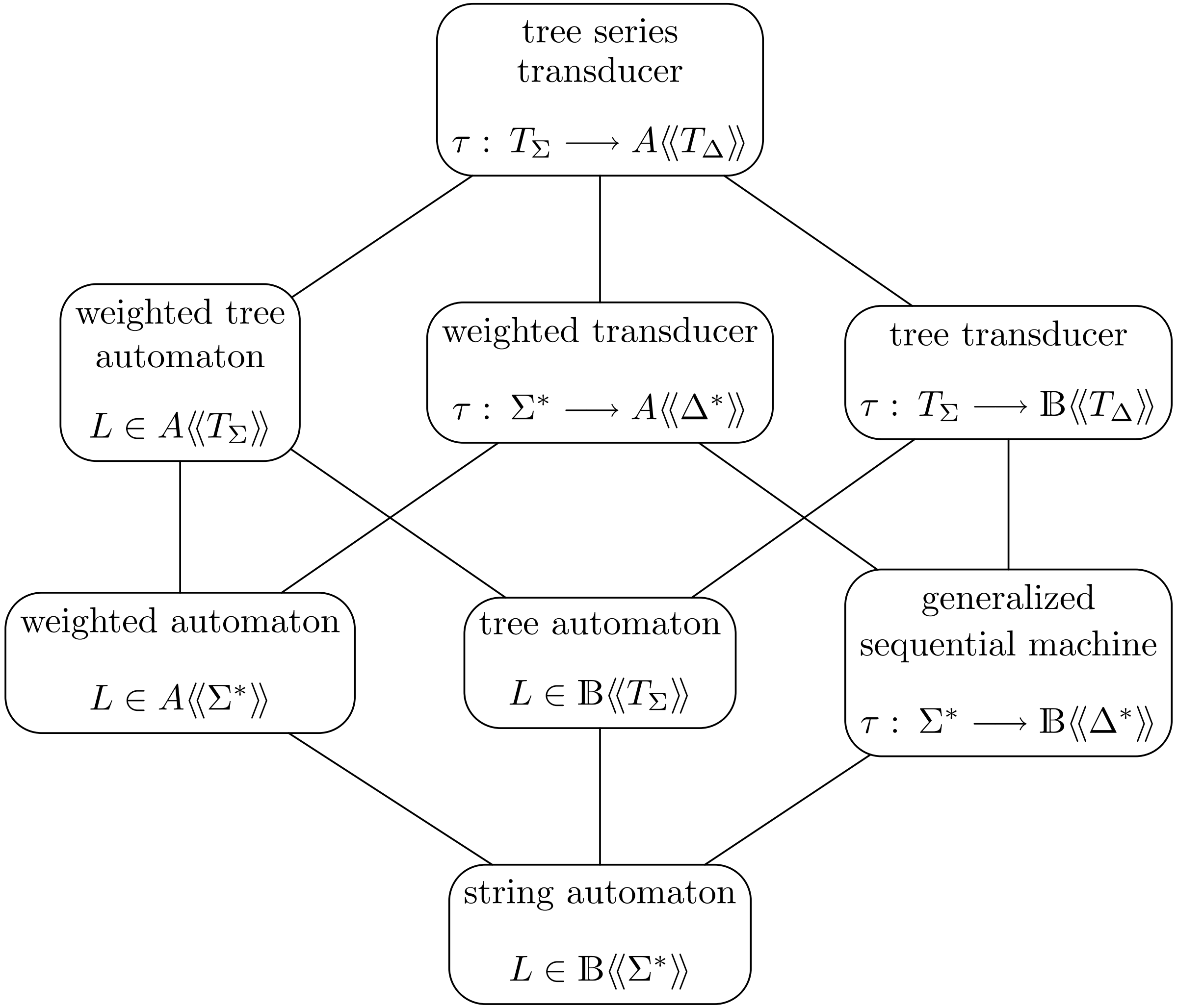
1. Motivation and Introductory Example
2. Semirings and DM-Monoids
3. Bottom-Up DM-Monoid Weighted Tree Automata
4. Establishing a Relationship

Generalization Hierarchy



Known Relations and Problems

- *String-based:*

  Theorem: Every gsm-mapping can be computed by a weighted automaton.

  Proof Idea: Extend monoid $(\Delta^*, \circ, \varepsilon)$ to semiring $(\mathcal{P}(\Delta^*), \cup, \circ, \emptyset, \{\varepsilon\})$

  Theorem: Weighted transductions can be computed by weighted automata.

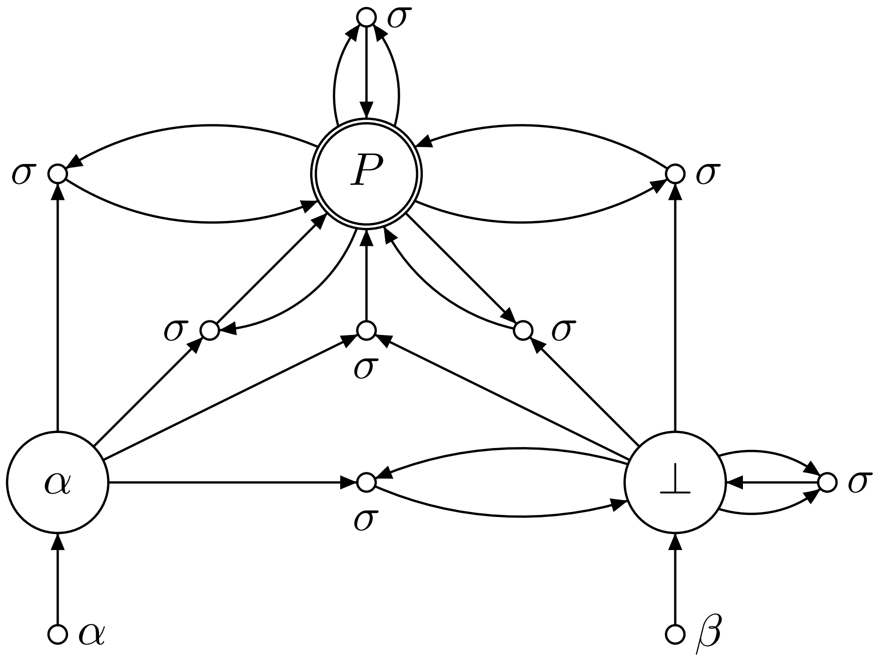
- *Tree-based:*

  Problem: Are tree transductions computable by weighted tree automata ?

  Problem: Are tree series transformations computable by weighted tree automata ?

Tree Pattern Matching

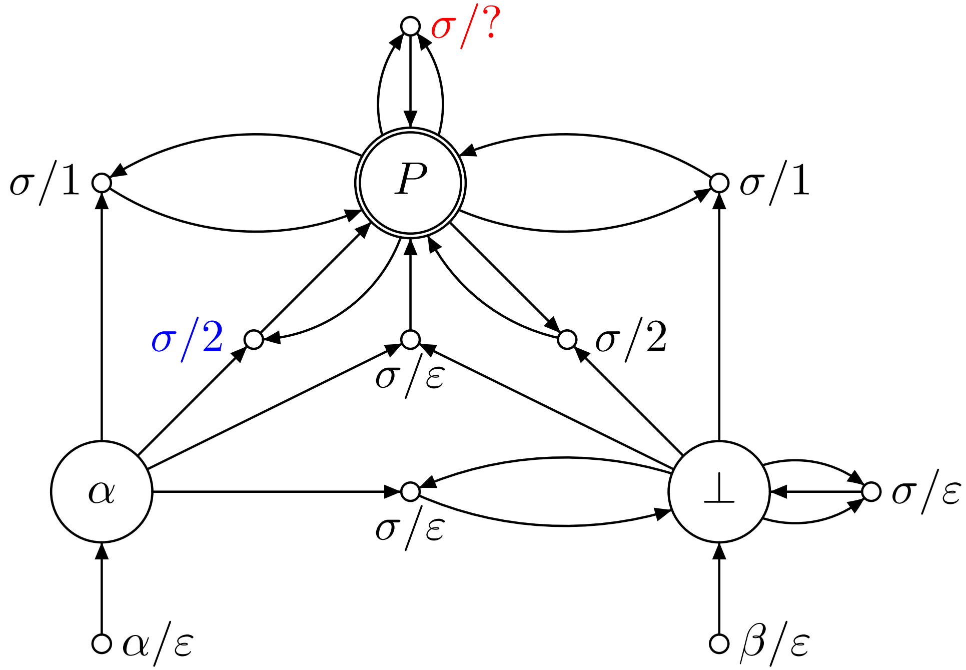
A deterministic (bottom-up) tree automaton matching the pattern $\sigma(\alpha, x)$



If pattern found, accepts tree. Otherwise reject.

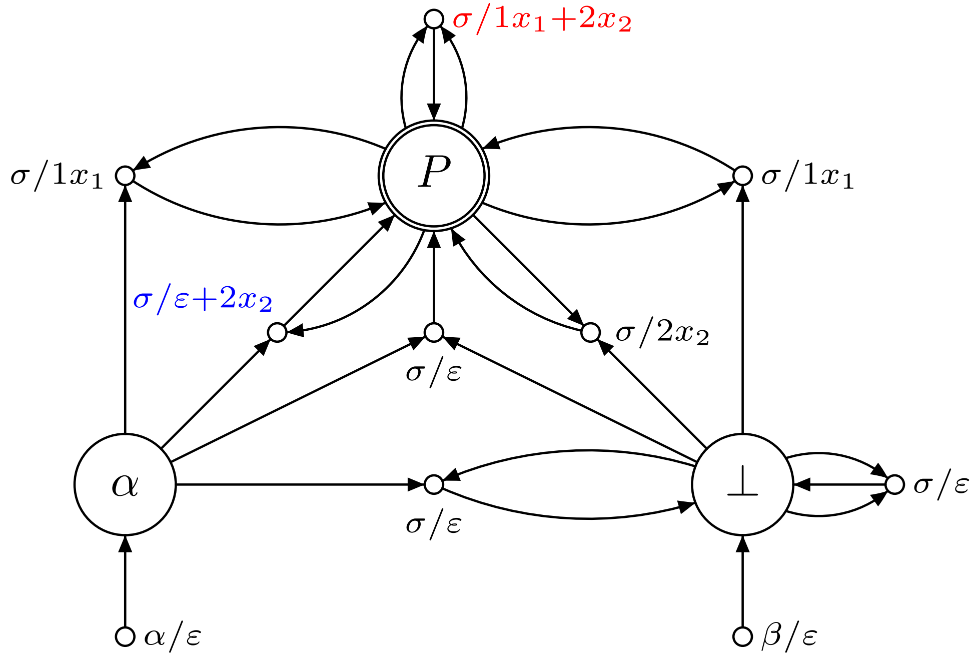
Extended Tree Pattern Matching

Towards a deterministic (bottom-up) weighted tree automaton computing the occurences of pattern $\sigma(\alpha, x)$



Extended Tree Pattern Matching

A deterministic tree transducer computing the occurences of pattern $\sigma(\alpha, x)$



Computes the set of occurences of $\sigma(\alpha, x)$ in input tree.

Complete Monoids

- $\mathcal{A} = (A, \bigoplus)$ complete monoid, iff

(C1) $\bigoplus_{i \in \{j\}} a_i = a_j$,

(C2) $\bigoplus_{j \in J}(\bigoplus_{i \in I_j} a_i) = \bigoplus_{i \in I} a_i$, if $I = \bigcup_{j \in J} I_j$ is a partition.

- $\mathcal{A}$ naturally ordered, iff $\sqsubseteq$ is partial order

$$a \sqsubseteq b \quad \Longleftrightarrow \quad (\exists c \in A):\ a \oplus c = b$$

- $\mathcal{A}$ continuous, iff $\mathcal{A}$ *naturally ordered* and *complete* and

$$\bigoplus_{i \in I} a_i \sqsubseteq a \Longleftrightarrow \bigoplus_{i \in E} a_i \sqsubseteq a \text{ for all finite } E \subseteq I$$

Semirings

- $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ semiring, iff
  (i) $(A, \oplus, \mathbf{0})$ commutative monoid,
  (ii) $(A, \odot, \mathbf{1})$ monoid,
  (iii) $\mathbf{0}$ absorbing element with respect to $\odot$, and
  (iv) $\odot$ (left and right) distributes over $\oplus$.

- $(A, \odot, \mathbf{0}, \mathbf{1}, \bigoplus)$ complete semiring, iff
  (S1) $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ semiring,
  (S2) $(A, \bigoplus)$ complete monoid, and
  (S3) $a \odot (\bigoplus_{i \in I} a_i) = \bigoplus_{i \in I} (a \odot a_i)$ and $(\bigoplus_{i \in I} a_i) \odot a = \bigoplus_{i \in I} (a_i \odot a)$.

Examples of Semirings

- complete natural numbers semiring $\mathbb{N}_\infty = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$,
- tropical semiring $\mathrm{Trop} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$,
- Boolean semiring $\mathbb{B} = (\{\bot, \top\}, \vee, \wedge, \bot, \top)$,
- formal language semiring $\mathrm{Lang}_\Sigma = (\mathcal{P}(\Sigma^*), \cup, \circ, \emptyset, \{\varepsilon\})$

Semiring	commutative	complete	naturally ordered	continuous
$\mathbb{N}_\infty$	yes	yes	yes	yes
Trop	yes	yes	yes	yes
$\mathbb{B}$	yes	yes	yes	yes
Lang_Σ	NO	yes	yes	yes

Excursion: Tree Series

$(A, \bigoplus)$ complete monoid, Σ ranked alphabet, and $X_k = \{x_1, \ldots, x_k\}$.

- Tree series is mapping $\psi : T_\Sigma(X_k) \longrightarrow A$
- $A\langle\langle T_\Sigma(X_k)\rangle\rangle$ set of all tree series
- Sum $(\bigoplus_{i\in I} \psi_i, t) = \bigoplus_{i\in I}(\psi_i, t)$
- $(A\langle\langle T_\Sigma(X_k)\rangle\rangle, \bigoplus)$ complete monoid

$(A, \odot, \mathbf{0}, \mathbf{1}, \bigoplus)$ complete semiring

- Tree series substiution of $\psi_1, \ldots, \psi_k \in A\langle\langle T_\Sigma\rangle\rangle$ into $\psi \in A\langle\langle T_\Sigma(X_k)\rangle\rangle$ is

$$\psi \longleftarrow (\psi_1, \ldots, \psi_k) = \bigoplus_{\substack{t\in T_\Sigma(X_k),\\ (\forall i\in[k]):t_i\in T_\Sigma}} \Big((\psi, t) \odot \bigodot_{i\in[k]}(\psi_i, t_i)\Big)\, t[t_1, \ldots, t_k]$$

Complete DM-Monoids

$(D, \sum)$ complete monoid, Ω ranked set

- $(D, \Omega, \sum)$ distributive multi-operator monoid (DM-monoid), iff

$$\omega\Big(\sum_{i_1 \in I_1} d_{i_1}, \ldots, \sum_{i_k \in I_k} d_{i_k}\Big) = \sum_{(\forall j \in [k]):\, i_j \in I_j} \omega(d_{i_1}, \ldots, d_{i_k}).$$

Examples:

- $(A, \bigoplus)$ complete monoid, $\Omega_{(k)} = \{\, \underline{a}_{(k)} \mid a \in A \,\}$ with

$$\underline{a}_{(k)}(d_1, \ldots, d_k) = a \odot d_1 \odot \cdots \odot d_k$$

  Then $(A, \Omega, \bigoplus)$ complete DM-monoid

- $(A, \odot, \mathbf{0}, \mathbf{1}, \bigoplus)$ complete semiring, $\Omega_{(k)} = \{\, \underline{\psi}_{(k)} \mid \psi \in A\langle\langle T_\Delta(X_k)\rangle\rangle \,\}$ with

$$\underline{\psi}_{(k)}(\psi_1, \ldots, \psi_k) = \psi \longleftarrow (\psi_1, \ldots, \psi_k)$$

  Then $(A\langle\langle T_\Delta\rangle\rangle, \Omega, \bigoplus)$ complete DM-monoid

DM-Monoid Weighted Tree Automata — Syntax

Σ ranked alphabet, I, Ω non-empty sets

- Tree representation over I, Σ, and Ω is $\mu = (\, \mu_k \mid k \in \mathbb{N} \,)$ such that

$$\mu_k : \Sigma_{(k)} \longrightarrow \Omega^{I \times I^k}$$

- $M = (I, \Sigma, \mathcal{D}, F, \mu)$ (bottom-up) DM-monoid weighted tree automaton (DM-wta), iff
  - I non-empty set of states,
  - Σ ranked alphabet of input symbols,
  - $\mathcal{D} = (D, \Omega, \sum)$ *complete DM-monoid*,
  - $F : I \longrightarrow \Omega_{(1)}$ final weight map, and
  - μ tree representation over I, Σ, and Ω such that $\mu_k : \Sigma_{(k)} \longrightarrow {\Omega_{(k)}}^{I \times I^k}$

DM-Monoid Weighted Tree Automata — Semantics

$\mathcal{D} = (D, \Omega, \sum)$ complete DM-monoid, $M = (I, \Sigma, \mathcal{D}, F, \mu)$ DM-wta.

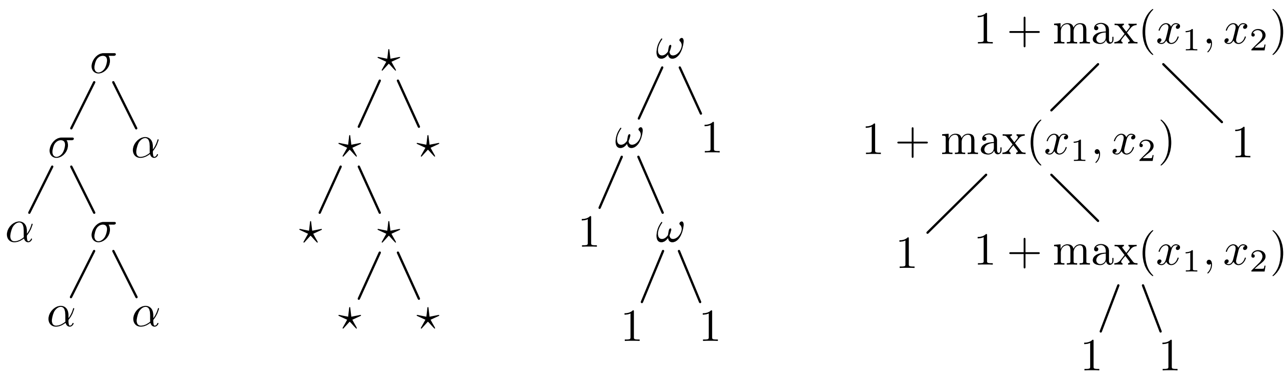
- Define $h_\mu : T_\Sigma \longrightarrow D^I$ by

$$h_\mu(\sigma(t_1, \ldots, t_k))_i = \sum_{i_1, \ldots, i_k \in I} \mu_k(\sigma)_{i,(i_1,\ldots,i_k)}\big(h_\mu(t_1)_{i_1}, \ldots, h_\mu(t_k)_{i_k}\big)$$

- $(\|M\|, t) = \sum_{i \in I} F_i(h_\mu(t)_i)$ is tree series recognized by M

Example DM-wta

- $\Sigma = \{\sigma, \alpha\}$ and $\Omega = \{\omega, \mathrm{id}, 1\}$ and $\omega(n_1, n_2) = 1 + \max(n_1, n_2)$,
- $\mathcal{N} = (\mathbb{N} \cup \{\infty\}, \Omega, \min)$ complete DM-monoid
- DM-wta $M_E = (\{\star\}, \Sigma, \mathcal{N}, F, \mu)$ with $F_\star = \mathrm{id}$, $\mu_0(\alpha)_\star = 1$, and $\mu_2(\sigma)_{\star,(\star,\star)} = \omega$



- $(\|M_E\|, t) = \mathrm{height}(t)$

Weighted Tree Automata & Tree Series Transducers

$M = (I, \Sigma, \mathcal{D}, F, \mu)$ DM-wta and $(A, \odot, \mathbf{0}, \mathbf{1}, \bigoplus)$ complete semiring

- M is weighted tree automaton (wta), iff $\mathcal{D} = (A, \Omega, \bigoplus)$ with $\Omega_{(k)} = \{\, \underline{a}_{(k)} \mid a \in A \,\}$ and

$$\underline{a}_{(k)}(d_1, \ldots, d_k) = a \odot d_1 \odot \cdots \odot d_k$$

- M is tree series transducer (tst), iff $\mathcal{D} = (A\langle\langle T_\Delta \rangle\rangle, \Omega, \bigoplus)$ with $\Omega_{(k)} = \{\, \underline{\psi}_{(k)} \mid \psi \in A\langle\langle T_\Delta(X_k) \rangle\rangle \,\}$ and

$$\underline{\psi}_{(k)}(\psi_1, \ldots, \psi_k) = \psi \longleftarrow (\psi_1, \ldots, \psi_k)$$

Constructing a Monoid (I)

$\mathcal{D} = (D, +, 0, \Omega)$ DM-monoid, $\Omega X = \{\, \overline{\omega}(x_1, \ldots, x_k) \mid k \in \mathbb{N}, \omega \in \Omega_{(k)} \,\}$

Theorem: There exists monoid $(B, \leftarrow, \varepsilon)$ such that $D \cup \Omega X \subseteq B$ and for all $d_1, \ldots, d_k \in D$

$$\omega(d_1, \ldots, d_k) = \overline{\omega}(x_1, \ldots, x_k) \leftarrow d_1 \leftarrow \cdots \leftarrow d_k.$$

Proof sketch: Let $\Omega' = \Omega \cup D$.

- Define $h : T_{\Omega'}(X) \longrightarrow T_{\Omega'}(X)$ for every $v \in D \cup X$ by

$$\begin{aligned} h(v) &= v \\ h(\omega(s_1, \ldots, s_k)) &= \begin{cases} \omega(h(s_1), \ldots, h(s_k)) & \text{, if } h(s_1), \ldots, h(s_k) \in D \\ \overline{\omega}(h(s_1), \ldots, h(s_k)) & \text{, otherwise} \end{cases} \end{aligned}$$

- $h(s) \in \widehat{T_{\Omega'}}(X_n)$, whenever $s \in \widehat{T_{\Omega'}}(X_n)$.

Constructing a Monoid (II)

- Let $s(\!|t|\!) = s[t, x_{k+1}, x_{k+2}, \ldots, x_{k+n-1}]$ for $s \in \widehat{T_\Sigma}(X_n)$ and $t \in \widehat{T_\Sigma}(X_k)$ (non-identifying tree substitution).
- $B = D^* \cup \bigcup_{n \in \mathbb{N}_+} D^* \cdot \widehat{T_{\Omega'}}(X_n)$.
- Define $\leftarrow : B^2 \longrightarrow B$ for every $a \in D^*$, $b \in B$, $s \in \widehat{T_{\Omega'}}(X_n)$, $t \in D \cup \widehat{T_{\Omega'}}(X_n)$ by

$$\begin{aligned} a \leftarrow b &= a \cdot b \\ a \cdot s \leftarrow \varepsilon &= a \cdot s \\ a \cdot s \leftarrow t \cdot b &= a \cdot (h(s(\!|t|\!))) \leftarrow b. \end{aligned}$$

- $(B, \leftarrow, \varepsilon)$ is a monoid.
- $\omega(d_1, \ldots, d_k) = \overline{\omega}(x_1, \ldots, x_k) \leftarrow d_1 \leftarrow \cdots \leftarrow d_k$.

From a Monoid to a Semiring (I)

$\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ semiring, DM-monoid $\mathcal{D} = (D, +, 0, \Omega)$ complete semimodule of $\mathcal{A}$, $\varphi_1, \ldots, \varphi_k \in A\langle\langle D \rangle\rangle$.

- Lift mapping $\leftarrow : \; B^2 \longrightarrow B$ to a mapping $\leftarrow : \; A\langle\langle B \rangle\rangle^2 \longrightarrow A\langle\langle B \rangle\rangle$ by

$$\psi_1 \leftarrow \psi_2 = \bigoplus_{b_1, b_2 \in B} \big((\psi_1, b_1) \odot (\psi_2, b_2)\big)\, (b_1 \leftarrow b_2).$$

- Define sum of a series $\varphi \in A\langle\langle D \rangle\rangle$ (summed in D) by $\sum : \; A\langle\langle D \rangle\rangle \longrightarrow D$

$$\sum \varphi = \sum_{d \in D} (\varphi, d) \cdot d.$$

- **Theorem:**

  (i) $\sum(\bigoplus_{i \in I} \varphi_i) = \sum_{i \in I} \sum \varphi_i$ for every family $(\, \varphi_i \mid i \in I \,)$ of series and

  (ii) $\omega(\sum \varphi_1, \ldots, \sum \varphi_k) = \sum\big(\overline{\omega}(x_1, \ldots, x_k) \leftarrow \varphi_1 \leftarrow \cdots \leftarrow \varphi_k\big)$.

From a Monoid to a Semiring (II)

$\mathcal{D} = (D, +, 0, \Omega)$ continuous DM-monoid. $M_1 = (I, \Sigma, \mathcal{D}, F_1, \mu_1)$ DM-wta.

- **Theorem:** There exists a semiring $(C, \oplus, \leftarrow, \widetilde{\mathbf{0}}, \varepsilon)$ such that $D \cup \Omega X \subseteq C$ and for all $d_1, \ldots, d_k \in D$

  (i) $\omega(d_1, \ldots, d_k) = \overline{\omega}(x_1, \ldots, x_k) \leftarrow d_1 \leftarrow \cdots \leftarrow d_k$,

  (ii) $\sum(\bigoplus_{i \in I} d_i) = \sum_{i \in I} d_i$.

  Proof sketch: Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ semiring such that $\mathcal{D}$ is a complete semimodule of $\mathcal{A}$. There exists a monoid $(B, \leftarrow, \varepsilon)$ such that (i) holds. Let $C = A\langle\langle B \rangle\rangle$ and $\leftarrow : C^2 \longrightarrow C$ be the extension of $\leftarrow$ on B.

- **Theorem:** There exists a wta $M = (I, \Sigma, \mathcal{B}, F, \mu)$ such that $\|M_1\| = \sum \|M\|$.

Establishing a Relationship

- **Theorem:** For every tst M_1, there exists a wta M such that $\sum \|M\| = \|M_1\|$.
- **Theorem:** For every deterministic tst M_2, there exists a deterministic wta M such that $\|M\| = \|M_2\|$.
- **Theorem:** For every tree transducer M_3, there exists a wta M such that $\|M\| = \|M_3\|$.
- **Theorem:** For every tst M_4 over an idempotent, continuous semiring, there exists a wta M such that $\|M\| = \|M_4\|$.

Pumping Lemma for DM-wta

$\mathcal{D} = (D, +, 0, \Omega)$ DM-monoid, $L \in \mathcal{L}^d_\Sigma(\mathcal{D})$, and $\Omega' = \Omega \cup D$.

Theorem: There exists $m \in \mathbb{N}$ such that for every $t \in \mathrm{supp}(L)$ with $\mathrm{height}(t) \geq m + 1$ there exist $C, C' \in \widehat{T_\Sigma}(X_1)$, $s \in T_\Sigma$, and $a, a' \in \widehat{T_{\Omega'}}(X_1)$, and $d \in D$ such that

- $t = C[C'[s]]$,
- $\mathrm{height}(C[s]) \leq m + 1$ and $C \neq x_1$, and
- $(L, C'[C^n[s]]) = a' \leftarrow a^n \leftarrow d$ for every $n \in \mathbb{N}$.

Conclusions

- the study of arbitrary weighted tree automata provides results for tree series transducers
- e.g., a pumping lemma for tree series transducers can be derived from a pumping lemma for weighted tree automata
- unfortunately, few results for weighted tree automata over non-commutative semirings exist

Thank You for Your Attention.