Simulations of Weighted Tree Automata

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Simulation of weighted string automata

Theorem (BÉAL, LOMBARDY, SAKAROVITCH 2005 & 2006)

For all equivalent weighted string automata over . . . there exists a chain of simulations connecting them.

- *a field*
- *the integers (more generally, an* EUCLIDIAN *domain)*
- *the natural numbers*
- *the* BOOLEAN *semiring*
- *(functional transducers)*

Consequence

Equivalence = Chain of Simulations

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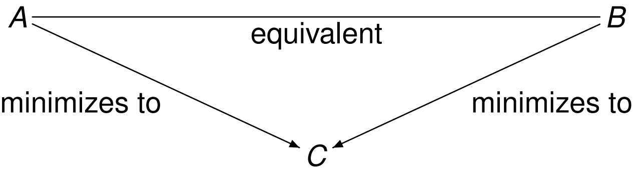
Minimization of weighted tree automata

In fields [SEIDL 1990, BOZAPALIDIS 1991]



- automata collapsed by equivalence relation
- the canonical homomorphism is a simulation

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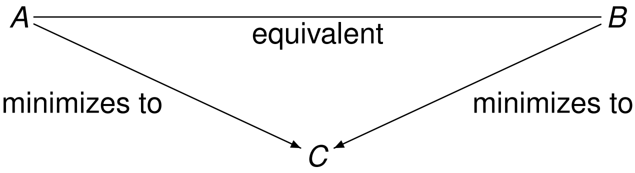

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Equivalence = Chain of Simulations

Contents

Semiring

Definition

A commutative semiring is an algebraic structure $(\mathbb{K}, +, \cdot, 0, 1)$ with

- commutative monoids $(\mathbb{K}, +, 0)$ and $(\mathbb{K}, \cdot, 1)$
- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- $a \cdot 0 = 0$

Example

- natural numbers
- tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$
- BOOLEAN semiring $(\{0, 1\}, \max, \min, 0, 1)$
- any commutative ring

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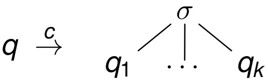
Weighted tree automaton

Definition (BERSTEL, REUTENAUER 1982)

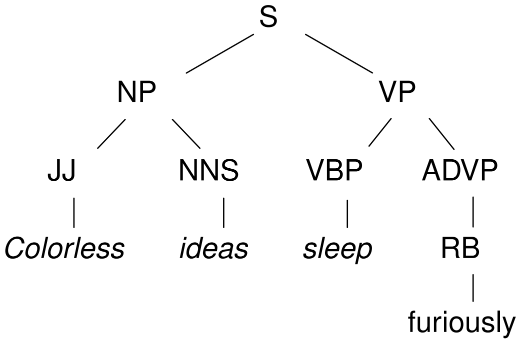
A weighted tree automaton (wta) is a tuple $A = (Q, \Sigma, \mathbb{K}, I, R)$ with rules of the form



where

- $q, q_1, \ldots, q_k \in Q$ are states
- $c \in \mathbb{K}$ is a weight (taken from a semiring)
- $\sigma \in \Sigma_k$ is an input symbol

Run



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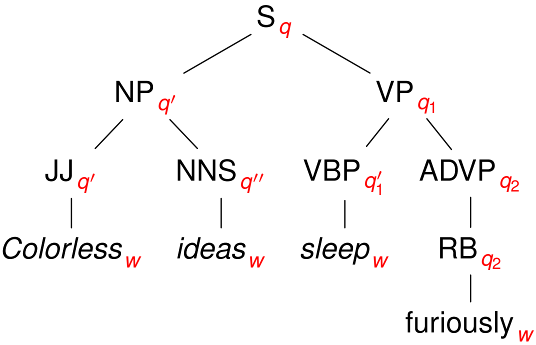
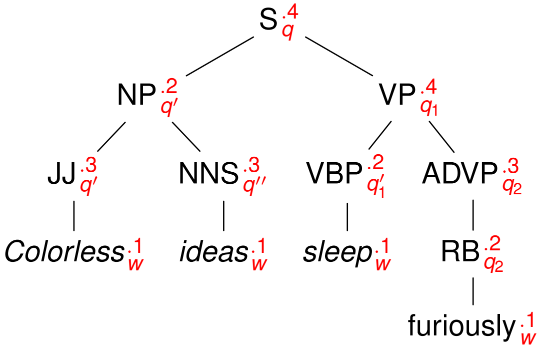

S q
NP q′
VP q1
JJ q′
NNS q′′
VBP q′1
ADVP q2
Colorless w
ideas w
sleep w
RB q2
furiously w

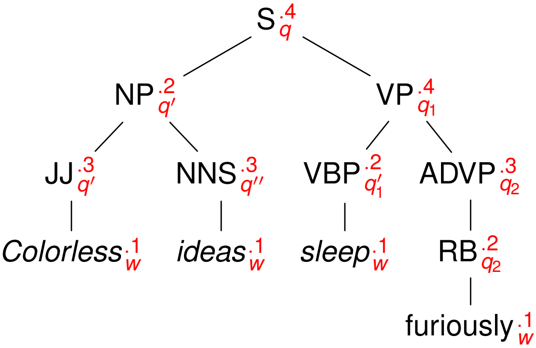
Run



Definition

The weight of a run is obtained by multiplying the weights in it.

Run



Weight of the run

$0.4 \cdot 0.2 \cdot 0.3 \cdot 0.1 \cdot 0.3 \cdot 0.1 \cdot 0.4 \cdot 0.2 \cdot 0.1 \cdot 0.3 \cdot 0.2 \cdot 0.1$

Semantics

Definition

The weight of an input tree t is

$$\text{weight}(t) = \sum_{r \text{ run on } t} I(\text{root}(r)) \cdot \text{weight}(r)$$

Definition

Two wta are equivalent if they assign the same weights to all trees.

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The weight of an input tree t is

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Definition

Two wta are equivalent if they assign the same weights to all trees.

Matrix representation

Definition

matrix presentation of a wta (Q, Σ, I, R)

$$R_k(\sigma)_{q_1 \cdots q_k, q} = c \qquad \Longleftrightarrow \qquad q \xrightarrow{c} \sigma(q_1, \ldots, q_k) \in R$$

Main definition

Definition (Bloom, Ésik 1993)

A wta (Q, Σ, I, R) **simulates** a wta (P, Σ, J, S) if there exists a matrix $X \in \mathbb{K}^{Q \times P}$ such that

- $I = XJ$

$$I(q) = \sum_{p \in P} X_{q,p} \cdot J(p)$$

- $R_k(\sigma) X = X^{k,\otimes} S_k(\sigma)$

$$\sum_{q \in Q} R_k(\sigma)_{q_1 \cdots q_k, q} \cdot X_{q,p} = \sum_{p_1 \cdots p_k \in P^k} X_{q_1,p_1} \cdot \ldots \cdot X_{q_k,p_k} \cdot S_k(\sigma)_{p_1 \cdots p_k, p}$$

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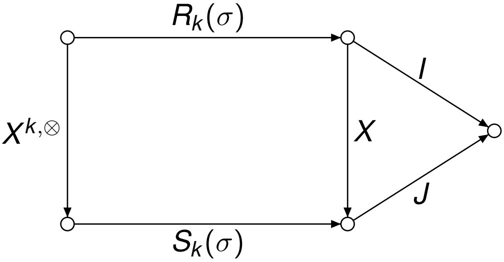

Relation to simple simulations

Definition

A matrix $X \in \mathbb{K}^{Q \times P}$ is

- relational if $X \in \{0, 1\}^{Q \times P}$
- functional if X is relational and induces a mapping
- surjective, injective, ...

Definition (Högberg, ∼, May 2007)

- wta A forward simulates wta B
  if A simulates B with a functional transfer matrix.
- wta A backward simulates wta B
  if B simulates A with transfer matrix X such that X^{T} is functional.

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Relation to simple simulations (cont'd)

Definition (Béal, Lombardy, Sakarovitch 2005)

The semiring $\mathbb{K}$ is equisubtractive if for every $a_1, a_2, b_1, b_2 \in \mathbb{K}$ with $a_1 + b_1 = a_2 + b_2$ there exist $c_1, c_2, d_1, d_2 \in \mathbb{K}$ such that

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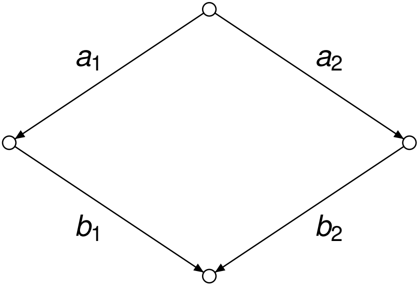

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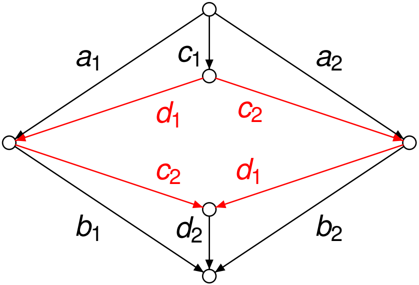
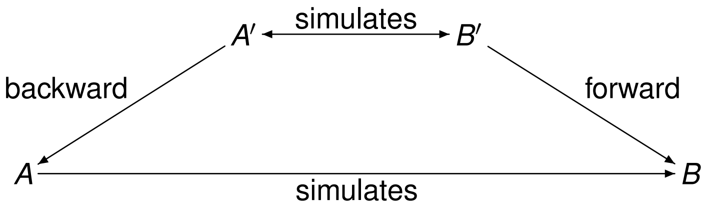


Relation to simple simulations (cont'd)

Theorem (Béal, Lombardy, Sakarovitch 2005)

Let $\mathbb{K}$ be equisubtractive and additively generated by units.
Then wta A simulates wta B iff there exist wta A' and B' such that

- *A' backward simulates A*
- *A' simulates B' with an invertable diagonal transfer matrix*
- *B' forward simulates B*



Main question

Theorem (Bloom, Ésik 1993)

Simulation is a pre-order that refines equivalence.

Question

Does the symmetric, transitive closure of simulation coincide with equivalence?

In other words

Are all equivalent wta joined by a finite chain of simulations?

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Proper semiring

Definition

The semiring $\mathbb{K}$ is proper if for all wta A and B

A and B are equivalent

iff there exists a finite chain of simulations that join A and B.

Example

- BOOLEAN semiring [BLOOM, ÉSIK 1993, KOZEN 1994]
- any commutative field [SEIDL 1990, BOZAPALIDIS 1991]

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NOETHERIAN semiring

Definition

A commutative monoid $(M, \oplus, 0)$ together with an action $\odot \colon \mathbb{K} \times M \to M$ is a $\mathbb{K}$-semimodule if

- $(a + b) \odot m = (a \odot m) \oplus (b \odot m)$
- $a \odot (m \oplus n) = (a \odot m) \oplus (a \odot n)$
- $(a \cdot b) \odot m = a \odot (b \odot m)$
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Definition

The semiring $\mathbb{K}$ is NOETHERIAN if all subsemimodules of every finitely generated $\mathbb{K}$-semimodule are again finitely generated.

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NOETHERIAN semiring (cont'd)

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All of the following are NOETHERIAN:

- fields
- finitely generated commutative rings
- finite semirings

Non-example

- natural numbers
- tropical semiring

NOETHERIAN semiring (cont'd)

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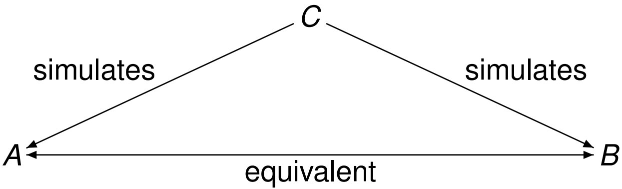
Main result

Theorem (cf. BÉAL, LOMBARDY, SAKAROVITCH 2006)

- *Every* NOETHERIAN *semiring is proper.*
- $\mathbb{N}$ *is proper.*

Note (on theorem)

There exists a single wta that simulates both wta.



Consequence

Theorem

Let $\mathbb{K}$ be proper and finitely and effectively presented. Then equivalence of wta is decidable.

Proof.

- Inequality is semi-decidable.
- Using the main result, equivalence is semi-decidable.

$\Rightarrow$ run in parallel $\Rightarrow$ equivalence decidable □

Corollary

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Consequence (cont'd)

Theorem

The tropical semiring is not proper.

Proof.

- Inequality is semi-decidable.
- If proper, then equivalence is semi-decidable.

$\Rightarrow$ Equivalence is decidable.

But Equivalence is undecidable by [Krob 1992].

□

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Thank you for your attention!