The Power of Tree Series Transducers of Type I and II

Andreas Maletti

July 6, 2005

1. Motivation
2. Tree Series Transducers
3. OI Tree Series Substitution
4. IO Tree Series Substitution

Tree Transducers

Applications:

- syntax-directed semantics
- functional program analysis and transformation
- analysis of cryptographic protocols

Form of the rules:

- bottom-up: $\sigma(q_1(z_1), \ldots, q_k(z_k)) \rightarrow q(t)$ where $t \in T_\Delta(Z_k)$
- top-down: $q(\sigma(x_1, \ldots, x_k)) \rightarrow t[z_i/q_i(x_{j_i})]$ where $t \in \widehat{T_\Delta}(Z_n)$
- general: $q(\sigma(x_1, \ldots, x_k)) \rightarrow t[z_i/q_i(x_{j_i})]$ where $t \in T_\Delta(Z_n)$

$Z_n = \{z_1, \ldots, z_n\}$, $T_\Delta(Z_n)$ set of Δ-trees indexed by Z_n, and $\widehat{T_\Delta}(Z_n)$ linear and nondeleting trees of $T_\Delta(Z_n)$

To Tree Series Transducers

general form of the rules:

- tree transducer: $q(\sigma(x_1, \ldots, x_k)) \to t[z_i/q_i(x_{j_i})]$ where $t \in T_\Delta(Z_n)$
- tree series transducer: $q(\sigma(x_1, \ldots, x_k)) \xrightarrow{c} t[z_i/q_i(x_{j_i})]$ where $t \in T_\Delta(Z_n)$

c is called *coefficient* (or *weight*), taken from a semiring

Definition: $(A, +, \cdot, 0, 1)$ *semiring*, if

- $(A, +, 0)$ commutative monoid,
- $(A, \cdot, 1)$ monoid,
- $\cdot$ distributes over $+$, and
- 0 acts as a zero wrt. $\cdot$ (i.e., $a \cdot 0 = 0 = 0 \cdot a$)

Applications of Tree Series Transducers

Natural Language Processing:

- data-oriented parsing (bottom-up)
- tree bank transformations (top-down)
- language translation (neither top-down nor bottom-up)

Example: (neither top-down nor bottom-up)

$$\sigma\big(\sigma(\alpha, x_2), \sigma(x_1, x_3)\big) \xrightarrow{c} \delta(x_1, x_2, x_3)$$

in natural language translation terms:

$$\mathrm{SENT}\big(\mathrm{FRONT}(\text{"does"}, PP3), \mathrm{VP}(VERB, OBJ)\big) \xrightarrow{c} \mathrm{QUES}(VERB, PP3, OBJ)$$

$\aleph_0$-complete Semirings

Definition: $(A, +, \cdot, 0, 1)$ $\aleph_0$-complete semiring, if for all I with $\mathrm{card}(I) \leq \aleph_0$ there exists $\sum\colon A^I \longrightarrow A$ such that:

- $\sum_{i \in I} a_i = a_{j_1} + a_{j_2}$, if $I = \{j_1, j_2\}$ with $j_1 \neq j_2$;
- $\sum_{i \in I} a_i = \sum_{j \in J} \left(\sum_{i \in I_j} a_i \right)$, if $I = \bigcup_{j \in J} I_j$ with $\mathrm{card}(J) \leq \aleph_0$ and $I_{j_1} \cap I_{j_2} = \emptyset$ for all $j_1 \neq j_2$; and
- $\left(\sum_{i \in I} a_i \right) \cdot \left(\sum_{j \in J} b_j \right) = \sum_{(i,j) \in I \times J} (a_i \cdot b_j)$.

Examples: (no non-trivial ring or field is $\aleph_0$-complete)

- *Natural numbers:* $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$,
- *Boolean semiring:* $(\{\bot, \top\}, \vee, \wedge, \bot, \top)$,
- *Tropical semiring:* $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$,
- *Language semiring:* $(\mathcal{P}(\Sigma^*), \cup, \circ, \emptyset, \{\varepsilon\})$.

Tree Series

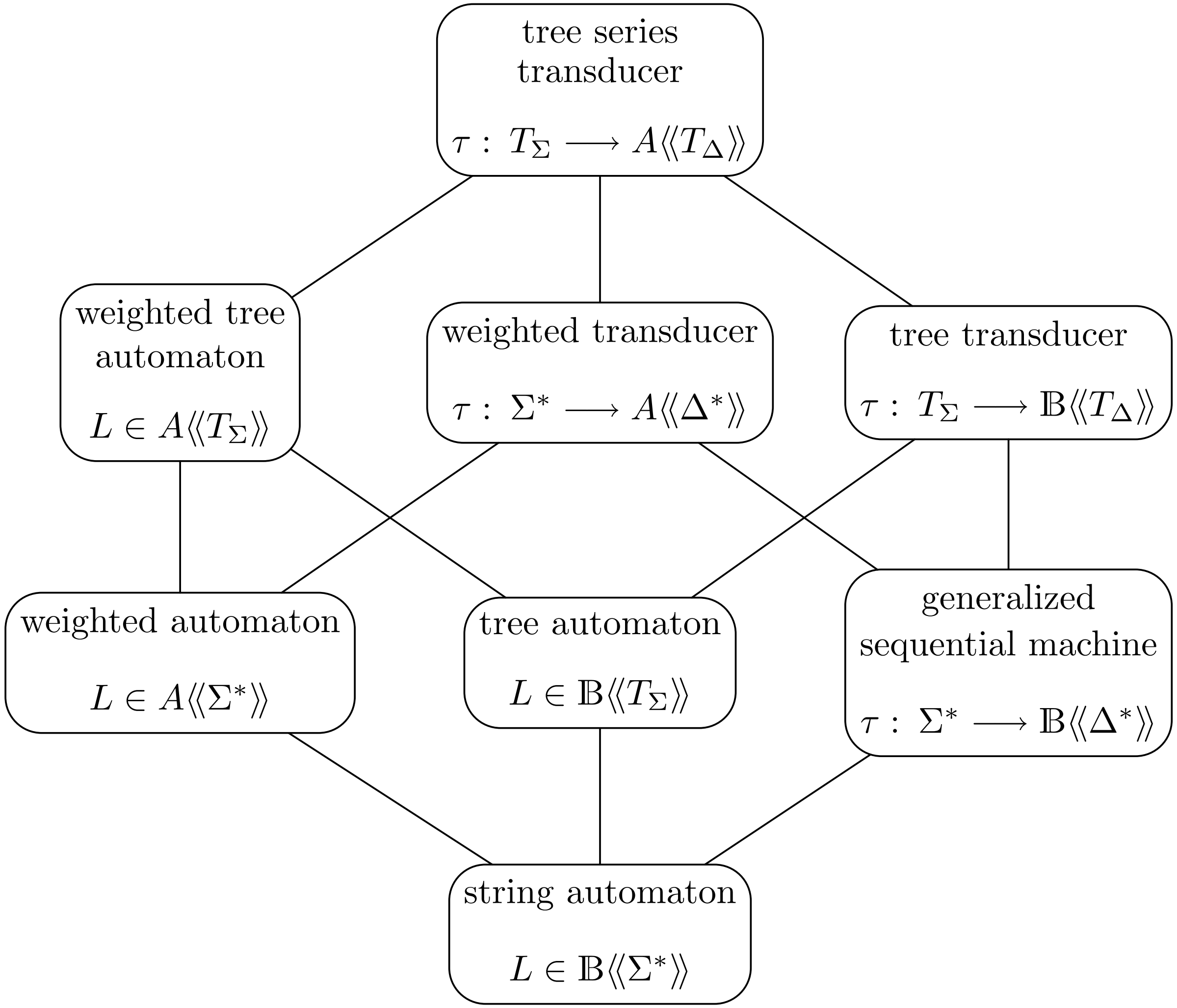
Definition: $\varphi\colon T_\Sigma(V) \longrightarrow A$ tree series, where $(A, +, \cdot, 0, 1)$ semiring

- $\mathrm{supp}(\varphi) = \{\, t \in T_\Sigma(V) \mid \varphi(t) \neq 0 \,\}$,
- φ polynomial, if $\mathrm{supp}(\varphi)$ finite,
- $A\langle\langle T_\Sigma(V)\rangle\rangle$ class of all tree series,
- $A\langle T_\Sigma(V)\rangle$ class of all polynomial tree series,
- $(\varphi + \psi)(t) = \varphi(t) + \psi(t)$,
- $(a \cdot \varphi)(t) = a \cdot \varphi(t)$.

Convention: We write (φ, t) instead of $\varphi(t)$ and $\sum_{t \in T_\Sigma(V)} (\varphi, t)\, t$ for φ.

Example: $\psi_{\mathrm{height}} = \sum_{t \in T_\Sigma} \mathrm{height}(t)\, t$,

Generalization Hierarchy



Tree Representations

$(A, +, \cdot, 0, 1)$ semiring, Σ, Δ ranked alphabets, Q finite set

Recall: $q(\sigma(x_1, \ldots, x_k)) \xrightarrow{c} t[z_i/q_i(x_{j_i})]$ where $t \in T_\Delta(Z_n)$

organized in matrices: $\mu_k(\sigma)_{q, q_1(x_{j_1}) \ldots q_n(x_{j_n})} = c\, t + \ldots$

Definition: $(\mu_k)_{k \in \mathbb{N}}$ of $\mu_k \colon \Sigma^{(k)} \longrightarrow A\langle\langle T_\Delta(Z)\rangle\rangle^{Q \times Q(X_k)^*}$ tree representation, if

- $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ for only finitely many $(q, w) \in Q \times Q(X_k)^*$,
- $\mu_k(\sigma)_{q,w} \in A\langle\langle T_\Delta(Z_{|w|})\rangle\rangle$.

Properties

Definition: μ is

- linear (resp., nondeleting), if for all (q, w) such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ both w is linear (resp., nondeleting) in X_k and $\mu_k(\sigma)_{q,w}$ is linear (resp., nondeleting) in $Z_{|w|}$,
- of type II (resp., top-down), if all $\mu_k(\sigma)_{q,w}$ are linear (resp., linear and nondeleting),
- bottom-up, if for all (q, w) such that $\mu_k(\sigma)_{q,w} \neq \widetilde{0}$ we have $w = q_1(x_1) \ldots q_k(x_k)$.

Example Tree Representation

Example: $\Sigma = \Delta = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$, $Q = \{\star, \bot\}$,
semiring $\mathrm{Arct} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$

$\mu_2(\sigma)_{\downarrow,\rightarrow}$	$\star(x_1)\bot(x_2)$	$\star(x_2)\bot(x_1)$	$\bot(x_1)\bot(x_2)$
$\star$	$1\ \sigma(z_1, z_2)$	$1\ \sigma(z_2, z_1)$	$\widetilde{-\infty}$
$\bot$	$\widetilde{-\infty}$	$\widetilde{-\infty}$	$0\ \sigma(z_1, z_2)$

$\mu_0(\alpha)_{\downarrow,\rightarrow}$	ε
$\star$	$1\ \alpha$
$\bot$	$0\ \alpha$

$\mu_1(\gamma)_{\downarrow,\rightarrow}$	$\star(x_1)$	$\bot(x_1)$
$\star$	$1\ \gamma(z_1)$	$\widetilde{-\infty}$
$\bot$	$\widetilde{-\infty}$	$0\ \gamma(z_1)$

is a linear and nondeleting top-down tree representation, but not bottom-up.

Tree Series Transducers

Definition: $(Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ tree series transducer, if

- Q finite set of *states*,
- Σ and Δ ranked alphabets of *input* and *output symbols*, resp.,
- $\mathcal{A} = (A, +, \cdot, 0, 1)$ $\aleph_0$-complete semiring,
- $F \in A\langle\langle \widehat{T_\Delta}(Z_1) \rangle\rangle^Q$,
- μ tree representation.

Convention: Tree series transducer inherits properties of its tree representation.

Example: $M_{\text{height}} = (\{\star, \bot\}, \Sigma, \Sigma, \text{Arct}, F, \mu)$ with μ from the previous example and $F_\bot = \widetilde{0}$ and $F_\star = 0\, z_1$ is a

linear and nondeleting top-down tree series transducer.

IO Tree Series Substitution

$\aleph_0$-complete semiring $(A, +, \cdot, 0, 1)$, ranked alphabet Δ

Definition: $\varphi \in A\langle\langle T_\Delta(Z_k)\rangle\rangle$, $\psi_1, \ldots, \psi_k \in A\langle\langle T_\Delta\rangle\rangle$

$$\varphi \longleftarrow (\psi_1, \ldots, \psi_k) = \sum_{\substack{t \in T_\Delta(Z_k), \\ t_1, \ldots, t_k \in T_\Delta}} (\varphi, t) \cdot (\psi_1, t_1) \cdot \ldots \cdot (\psi_k, t_k)\, t[t_1, \ldots, t_k]$$

Example: $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$

$$2\,\sigma(z_1, z_1) \longleftarrow \big(2\,\alpha\,,\ 3\,\gamma(\alpha)\big) = 12\,\sigma(\alpha, \alpha)$$

IO Tree Series Transformations

$M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ tree series transducer

- Consider the Σ-algebra $(A^Q, (\,\mu(\sigma)\,)_{\sigma\in\Sigma})$ with

$$\mu(\sigma)(V_1, \ldots, V_k)_q = \sum_{\substack{w\in Q(X_k)^*,\\ w=q_1(x_{i_1})\ldots q_n(x_{i_n})}} \mu_k(\sigma)_{q,w} \longleftarrow \big((V_{i_1})_{q_1}, \ldots, (V_{i_n})_{q_n}\big)$$

- Let h_μ be the unique homomorphism from T_Σ to A^Q.

Definition: IO tree series transformation induced by M is $\|M\|\colon A\langle\!\langle T_\Sigma\rangle\!\rangle \longrightarrow A\langle\!\langle T_\Delta\rangle\!\rangle$

$$\|M\|(\varphi) = \sum_{t\in T_\Sigma} (\varphi, t) \cdot \sum_{q\in Q} F_q \longleftarrow \big(h_\mu(t)_q\big)$$

Example: $\|M_{\text{height}}\|\big(\sum_{t\in T_\Sigma} 0\,t\big) = \psi_{\text{height}}$

Decomposition

p... polynomial, b ... boolean, l ... linear, h ... homomorphism (1-state deterministic)

x–GST$(\mathcal{A})$	tree series transformations computed by x tree series transducers of type I (generalized sequential transducer)
x–TOP$^{\mathrm{R}}(\mathcal{A})$	... of type II (top-down with look-ahead)
x–BOT$(\mathcal{A})$	... x bottom-up tree series transducers

Theorem: Let $\mathcal{A}$ be a commutative, $\aleph_0$-complete semiring.

$$\text{[p][b][l]–GST}(\mathcal{A}) \subseteq \text{[l]bh–TOP}(\mathcal{A}) \,;\, \text{[p][b][l]–BOT}(\mathcal{A})$$

$$\text{[p][b][l]–TOP}^{\mathrm{R}}(\mathcal{A}) \subseteq \text{[l]bh–TOP}(\mathcal{A}) \,;\, \text{[p][b]l–BOT}(\mathcal{A})$$

Decomposition Theorem — Proof

Proof:

- $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ tree series transducer, construct homomorphism top-down tree series transducer M_1 and bottom-up tree series transducer M_2
- $\mathrm{mx} = \max\{\, |w|_{x_j} \mid q \in Q, k \in \mathbb{N}, \sigma \in \Sigma_{(k)}, j \in [k], w \in Q(X_k)^*, \mu_k(\sigma)_{q,w} \neq \widetilde{0}\,\}$ and $\Gamma^{(k \cdot \mathrm{mx})} = \Sigma^{(k)}$ and $\Gamma^{(n)} = \emptyset$ otherwise.
- construct $M_1 = (\{\star\}, \Sigma, \Gamma, \mathcal{A}, F_1, \mu_1)$ with $(F_1)_\star = 1\; z_1$

$$(\mu_1)_k(\sigma)_{\star, \underbrace{\star(x_1)\ldots\star(x_1)}_{\mathrm{mx}\text{ times}}\ldots\underbrace{\star(x_k)\ldots\star(x_k)}_{\mathrm{mx}\text{ times}}} = 1\; \sigma(z_1, \ldots, z_{k\cdot\mathrm{mx}}) \enspace .$$

- $Q' = Q \cup \{\bot\}$, $M_2' = (Q', \Gamma, \Delta, \mathcal{A}, F_2, \mu_2')$ with $(F_2)_q = F_q$ and $(F_2)_\bot = \widetilde{0}$ and

$$(\mu_2')_{k\cdot\mathrm{mx}}(\sigma)_{q,\mathrm{ren}(w,I)} = \mu_k(\sigma)_{q,w}$$
$$(\mu_2')_{k\cdot\mathrm{mx}}(\sigma)_{\bot,\bot(x_1)\ldots\bot(x_{k\cdot\mathrm{mx}})} = 1\; \sigma(z_1, \ldots, z_{k\cdot\mathrm{mx}})$$

Composition

Theorem: Let $\mathcal{A}$ be a commutative and $\aleph_0$-complete semiring.

$$\text{[l]h–TOP}(\mathcal{A})\ ;\ \text{[p][l][h]–BOT}(\mathcal{A}) \subseteq \text{[p][l][h]–GST}(\mathcal{A})$$

$$\text{[l]h–TOP}(\mathcal{A})\ ;\ \text{[p][h]l–BOT}(\mathcal{A}) \subseteq \text{[p][l][h]–TOP}^{\mathrm{R}}(\mathcal{A})$$

$$\text{[l]h–TOP}(\mathcal{A})\ ;\ \text{[p][h]nl–BOT}(\mathcal{A}) \subseteq \text{[p][l][h]–TOP}(\mathcal{A})$$

Proof:

- $M_1 = (\{\star\}, \Sigma, \Gamma, \mathcal{A}, F_1, \mu_1)$ homomorphism top-down tree series transducer and $M_2 = (Q, \Gamma, \Delta, \mathcal{A}, F, \mu_2)$ bottom-up tree series transducer
- construct tree series transducer $M = (Q, \Sigma, \Delta, \mathcal{A}, F, \mu)$ for $w = q_1(x_{i_1}) \dots q_n(x_{i_n}) \in Q(X_k)^*$ set

$$\mu_k(\sigma)_{q,w} = h_{\mu_2}^{q_1 \cdots q_n}((\mu_1)_k(\sigma)_{\star, \star(x_{i_1}) \dots \star(x_{i_n})})_q$$

Characterization Theorem

Theorem: Let $\mathcal{A}$ be a commutative and $\aleph_0$-complete semiring.

$$\text{[p][l]–GST}(\mathcal{A}) = \text{[l]bh–TOP}(\mathcal{A}) \; ; \; \text{[p][l]–BOT}(\mathcal{A})$$

$$\text{[p][l]–TOP}^{\mathrm{R}}(\mathcal{A}) = \text{[l]bh–TOP}(\mathcal{A}) \; ; \; \text{[p]l–BOT}(\mathcal{A})$$

OI Tree Series Substitution

$\aleph_0$-complete semiring $(A, +, \cdot, 0, 1)$, ranked alphabet Δ

Definition: $\varphi \in A\langle\langle T_\Delta(Z_k)\rangle\rangle$, $\psi_1, \ldots, \psi_k \in A\langle\langle T_\Delta\rangle\rangle$

$$z_j[\psi_1, \ldots, \psi_k] = \psi_j$$
$$\sigma(t_1, \ldots, t_n)[\psi_1, \ldots, \psi_k] = \sigma(t_1[\psi_1, \ldots, \psi_k], \ldots, t_n[\psi_1, \ldots, \psi_k])$$

where

$$\sigma(\psi_1, \ldots, \psi_k) = \sum_{t_1, \ldots, t_k \in T_\Delta} (\psi_1, t_1) \cdot \ldots \cdot (\psi_k, t_k)\, \sigma(t_1, \ldots, t_k).$$

Example: $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$

$$2\,\sigma(z_1, z_1) \longleftarrow \big(2\,\alpha\;,\; 3\,\gamma(\alpha)\big) = 8\,\sigma(\alpha, \alpha)$$

Considering Linearity and Deletion

n ... nondeleting, d ... deterministic

Theorem: For every $\aleph_0$-complete semiring $\mathcal{A}$

$$\begin{aligned} \text{[b][l][n][d][h]p–TOP}^{\mathrm{R}}_{\mathsf{OI}}(\mathcal{A}) &= \text{[b][l][n][d][h]p–GST}_{\mathsf{OI}}(\mathcal{A}) \;, \\ \text{[p][b][l][n][d][h]–TOP}^{\mathrm{R}}_{\mathsf{OI}}(\mathcal{A}) &= \text{[p][b][l][n][d][h]–TOP}_{\mathsf{OI}}(\mathcal{A}) \;. \end{aligned}$$

Theorem: For every $\aleph_0$-complete semiring $\mathcal{A}$

$$\text{p–TOP}(\mathcal{A}) = \text{p–TOP}_{\mathsf{OI}}(\mathcal{A}) = \text{p–TOP}^{\mathrm{R}}_{\mathsf{OI}}(\mathcal{A}) = \text{p–GST}_{\mathsf{OI}}(\mathcal{A}) \;.$$

The End

Thank you for your attention.