The Power of Tree Series Transducers of Type I and II

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Semirings

Definition: $(A, +, \cdot, 0, 1)$ semiring, if

- $(A, +, 0)$ commutative monoid,
- $(A, \cdot, 1)$ monoid,
- $\cdot$ distributes over $+$, and
- 0 is absorbing wrt. $\cdot$ (i.e., $a \cdot 0 = 0 = 0 \cdot a$).

Definition: $(A, +, \cdot, 0, 1)$ commutative semiring, if $(A, \cdot, 1)$ commutative.

Examples: (all rings and fields are semirings)

- *Natural numbers:* $(\mathbb{N}, +, \cdot, 0, 1)$ commutative,
- *Boolean semiring:* $(\{\bot, \top\}, \vee, \wedge, \bot, \top)$ commutative,
- *Tropical semiring:* $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ commutative,
- *Language semiring:* $(\mathcal{P}(\Sigma^*), \cup, \circ, \emptyset, \{\varepsilon\})$ non-commutative.

$\aleph_0$-complete Semirings

Definition: $(A, +, \cdot, 0, 1)$ $\aleph_0$-complete semiring, if for all I with $\mathrm{card}(I) \leq \aleph_0$ there exists $\sum_I : A^I \longrightarrow A$ such that

- $\sum_I (\, a_i \,)_{i \in I} = a_j$, if $I = \{j\}$,
- $\sum_I (\, a_i \,)_{i \in I} = a_{j_1} + a_{j_2}$, if $I = \{j_1, j_2\}$ with $j_1 \neq j_2$, and
- $\sum_I (\, a_i \,)_{i \in I} = \sum_J \Big(\sum_{I_j} (\, a_i \,)_{i \in I_j} \Big)$, if $I = \bigcup_{j \in J} I_j$ with $\mathrm{card}(J) \leq \aleph_0$ and $I_{j_1} \cap I_{j_2} = \emptyset$ for all $j_1 \neq j_2$.

Convention: We write $\sum_{i \in I} a_i$ for $\sum_I (\, a_i \,)_{i \in I}$.

Examples: (no non-trivial ring or field is $\aleph_0$-complete)

- *Natural numbers:* $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$,
- *Boolean semiring:* $(\{\bot, \top\}, \vee, \wedge, \bot, \top)$,
- *Tropical semiring:* $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$,
- *Language semiring:* $(\mathcal{P}(\Sigma^*), \cup, \circ, \emptyset, \{\varepsilon\})$.

Trees

Definition:

- $T_\Sigma(V)$ set of Σ-trees indexed by V,
- $T_\Sigma = T_\Sigma(\emptyset)$,
- $t \in T_\Sigma(V)$ linear (resp., nondeleting) in U, if every $u \in U$ occurs at most (resp., at least) once in t,
- $\widehat{T_\Sigma}(V)$ set of linear and nondeleting Σ-trees indexed by V.

Examples: $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$ and $V = \{x_1, x_2\}$

- $\sigma(\alpha, \gamma(\alpha)) \in T_\Sigma$,
- $\sigma(x_1, x_1) \in T_\Sigma(V)$ linear in $\{x_2\}$ and nondeleting in $\{x_1\}$,
- $\sigma(x_1, x_2) \in \widehat{T_\Sigma}(V)$.

Tree Series

Definition: $\varphi : T_\Sigma(V) \longrightarrow A$ tree series, where $(A, +, \cdot, 0, 1)$ semiring

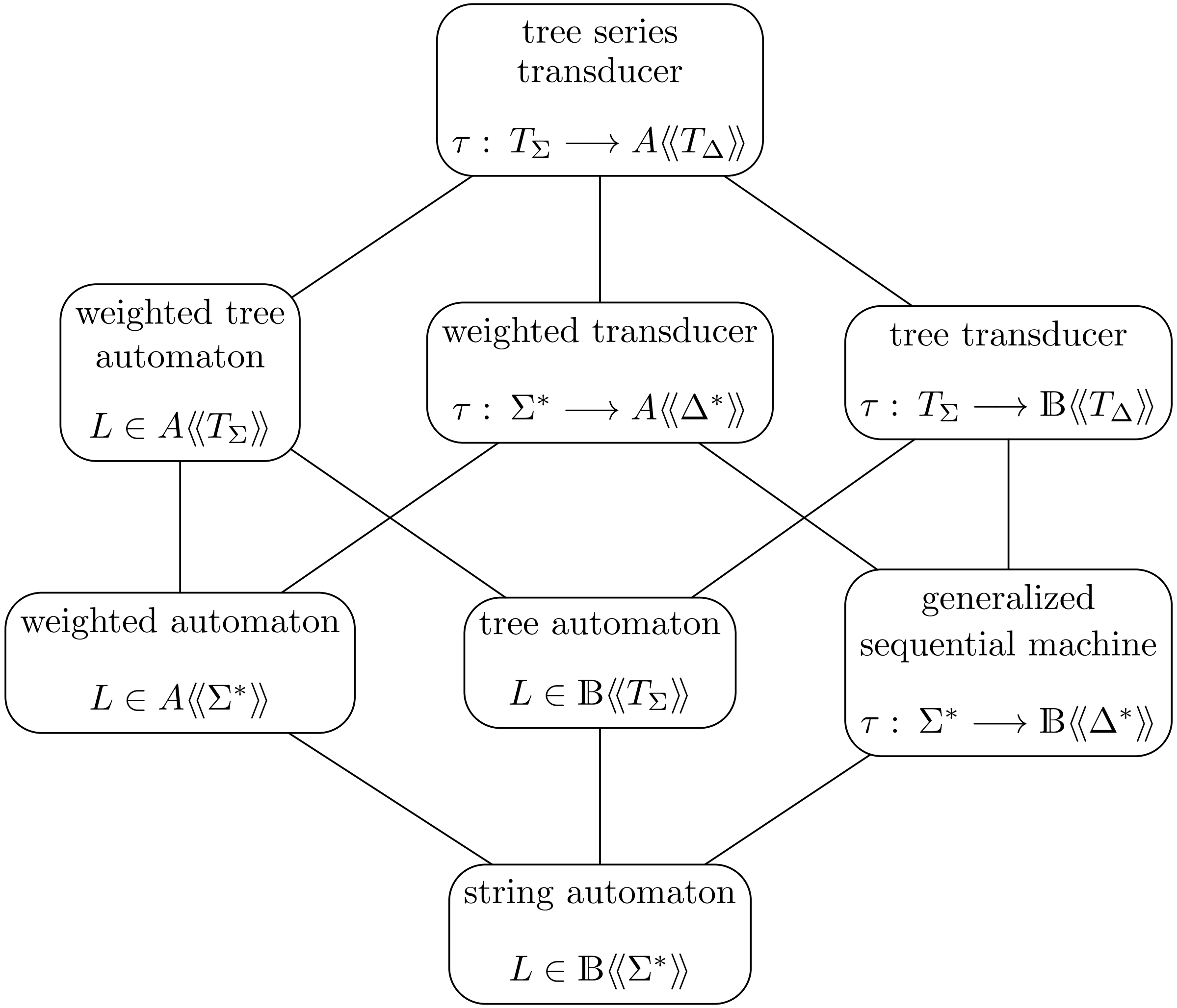
- $A\langle\langle T_\Sigma(V)\rangle\rangle$ class of all tree series,
- $\mathrm{supp}(\varphi) = \{\, t \in T_\Sigma(V) \mid \varphi(t) \neq 0 \,\}$,
- φ polynomial, if $\mathrm{supp}(\varphi)$ finite,
- $A\langle T_\Sigma(V)\rangle$ class of all polynomial tree series,
- $(\varphi + \psi)(t) = \varphi(t) + \psi(t)$,
- $(a \cdot \varphi)(t) = a \cdot \varphi(t)$.

Convention: We write (φ, t) instead of $\varphi(t)$ and $\sum_{t \in T_\Sigma(V)} (\varphi, t)\, t$ for φ.

Example:

- $\psi_{\mathrm{height}} = \sum_{t \in T_\Sigma} \mathrm{height}(t)\, t$,
- $\big(\psi_{\mathrm{height}}, \sigma(\gamma(\gamma(\alpha)), \alpha)\big) = 4$.

Generalization Hierarchy



Tree Representations

$(A, +, \cdot, 0, 1)$ semiring, Σ, Δ ranked alphabets, Q finite set

Definition: Family $(\mu_k)_{k \in \mathbb{N}}$ of $\mu_k : \Sigma^{(k)} \longrightarrow A\langle\langle T_\Delta(X)\rangle\rangle^{Q \times Q(X_k)^*}$ tree representation, if

- $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ for only finitely many $(q, w) \in Q \times Q(X_k)^*$,
- $\mu_k(\sigma)_{q,w} \in A\langle\langle T_\Delta(X_{|w|})\rangle\rangle$.

Convention: All entries left unspecified are assumed to be $\widetilde{0}$.

Definition: μ is

- linear (resp., nondeleting), if for all (q, w) such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ both w is linear (resp., nondeleting) in X_k and $\mu_k(\sigma)_{q,w}$ is linear (resp., nondeleting) in $X_{|w|}$,
- of type II (resp., top-down), if all $\mu_k(\sigma)_{q,w}$ are linear (resp., linear and nondeleting),
- bottom-up, if for all (q, w) such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ we have $w = q_1(x_1) \ldots q_k(x_k)$.

Example Tree Representation

Example: $\Sigma = \Delta = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$, $Q = \{\star, \bot\}$, semiring $\mathrm{Arct} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$

$\mu_2(\sigma)_{\downarrow,\rightarrow}$	$\star(x_1)\bot(x_2)$	$\star(x_2)\bot(x_1)$	$\bot(x_1)\bot(x_2)$
$\star$	$1\ \sigma(x_1, x_2)$	$1\ \sigma(x_2, x_1)$	$\widetilde{-\infty}$
$\bot$	$\widetilde{-\infty}$	$\widetilde{-\infty}$	$0\ \sigma(x_1, x_2)$

$\mu_0(\alpha)_{\downarrow,\rightarrow}$	ε
$\star$	$1\ \alpha$
$\bot$	$0\ \alpha$

$\mu_1(\gamma)_{\downarrow,\rightarrow}$	$\star(x_1)$	$\bot(x_1)$
$\star$	$1\ \gamma(x_1)$	$\widetilde{-\infty}$
$\bot$	$\widetilde{-\infty}$	$0\ \gamma(x_1)$

is a linear and nondeleting top-down tree representation, but not bottom-up.

Tree Series Transducers

Definition: $(Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ tree series transducer, if

- Q finite set of *states*,
- Σ and Δ ranked alphabets of *input* and *output symbols*, resp.,
- $\mathcal{A} = (A, +, \cdot, 0, 1)$ semiring,
- $F \in A\langle\langle \widehat{T_\Delta}(X_1) \rangle\rangle^Q$,
- μ tree representation (over Q, Σ, Δ, and A).

Convention: Tree series transducer inherits properties of its tree representation.

Example: $M_{\text{height}} = (\{\star, \bot\}, \Sigma, \Sigma, \text{Arct}, F, \mu)$ with μ from the previous example and $F_\bot = \widetilde{0}$ and $F_\star = 0\, x_1$ is a

linear and nondeleting top-down tree series transducer.

IO Tree Series Substitution

$\aleph_0$-complete semiring $(A, +, \cdot, 0, 1)$, ranked alphabet Δ

Definition: $\varphi \in A\langle\langle T_\Delta(X_k)\rangle\rangle$, $\psi_1, \ldots, \psi_k \in A\langle\langle T_\Delta\rangle\rangle$

$$\varphi \longleftarrow (\psi_1, \ldots, \psi_k) = \sum_{\substack{t \in T_\Delta(X_k),\\ t_1, \ldots, t_k \in T_\Delta}} (\varphi, t) \cdot (\psi_1, t_1) \cdot \ldots \cdot (\psi_k, t_k)\, t[t_1, \ldots, t_k]$$

Example: $(\mathbb{N}, +, \cdot, 0, 1)$

$$2\,\sigma(x_1, x_1) \longleftarrow \big(2\,\alpha\ ,\ 3\,\gamma(\alpha)\big) = 12\,\sigma(\alpha, \alpha)$$

IO Tree Series Transformations

$M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ tree series transducer

- Consider the Σ-algebra $(A^Q, (\, \mu(\sigma)\,)_{\sigma\in\Sigma})$ with

$$\mu(\sigma)(V_1,\ldots,V_k)_q = \sum_{\substack{w\in Q(X_k)^*,\\ w=q_1(x_{i_1})\ldots q_n(x_{i_n})}} \mu_k(\sigma)_{q,w} \longleftarrow \big((V_{i_1})_{q_1},\ldots,(V_{i_n})_{q_n}\big)$$

- Let h_μ be the unique homomorphism from T_Σ to A^Q.

Definition: IO tree series transformation induced by M is $\|M\| : A\langle\langle T_\Sigma\rangle\rangle \longrightarrow A\langle\langle T_\Delta\rangle\rangle$

$$\|M\|(\varphi) = \sum_{t\in T_\Sigma} (\varphi, t) \cdot \sum_{q\in Q} F_q \longleftarrow \big(h_\mu(t)_q\big)$$

Example: $\|M_{\mathrm{height}}\|\big(\sum_{t\in T_\Sigma} 0\, t\big) = \psi_{\mathrm{height}}$

Decomposition

Theorem: Let $\mathcal{A}$ be a commutative, $\aleph_0$-complete semiring.

$$\text{[p][b][l]–GST}(\mathcal{A}) \subseteq \text{[l]bh–TOP}(\mathcal{A}) \circ \text{[p][b][l]–BOT}(\mathcal{A})$$

$$\text{[p][b][l]–TOP}_+(\mathcal{A}) \subseteq \text{[l]bh–TOP}(\mathcal{A}) \circ \text{[p][b]l–BOT}(\mathcal{A})$$

Decomposition Theorem — Proof

Proof:

- $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ tree series transducer, construct td-homomorphism top-down tree series transducer M_1 and bottom-up tree series transducer M_2
- $\mathrm{mx} = \max\{\, |w|_{x_j} \mid q \in Q, k \in \mathbb{N}, \sigma \in \Sigma_{(k)}, j \in [k], w \in Q(X_k)^*, \mu_k(\sigma)_{q,w} \neq \widetilde{0}\,\}$ and $\Gamma^{(k \cdot \mathrm{mx})} = \Sigma^{(k)}$ and $\Gamma^{(n)} = \emptyset$ otherwise.
- construct $M_1 = (\{\star\}, \Sigma, \Gamma, \mathcal{A}, F_1, \mu_1)$ with $(F_1)_\star = 1\, x_1$

$$(\mu_1)_k(\sigma)_{\star, \underbrace{\star(x_1)\dots\star(x_1)}_{\mathrm{mx}\text{ times}} \dots \underbrace{\star(x_k)\dots\star(x_k)}_{\mathrm{mx}\text{ times}}} = 1\, \sigma(x_1, \dots, x_{k\cdot\mathrm{mx}}) \ .$$

- $Q' = Q \cup \{\bot\}$, $M_2' = (Q', \Gamma, \Delta, \mathcal{A}, F_2, \mu_2')$ with $(F_2)_q = F_q$ and $(F_2)_\bot = \widetilde{0}$ and

$$(\mu_2')_{k\cdot\mathrm{mx}}(\sigma)_{q, \mathrm{ren}(w, I)} = \mu_k(\sigma)_{q,w}$$
$$(\mu_2')_{k\cdot\mathrm{mx}}(\sigma)_{\bot, \bot(x_1)\dots\bot(x_{k\cdot\mathrm{mx}})} = 1\, \sigma(x_1, \dots, x_{k\cdot\mathrm{mx}})$$

Composition

Theorem: Let $\mathcal{A}$ be a commutative and $\aleph_0$-complete semiring.

$$[l]h\text{–}TOP(\mathcal{A}) \circ [p][l][h]\text{–}BOT(\mathcal{A}) \subseteq [p][l][h]\text{–}GST(\mathcal{A})$$
$$[l]h\text{–}TOP(\mathcal{A}) \circ [p][h]l\text{–}BOT(\mathcal{A}) \subseteq [p][l][h]\text{–}TOP_+(\mathcal{A})$$
$$[l]h\text{–}TOP(\mathcal{A}) \circ [p][h]nl\text{–}BOT(\mathcal{A}) \subseteq [p][l][h]\text{–}TOP(\mathcal{A})$$

Proof:

- $M_1 = (\{\star\}, \Sigma, \Gamma, \mathcal{A}, F_1, \mu_1)$ homomorphism top-down tree series transducer and $M_2 = (Q, \Gamma, \Delta, \mathcal{A}, F, \mu_2)$ bottom-up tree series transducer
- construct tree series transducer $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ for $w = q_1(x_{i_1}) \ldots q_n(x_{i_n}) \in Q(X_k)^*$ set

$$\mu_k(\sigma)_{q,w} = h_{\mu_2}^{q_1 \cdots q_n}\big((\mu_1)_k(\sigma)_{\star, \star(x_{i_1}) \ldots \star(x_{i_n})}\big)_q$$

Characterization Theorem

Theorem: Let $\mathcal{A}$ be a commutative and $\aleph_0$-complete semiring.

$$\text{[p][l]–GST}(\mathcal{A}) = \text{[l]bh–TOP}(\mathcal{A}) \circ \text{[p][l]–BOT}(\mathcal{A})$$
$$\text{[p][l]–TOP}_{+}(\mathcal{A}) = \text{[l]bh–TOP}(\mathcal{A}) \circ \text{[p]l–BOT}(\mathcal{A})$$

OI Tree Series Substitution

$\aleph_0$-complete semiring $(A, +, \cdot, 0, 1)$, ranked alphabet Δ

Definition: $\varphi \in A\langle\langle T_\Delta(X_k)\rangle\rangle$, $\psi_1, \ldots, \psi_k \in A\langle\langle T_\Delta\rangle\rangle$

$$x_j[\psi_1, \ldots, \psi_k] = \psi_j$$
$$\sigma(t_1, \ldots, t_n)[\psi_1, \ldots, \psi_k] = \sigma(t_1[\psi_1, \ldots, \psi_k], \ldots, t_n[\psi_1, \ldots, \psi_k])$$

where

$$\sigma(\psi_1, \ldots, \psi_k) = \sum_{t_1, \ldots, t_k \in T_\Delta} (\psi_1, t_1) \cdot \ldots \cdot (\psi_k, t_k)\, \sigma(t_1, \ldots, t_k).$$

Example: $(\mathbb{N}, +, \cdot, 0, 1)$

$$2\,\sigma(x_1, x_1) \longleftarrow \big(2\,\alpha\,,\ 3\,\gamma(\alpha)\big) = 8\,\sigma(\alpha, \alpha)$$

OI Tree Series Transformations

$M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ tree series transducer

- Consider the Σ-algebra $(A^Q, (\,\mu^{\mathrm{OI}}(\sigma)\,)_{\sigma\in\Sigma})$ with

$$\mu^{\mathrm{OI}}(\sigma)(V_1, \dots, V_k)_q = \sum_{\substack{w\in Q(X_k)^*,\\ w=q_1(x_{i_1})\dots q_n(x_{i_n})}} \mu_k(\sigma)_{q,w}\big[(V_{i_1})_{q_1}, \dots, (V_{i_n})_{q_n}\big]$$

- Let h_μ^{OI} be the unique homomorphism from T_Σ to A^Q.

Definition: OI tree series transformation induced by M is $\|M\|^{\mathrm{OI}} : A\langle\!\langle T_\Sigma\rangle\!\rangle \longrightarrow A\langle\!\langle T_\Delta\rangle\!\rangle$

$$\|M\|^{\mathrm{OI}}(\varphi) = \sum_{t\in T_\Sigma} (\varphi, t) \cdot \sum_{q\in Q} F_q \longleftarrow \big(h_\mu^{\mathrm{OI}}(t)_q\big)$$

Considering Linearity and Deletion

Theorem: For every $\aleph_0$-complete semiring $\mathcal{A}$

$$\text{[b][l][n][d][h]p–TOP}_+^{\text{OI}}(\mathcal{A}) = \text{[b][l][n][d][h]p–GST}^{\text{OI}}(\mathcal{A}) \ .$$

Theorem: For every $\aleph_0$-complete semiring $\mathcal{A}$

$$\text{[p][b][l][n][d][h]–TOP}_+^{\text{OI}}(\mathcal{A}) = \text{[p][b][l][n][d][h]–TOP}^{\text{OI}}(\mathcal{A}) \ .$$

Considering Deletion — Proof

Proof:

- $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ tree series transducer
- $j \in \mathbb{N}_+$ maximal s.t. there exist $k \in \mathbb{N}$, $\sigma \in \Sigma_{(k)}$, $q \in Q$, $w \in Q(X)^*$, and $t \in \mathrm{supp}(\mu_k(\sigma)_{q,w})$ such that $j \leq |w|$ and $|t|_{x_j} = 0$.
- construct tree series transducer $M' = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu')$ with $w = q_1(x_{i_1}) \ldots q_n(x_{i_n}) \in Q(X_k)^*$ such that $j > n$ set $\mu'_k(\sigma)_{q,w} = \mu_k(\sigma)_{q,w}$ and otherwise

$$\mu'_k(\sigma)_{q,w} = \sum_{\substack{t \in T_\Delta(X), \\ |t|_{x_j} \geq 1}} (\mu_k(\sigma)_{q,w}, t)\, t + \sum_{\substack{w' \in Q(X_k)^{n+1}, \\ w = w'_1 \ldots w'_{j-1} w'_{j+1} \ldots w'_{n+1}, \\ t \in T_\Delta(X \setminus \{x_j\})}} (\mu_k(\sigma)_{q,w'}, t)\, t[x_1, \ldots, x_j, x_j, \ldots, x_n] \ .$$

Characterization Theorem

Theorem: For every $\aleph_0$-complete semiring $\mathcal{A}$

$$\text{[b][l][n][d][h]p–TOP}(\mathcal{A}) = \text{[b][l][n][d][h]p–TOP}^{\text{OI}}(\mathcal{A})$$
$$= \text{[b][l][n][d][h]p–TOP}_{+}^{\text{OI}}(\mathcal{A}) = \text{[b][l][n][d][h]p–GST}^{\text{OI}}(\mathcal{A}) \ .$$