Does O-Substitution Preserve Recognizability?

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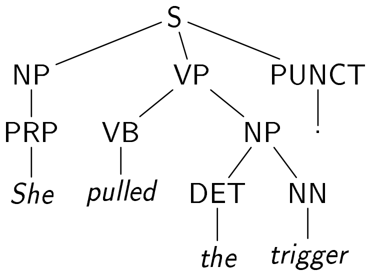
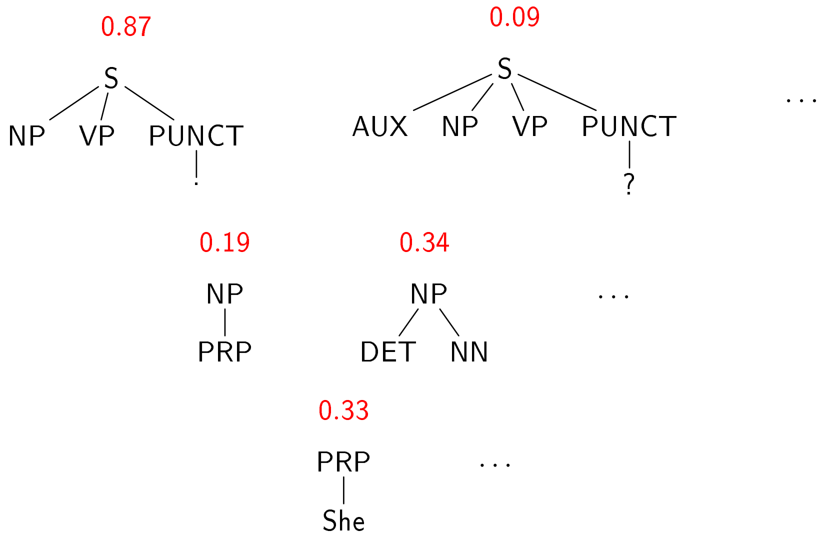
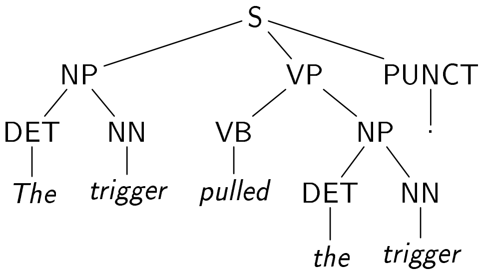
Phrase Structure



Grammar Rules



Plugging Blocks



Probability: $0.87 \cdot 0.34^2 \cdot \ldots$

Semiring

Definition

$(A, +, \cdot, 0, 1)$ semiring, if

- $(A, +, 0)$ commutative monoid
- $(A, \cdot, 1)$ monoid
- $\cdot$ distributes (both sided) over $+$
- 0 is absorbing for $\cdot$ $(a \cdot 0 = 0 = 0 \cdot a)$

Example

- Natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$
- Probabilities $([0, 1], \max, \cdot, 0, 1)$
- Subsets $(\mathfrak{P}(A), \cup, \cap, \emptyset, A)$
- any ring and field

Tree Series

Definition

$(A, +, \cdot, 0, 1)$ semiring, Σ ranked alphabet, X set

- Tree series is mapping $\psi\colon T_\Sigma(X) \to A$
- $A\langle\!\langle T_\Sigma(X)\rangle\!\rangle$ Set of tree series
- $\mathrm{supp}(\psi) = \{t \in T_\Sigma(X) \mid (\psi, t) \neq 0\}$

Conventions

- $\widetilde{0}$ is tree series that maps every tree to 0
- $\psi(t)$ written as (ψ, t)
- ψ written as $\sum_{t \in \mathrm{supp}(\psi)} (\psi, t)\, t$ [Example: $\psi = 5\,\alpha + 10\,\sigma(\alpha, \alpha)$]
- $(\psi + \varphi, t) = (\psi, t) + (\varphi, t)$
- $(a \cdot \psi, t) = a \cdot (\psi, t)$

O-Substitution

Definition (Fülöp, Vogler 03)

$\psi, \psi_1, \ldots, \psi_k \in A\langle\!\langle T_\Sigma(X) \rangle\!\rangle$

$$\psi \overset{\text{o}}{\leftarrow} (\psi_1, \ldots, \psi_k) = \sum_{\substack{t, t_1, \ldots, t_k \in T_\Sigma(X), \\ (\psi_i, t_i) \neq 0}} (\psi, t) \cdot \prod_{i=1}^{k} (\psi_i, t_i)^{|t|_{x_i}} \; t[t_1, \ldots, t_k]$$

Example

Natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ and

$$\psi = 2\,\sigma(x_1, x_1) + 3\,\sigma(\alpha, x_1) + 4\,\sigma(x_1, \alpha) \quad \text{and} \quad \psi' = 3\,\alpha$$

Then

$$\psi \overset{\text{o}}{\leftarrow} (\psi') = (2 \cdot 3^2)\,\sigma(\alpha, \alpha) + (3 \cdot 3)\,\sigma(\alpha, \alpha) + (4 \cdot 3)\,\sigma(\alpha, \alpha)$$

Illustration

Semiring $([0,1], \max, \cdot, 0, 1)$

$$\psi = \max\{0.87\,\mathrm{S}(x_1, x_2, \mathrm{PUNCT}(.))\,,\ 0.09\,\mathrm{S}(x_3, x_1, x_2, \mathrm{PUNCT}(.))\,,\ \dots\}$$

$$\psi_1 = \max\{0.0627\,\mathrm{NP}(\mathrm{PRP}(\mathit{she}))\,,\ 0.012\,\mathrm{NP}(\mathrm{DET}(\mathit{the}), \mathrm{NN}(\mathit{trigger}))\,,\ \dots\}$$

$$\psi_2 = \dots$$

$$\psi_3 = \dots$$

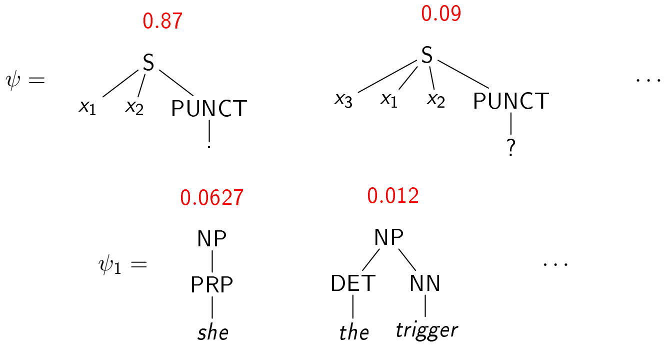
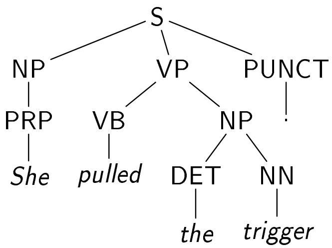


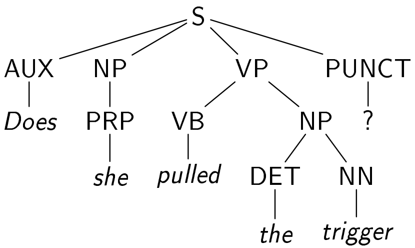
Illustration (cont'd)

$\psi \overset{o}{\leftarrow} (\psi_1, \psi_2, \psi_3) =$

$0.87 \cdot 0.0627 \cdot 0.42^0 \cdot \ldots$



$0.09 \cdot 0.0627 \cdot 0.42 \cdot \ldots$



$\ldots$

Nondeletion and Linearity

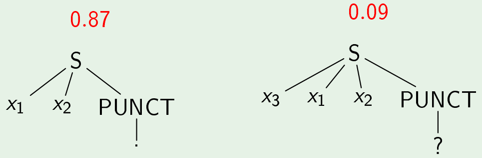
Definition

$\psi \in A\langle\langle T_\Sigma(X)\rangle\rangle$ and $V \subseteq X$

- ψ **nondeleting in V**, if every $v \in V$ occurs in every $t \in \mathrm{supp}(\psi)$
- ψ **linear in V**, if every $v \in V$ occurs at most once in every $t \in \mathrm{supp}(\psi)$

Example

$\psi =$



$\cdots$

Linear in $\{x_1, x_2, x_3\}$ but not nondeleting in $\{x_1, x_2, x_3\}$.

Weighted Tree Automaton

Definition

$(Q, \Sigma, \mathcal{A}, I, \mu)$ weighted tree automaton (wta), if

- Q finite set (of *states*)
- Σ ranked alphabet (of *input symbols*)
- $\mathcal{A} = (A, +, \cdot, 0, 1)$ semiring
- $I \subseteq Q$ (set of *initial states*)
- $\mu = (\mu_k)_{k \in \mathbb{N}}$ with

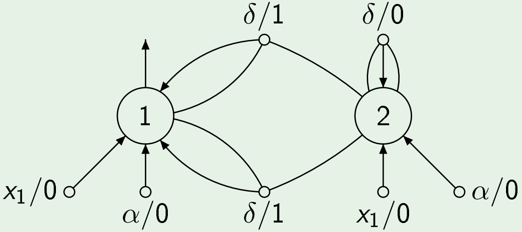
$$\mu_k \colon \Sigma^{(k)} \to A^{Q \times Q^k}$$

Intuition

$$(q, \sigma, a, q_1, \ldots, q_k) \in \delta_k \qquad \Longleftrightarrow \qquad \mu_k(\sigma)_{q,q_1,\ldots,q_k} = a$$

Weighted Tree Automaton

Example



$(1, x_1, 0, \varepsilon)$, $(2, x_1, 0, \varepsilon)$ $\quad \mu_0(x_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(1, \alpha, 0, \varepsilon)$, $(2, \alpha, 0, \varepsilon)$ $\quad \mu_0(\alpha) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(1, \delta, 1, 12)$, $(1, \delta, 1, 21)$, $(2, \delta, 0, 22)$ $\quad \mu_2(\delta) = \begin{pmatrix} -\infty & 1 & 1 & -\infty \\ -\infty & -\infty & -\infty & 0 \end{pmatrix}$

Semantics of WTA

Definition

$M = (Q, \Sigma, \mathcal{A}, I, \mu)$ wta over $\mathcal{A} = (A, +, \cdot, 0, 1)$

- $h_\mu \colon T_\Sigma \to A^Q$

$$h_\mu(\sigma(t_1, \dots, t_k))_q = \sum_{q_1, \dots, q_k \in Q} \mu_k(\sigma)_{q, q_1, \dots, q_k} \cdot h_\mu(t_1)_{q_1} \cdot \ldots \cdot h_\mu(t_k)_{q_k}$$

- $\|M\| \in A\langle\langle T_\Sigma \rangle\rangle$

$$(\|M\|, t) = \sum_{q \in I} h_\mu(t)_q$$

Definition

- $\psi \in A\langle\langle T_\Sigma \rangle\rangle$ recognized by M, if $\|M\| = \psi$
- $A^{\mathrm{rec}}\langle\langle T_\Sigma \rangle\rangle$ set of all recognizable tree series

1. Motivation

2. The Basics

3. Recognizable Tree Series

4. Preservation of Recognizability

5. Tree Series Transducers

6. Conclusion

The Main Problem

Question

Given $\psi, \psi_1, \dots, \psi_k \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$

$$\text{Is} \quad \psi \overset{\mathrm{o}}{\leftarrow} (\psi_1, \dots, \psi_k) \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle \quad \text{or not?}$$

Partial Solution

Given $\psi, \psi_1, \dots, \psi_k \in \mathbb{B}^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$ with ψ linear in X_k
($\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ boolean semiring)

$$\psi \overset{\mathrm{o}}{\leftarrow} (\psi_1, \dots, \psi_k) \in \mathbb{B}^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$$

The Main Problem (cont'd)

Partial Solution [Kuich 99]

Given $\psi, \psi_1, \ldots, \psi_k \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$ with ψ nondeleting and linear in X_k ($\mathcal{A}$ commutative and continuous)

$$\psi \overset{\mathrm{o}}{\leftarrow} (\psi_1, \ldots, \psi_k) \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$$

Contribution

Given $\psi, \psi_1, \ldots, \psi_k \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$ with ψ linear in X_k and $\mathcal{A}$ commutative, idempotent, and continuous

$$\psi \overset{\mathrm{o}}{\leftarrow} (\psi_1, \ldots, \psi_k) \in A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$$

Proof Sketch

Lemma (Kuich 99)

$A^{\mathrm{rec}}\langle\langle T_\Sigma(X)\rangle\rangle$ *is closed under relabeling.*

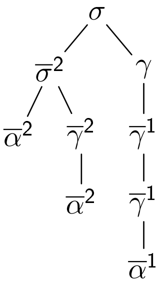
Proof Sketch of Main Theorem

How to construct a wta recognizing $\psi \stackrel{\mathrm{o}}{\leftarrow} (\psi_1, \ldots, \psi_k)$?

1. Make alphabets of $\psi, \psi_1, \ldots, \psi_k$ pairwise disjoint (the substitution is explicit because decomposition is given)
2. Perform standard concatenation
3. Relabel the result

Proof Sketch (cont'd)

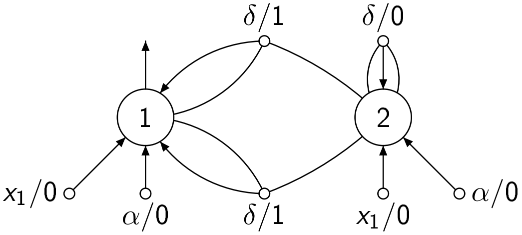
Relabeling example:



ψ: $\sigma(x_2, \gamma(x_1))$
ψ_1: $\gamma(\gamma(\alpha))$
ψ_2: $\sigma(\alpha, \gamma(\alpha))$

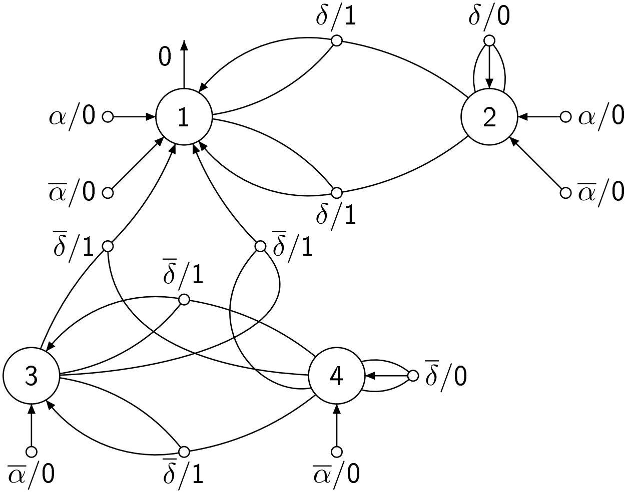
Proof Sketch (cont'd)

Input wta:



Proof Sketch (cont'd)

Concatenation of wta:



1 Motivation

2 The Basics

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6 Conclusion

Tree Series Transducer [Engelfriet et al 02]

Definition

$(Q, \Sigma, \Delta, \mathcal{A}, F, R)$ is bottom-up tree series transducer, if

- Q finite set (of *states*)
- Σ and Δ ranked alphabets (of *input* and *output symbols*)
- $\mathcal{A} = (A, +, \cdot, 0, 1)$ semiring
- $F \subseteq Q$ (set of *final states*)
- R finite set of rules of the form

$$\sigma(q_1(x_1), \ldots, q_k(x_k)) \xrightarrow{a} q(t)$$

where $t \in T_\Delta(X_k)$

Application to tree series transducers

Theorem

$M = (Q, \Sigma, \Delta, \mathcal{A}, F, R)$ linear bottom-up tree series transducer
$\mathcal{A}$ commutative, continuous, and idempotent

For every $t \in T_\Sigma$

$$\|M\|(t) \in A^{\mathrm{rec}}\langle\langle T_\Delta \rangle\rangle$$

Conclusion

Summary

- O-Substitution preserves recognizability in idempotent semirings
- Output series of linear tree series transducers is pointwise recognizable

Open Problems

- What about OI-Substitution?
- Is the image of a recognizable series under linear tree-series-transducer-transformations recognizable?

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