TECHNISCHE
UNIVERSITÄT
DRESDEN

Myhill-Nerode Theorem for Recognizable Tree Series — Revisited

Andreas Maletti

Mondello - Palermo, 22 June 2007

00 Motivation

Goal

- Given $\psi\colon T_\Sigma \to A$ with $(A, +, \cdot, 0, 1)$ commutative semiring
- Is ψ (deterministically) recognizable?

Answers

- If A field, then ψ recognizable iff $\dim V_\psi$ finite [Bozapalidis et.al. '83]

00 Motivation

Goal

- Given $\psi : T_\Sigma \to A$ with $(A, +, \cdot, 0, 1)$ commutative semiring
- Is ψ (deterministically) recognizable?

Answers

- If A field, then ψ recognizable iff $\dim V_\psi$ finite [Bozapalidis et.al. '83]
- If A semifield, then ψ deterministically recognizable iff $\equiv_\psi$ finite index [Borchardt '03]

00 Motivation

Goal

- Given $\psi \colon T_\Sigma \to A$ with $(A, +, \cdot, 0, 1)$ commutative semiring
- Is ψ (deterministically) recognizable?

Answers

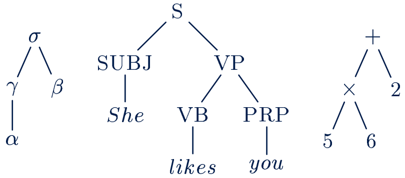
- If A field, then ψ recognizable iff $\dim V_\psi$ finite [Bozapalidis et.al. '83]
- If A semifield, then ψ deterministically recognizable iff $\equiv_\psi$ finite index [Borchardt '03]
- If A semifield, then ψ deterministically all-accepting recognizable iff $\equiv_\psi$ finite index and ψ subtree-closed (basically [Drewes, Vogler '07])

00 Table of Contents

01 Notation

Trees

- T_Σ: trees over ranked alphabet Σ

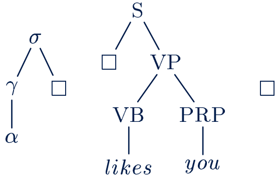


- C_Σ: contexts (trees with exactly one occurrence of $\square$) over Σ
- $\text{size}(t)$: number of nodes of a tree t

01 Notation

Trees

- T_Σ: trees over ranked alphabet Σ
- C_Σ: contexts (trees with exactly one occurrence of $\Box$) over Σ

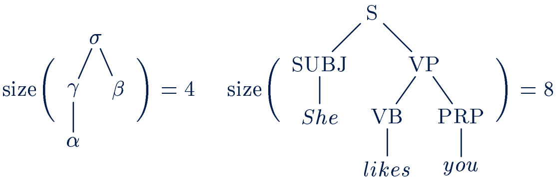


- $\text{size}(t)$: number of nodes of a tree t

01 Notation

Trees

- T_Σ: trees over ranked alphabet Σ
- C_Σ: contexts (trees with exactly one occurrence of $\square$) over Σ
- $\mathrm{size}(t)$: number of nodes of a tree t



01 Notation

Tree series

- tree series: mapping of type $T_\Sigma \to A$
  (e.g., $\mathrm{size}\colon T_\Sigma \to \mathbb{N}$, $\mathrm{height}\colon T_\Sigma \to \mathbb{N}$, $\mathrm{yield}\colon T_\Sigma \to \Sigma^*$, etc.)
- we write (ψ, t) for $\psi(t)$ with $\psi\colon T_\Sigma \to A$
- $A\langle\langle T_\Sigma \rangle\rangle$: set of all mappings of type $T_\Sigma \to A$

02 Table of Contents

02 Syntax

Definition (Borchardt and Vogler '03)

(Q, Σ, A, μ, F) weighted tree automaton (wta)

- Q finite nonempty set (states)
- Σ ranked alphabet (of input symbols)
- $A = (A, +, \cdot, 0, 1)$ commutative semiring (of weights)
- $\mu = (\mu_k)_{k \geq 0}$ with $\mu_k : \Sigma^{(k)} \to A^{Q^k \times Q}$ (called tree representation)
- $F : Q \to A$ (final distribution)

02 Syntax

Definition (Borchardt and Vogler '03)

(Q, Σ, A, μ, F) weighted tree automaton (wta)

- Q finite nonempty set (states)
- Σ ranked alphabet (of input symbols)
- $A = (A, +, \cdot, 0, 1)$ commutative semiring (of weights)
- $\mu = (\mu_k)_{k \geq 0}$ with $\mu_k \colon \Sigma^{(k)} \to A^{Q^k \times Q}$ (called tree representation)
- $F \colon Q \to A$ (final distribution)

Definition

wta (Q, Σ, A, μ, F) deterministic if for every $\sigma \in \Sigma^{(k)}$ and $w \in Q^k$ there exists at most one $q \in Q$ such that $\mu_k(\sigma)_{w,q} \neq 0$.

02 Semantics

Let $M = (Q, \Sigma, A, \mu, F)$ wta.

Definition

Define $h_\mu : T_\Sigma \to A^Q$ by

$$h_\mu(\sigma(t_1, \ldots, t_k))_q = \sum_{q_1 \cdots q_k \in Q^k} \mu_k(\sigma)_{q_1 \cdots q_k, q} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i}$$

02 Semantics

Let $M = (Q, \Sigma, A, \mu, F)$ wta.

Definition

Define $h_\mu : T_\Sigma \to A^Q$ by

$$h_\mu(\sigma(t_1, \ldots, t_k))_q = \sum_{q_1 \cdots q_k \in Q^k} \mu_k(\sigma)_{q_1 \cdots q_k, q} \cdot \prod_{i=1}^{k} h_\mu(t_i)_{q_i}$$

Semantics of M given by

$$(\|M\|, t) = \sum_{q \in Q} F(q) \cdot h_\mu(t)_q$$

03 Table of Contents

03 Recognizability

Let $\psi \in A\langle\langle T_\Sigma \rangle\rangle$.

Definition

ψ recognizable if there exists wta M such that $||M|| = \psi$.

Definition

For every $t \in T_\Sigma$ let $t^{-1}\psi \in A\langle\langle C_\Sigma \rangle\rangle$ with

$$(t^{-1}\psi, c) = (\psi, c[t]) \ .$$

V_ψ sub-(vector)-space of $A\langle\langle C_\Sigma \rangle\rangle$ generated by $t^{-1}\psi$ for all $t \in T_\Sigma$.

03 Recognizability

Example

Consider $\psi = \mathrm{size}$ over the reals $(\mathbb{R}, +, \cdot, 0, 1)$

$$(t_1^{-1}\,\mathrm{size}, c) = (\mathrm{size}, c[t_1]) = \mathrm{size}(c[t_1]) = \mathrm{size}(c) - 1 + \mathrm{size}(t_1)$$
$$(t_2^{-1}\,\mathrm{size}, c) = (\mathrm{size}, c[t_2]) = \mathrm{size}(c[t_2]) = \mathrm{size}(c) - 1 + \mathrm{size}(t_2)$$

03 Recognizability

Example

Consider $\psi = \text{size}$ over the reals $(\mathbb{R}, +, \cdot, 0, 1)$

$$(t_1^{-1}\,\text{size}, c) = (\text{size}, c[t_1]) = \text{size}(c[t_1]) = \text{size}(c) - 1 + \text{size}(t_1)$$
$$(t_2^{-1}\,\text{size}, c) = (\text{size}, c[t_2]) = \text{size}(c[t_2]) = \text{size}(c) - 1 + \text{size}(t_2)$$

Hence $(t_1^{-1}\,\text{size}) = (t_2^{-1}\,\text{size}) + \widetilde{a}$ where $a = \text{size}(t_1) - \text{size}(t_2)$

03 Recognizability

Example

Consider $\psi = \text{size}$ over the reals $(\mathbb{R}, +, \cdot, 0, 1)$

$$(t_1^{-1}\,\text{size}, c) = (\text{size}, c[t_1]) = \text{size}(c[t_1]) = \text{size}(c) - 1 + \text{size}(t_1)$$
$$(t_2^{-1}\,\text{size}, c) = (\text{size}, c[t_2]) = \text{size}(c[t_2]) = \text{size}(c) - 1 + \text{size}(t_2)$$

Hence $(t_1^{-1}\,\text{size}) = (t_2^{-1}\,\text{size}) + \widetilde{a}$ where $a = \text{size}(t_1) - \text{size}(t_2)$

$\Longrightarrow (t_1^{-1}\,\text{size})$ and $\widetilde{1}$ are basis of V_{size} and $\dim V_{\text{size}} = 2$

03 Recognizability

Theorem (Bozapalidis, Louscou-Bozapalidou ’83)

Let A field and $\psi \in A\langle\langle T_\Sigma \rangle\rangle$. Then

$$\psi \textit{ recognizable} \iff \dim V_\psi \textit{ finite} .$$

Notes

- Exact requirements for either direction?
- Might lead to nice necessary and/or sufficient conditions of recognizability
- not considered here, but most likely:
  If ψ recognizable, then V_ψ finite basis.

03 Deterministic recognizability

Let $\psi \in A\langle\langle T_\Sigma \rangle\rangle$.

Definition

ψ deterministically recognizable if there is deterministic wta M such that $||M|| = \psi$.

Definition

Define $\equiv_\psi \subseteq T_\Sigma \times T_\Sigma$ by $t \equiv_\psi u$ iff there is $a \in A \setminus \{0\}$ such that

$$t^{-1}\psi = a \cdot (u^{-1}\psi) \ .$$

03 Deterministic recognizability

Example

Again consider $\psi = \text{size}$ over the reals $(\mathbb{R}, +, \cdot, 0, 1)$

$$(t_1^{-1}\,\text{size}, c) = (\text{size}, c[t_1]) = \text{size}(c[t_1]) = \text{size}(c) - 1 + \text{size}(t_1)$$
$$(t_2^{-1}\,\text{size}, c) = (\text{size}, c[t_2]) = \text{size}(c[t_2]) = \text{size}(c) - 1 + \text{size}(t_2)$$

03 Deterministic recognizability

Example

Again consider $\psi = \mathrm{size}$ over the reals $(\mathbb{R}, +, \cdot, 0, 1)$

$$(t_1^{-1}\,\mathrm{size}, c) = (\mathrm{size}, c[t_1]) = \mathrm{size}(c[t_1]) = \mathrm{size}(c) - 1 + \mathrm{size}(t_1)$$
$$(t_2^{-1}\,\mathrm{size}, c) = (\mathrm{size}, c[t_2]) = \mathrm{size}(c[t_2]) = \mathrm{size}(c) - 1 + \mathrm{size}(t_2)$$

Hence $t_1 \equiv_{\mathrm{size}} t_2$ iff $\mathrm{size}(t_1) = \mathrm{size}(t_2)$

03 Deterministic recognizability

Example

Again consider $\psi = \text{size}$ over the reals $(\mathbb{R}, +, \cdot, 0, 1)$

$$(t_1^{-1}\,\text{size}, c) = (\text{size}, c[t_1]) = \text{size}(c[t_1]) = \text{size}(c) - 1 + \text{size}(t_1)$$
$$(t_2^{-1}\,\text{size}, c) = (\text{size}, c[t_2]) = \text{size}(c[t_2]) = \text{size}(c) - 1 + \text{size}(t_2)$$

Hence $t_1 \equiv_{\text{size}} t_2$ iff $\text{size}(t_1) = \text{size}(t_2)$

$\Longrightarrow$ index of $\equiv_{\text{size}}$ infinite

03 Deterministic recognizability

Theorem (Borchardt '03)

Let A semifield and $\psi \in A\langle\langle T_\Sigma \rangle\rangle$. Then

$$\psi \text{ deterministically recognizable} \quad \Longleftrightarrow \quad \equiv_\psi \text{ finite index} \; .$$

Notes

- $\equiv_\psi$ finite index iff
  scalar-multiplication closed subset generated by $t^{-1}\psi$ has finite generator
- The latter yields finite basis for V_ψ
- So conceptionally both share the same idea

TECHNISCHE UNIVERSITÄT DRESDEN

03 Deterministic recognizability

Example

Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. Define $\mathrm{zigzag}\colon T_\Sigma \to \mathbb{N}$ over $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$

$$\begin{aligned}\mathrm{zigzag}(\alpha) &= 1\\ \mathrm{zigzag}(\sigma(\alpha, t)) &= 2\\ \mathrm{zigzag}(\sigma(\sigma(t, u), v)) &= 2 + \mathrm{zigzag}(u)\end{aligned}$$

03 Deterministic recognizability

Example

Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. Define $\mathrm{zigzag}\colon T_\Sigma \to \mathbb{N}$ over $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$

$$\begin{aligned} \mathrm{zigzag}(\alpha) &= 1 \\ \mathrm{zigzag}(\sigma(\alpha, t)) &= 2 \\ \mathrm{zigzag}(\sigma(\sigma(t, u), v)) &= 2 + \mathrm{zigzag}(u) \end{aligned}$$

By theorem, zigzag not det. recognizable over $(\mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0)$

03 Deterministic recognizability

Example

Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. Define $\mathrm{zigzag}\colon T_\Sigma \to \mathbb{N}$ over $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$

$$\begin{aligned} \mathrm{zigzag}(\alpha) &= 1 \\ \mathrm{zigzag}(\sigma(\alpha, t)) &= 2 \\ \mathrm{zigzag}(\sigma(\sigma(t, u), v)) &= 2 + \mathrm{zigzag}(u) \end{aligned}$$

By theorem, zigzag not det. recognizable over $(\mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0)$

$\Longrightarrow$ zigzag not det. recognizable over $(\mathbb{N}, \dots)$ because it embeds into $(\mathbb{Z}, \dots)$

03 Deterministic recognizability

Let $\psi \in A\langle\langle T_\Sigma \rangle\rangle$.

Definition (Borchardt '05)

Define $\equiv_\psi \subseteq T_\Sigma \times T_\Sigma$ by $t \equiv_\psi u$ if there exist $a, b \in A \setminus \{0\}$ such that

$$a \cdot (t^{-1}\psi) = b \cdot (u^{-1}\psi) \enspace .$$

03 Deterministic recognizability

Let $\psi \in A\langle\langle T_\Sigma \rangle\rangle$.

Definition (Borchardt '05)

Define $\equiv_\psi \subseteq T_\Sigma \times T_\Sigma$ by $t \equiv_\psi u$ if there exist $a, b \in A \setminus \{0\}$ such that

$$a \cdot (t^{-1}\psi) = b \cdot (u^{-1}\psi) \ .$$

Definition

A zero-divisor free if $a \cdot b = 0$ implies $0 \in \{a, b\}$.

03 Deterministic recognizability

Let $\psi \in A\langle\langle T_\Sigma \rangle\rangle$.

Definition (Borchardt '05)

Define $\equiv_\psi \subseteq T_\Sigma \times T_\Sigma$ by $t \equiv_\psi u$ if there exist $a, b \in A \setminus \{0\}$ such that

$$a \cdot (t^{-1}\psi) = b \cdot (u^{-1}\psi) \enspace .$$

Definition

A zero-divisor free if $a \cdot b = 0$ implies $0 \in \{a, b\}$.

Lemma

If A zero-divisor free, then $\equiv_\psi$ congruence of (T_Σ, Σ).

03 Deterministic recognizability

Theorem

Let A zero-divisor free and $\psi \in A\langle\langle T_\Sigma \rangle\rangle$. Then

$$\psi \text{ deterministically recognizable} \quad \Longrightarrow \quad \equiv_\psi \text{ finite index} \ .$$

03 Deterministic recognizability

Theorem

Let A zero-divisor free and $\psi \in A\langle\langle T_\Sigma \rangle\rangle$. Then

$$\psi \text{ deterministically recognizable} \quad \Longrightarrow \quad \equiv_\psi \text{ finite index} \;.$$

Lemma

If A zero-divisor free and $\psi \in A\langle\langle T_\Sigma \rangle\rangle$, then every deterministic (and complete) wta recognizing ψ has at least $\mathrm{index}(\equiv_\psi)$ states.

03 Deterministic recognizability

Theorem

Let A zero-divisor free and $\psi \in A\langle\langle T_\Sigma \rangle\rangle$. Then

$$\psi \text{ deterministically recognizable} \quad \Longrightarrow \quad \equiv_\psi \text{ finite index} .$$

Lemma

If A zero-divisor free and $\psi \in A\langle\langle T_\Sigma \rangle\rangle$, then every deterministic (and complete) wta recognizing ψ has at least $\mathrm{index}(\equiv_\psi)$ states.

Corollary

height *not deterministically recognizable over* $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$

03 Deterministic recognizability

Question

What about

$$\psi \text{ deterministically recognizable} \quad \Longleftarrow \quad \equiv_\psi \text{ finite index } ?$$

03 Deterministic recognizability

Question

What about

$$\psi \text{ deterministically recognizable} \quad \Longleftarrow \quad \equiv_\psi \text{ finite index } ?$$

Notes

- Solved for strings [Eisner '03]: requires certain cancellative semirings
- Open for trees

03 All-accepting recognizability

Let $\psi \in A\langle\langle T_\Sigma \rangle\rangle$.

Definition

Deterministic wta $(Q, \Sigma, \mathcal{A}, \mu, F)$ all-accepting if $F = Q$.

03 All-accepting recognizability

Let $\psi \in A\langle\langle T_\Sigma \rangle\rangle$.

Definition

Deterministic wta $(Q, \Sigma, \mathcal{A}, \mu, F)$ all-accepting if $F = Q$.

Definition

ψ subtree-closed if $(\psi, t) \neq 0$ implies $(\psi, u) \neq 0$ for all subtrees u of t.

03 All-accepting recognizability

Let $\psi \in A\langle\langle T_\Sigma \rangle\rangle$.

Definition

Deterministic wta $(Q, \Sigma, \mathcal{A}, \mu, F)$ all-accepting if $F = Q$.

Definition

ψ subtree-closed if $(\psi, t) \neq 0$ implies $(\psi, u) \neq 0$ for all subtrees u of t.

Lemma

ψ deterministically all-accepting recognizable iff
ψ deterministically recognizable and subtree-closed.

03 All-accepting recognizability

Let $\psi \in A\langle\langle T_\Sigma \rangle\rangle$.

Theorem

If A cancellative, then

ψ *all-accepting det. recognizable* $\iff$ $\equiv_\psi$ *finite index and* ψ *subtree-closed*

03 All-accepting recognizability

Let $\psi \in A\langle\langle T_\Sigma \rangle\rangle$.

Theorem

If A cancellative, then

ψ *all-accepting det. recognizable* $\iff$ $\equiv_\psi$ *finite index and* ψ *subtree-closed*

Notes

- Other interesting classes?
- Suitable implementability conditions

03 Thank you for your attention!

References

- Bozapalidis, Louscou-Bozapalidou: The rank of a formal tree power series. Theoret. Comput. Sci. 27: 211–215, 1983
- Borchardt, Vogler: Determinization of finite state weighted tree automata. J. Autom. Lang. Combin. 8(3):417–463, 2003
- Borchardt: The Myhill-Nerode theorem for recognizable tree series. Proc. 7th Int. Conf. Developments in Language Theory, LNCS 2710, p. 146–158, Springer 2003
- Drewes, Vogler: Learning deterministically recognizable tree series. J. Autom. Lang. Combin., to appear, 2007