Cut Sets as Recognizable Tree Languages

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Abstract

A tree series over a semiring with partially ordered carrier set can be considered as a fuzzy set. We investigate conditions under which it can also be understood as a fuzzified recognizable tree language. In this sense, sufficient conditions are presented which, when imposed, ensure that every cut set, i.e., the pre-image of a prime filter of the carrier set, is a recognizable tree language. Moreover, such conditions are also presented for cut sets of recognizable tree series.

1 Introduction

There are two sources for the investigations in this paper, namely (i) fuzzy sets and (ii) tree series and recognizable tree series, in particular. Both sources

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are derivatives of the concept of characteristic functions, where as usual, given a set S every characteristic function $\chi : S \to \{0,1\}$ on S identifies the subset $\{ s \in S \mid \chi(s) = 1 \}$ of S.

The first source is the concept of fuzzy sets. In most common settings (introduced at the beginning of fuzzy set theory by ZADEH), a fuzzy set is a mapping from a set into the unit interval $[0,1]$, ordered as usual by $\leq$ for real numbers. Under this ordering the unit interval is a partially ordered set (for short: poset), moreover it is a lattice. This fact is used in other, more general definitions of fuzzy sets in which the co-domain is a lattice (GOGUEN [1]) or simply a poset (see [2], [3] and references given there). In our approach a fuzzy set is a mapping $\varphi : S \to A$ with $(A, \leq)$ being a poset, usually with the top and the bottom element. Fuzzy sets generalize the notion of characteristic functions by shifting the correspondence "characteristic function—subset" to the correspondence "fuzzy set—family of cut sets", where for every $a \in A$, the a-cut of the fuzzy set φ is the set $\varphi_a = \{ s \in S \mid \varphi(s) \geq a \}$. These cut sets or a-cuts are among the basic tools for investigating fuzzy structures. Indeed, many important properties of fuzzy structures are cut-worthy, thereby representing a bridge between fuzzy world and crisp structures [4,2]. In addition, if a fuzzy set is defined over some algebraic structure, then the notion of the corresponding fuzzy algebra is obtained. Cut sets in this case are crisp (ordinary) subalgebras of the starting structure.

The second source is the concept of (recognizable) tree series. Here, the connection to the concept of characteristic functions can be explained by performing one restriction and one generalization as follows. The set S is restricted to the set T_Σ of terms over a finite, nonempty operator domain Σ; such terms and operator domains are called *trees* and *ranked alphabets*, respectively (cf. page 87 of [5]). The generalization amounts to a consideration of an arbitrary carrier set A of a semiring instead of the set $\{0,1\}$ of truth values. That is, a tree series is a mapping $\varphi : T_\Sigma \to A$ where $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring. The tree series φ is recognizable, if there exists a bottom-up finite state weighted tree automaton M which accepts φ (cf., e.g., Definition 3.3 of [6]). By now, the concept of recognizable tree series has been intensively studied [7–13]. If the chosen semiring $\mathcal{A}$ is the Boolean semiring $\mathrm{Bool} = (\{0,1\}, \vee, \wedge, 0, 1)$, then a recognizable tree series is the characteristic function of a recognizable tree language (cf. [5,14] for a survey on the theory of recognizable tree languages).

In summary, fuzzy sets (structures) and (recognizable) tree series generalize characteristic functions with domain T_Σ by replacing the set of truth values by a poset $(A, \leq)$ and by a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, respectively. We note that the connection between fuzzy sets and (finite-state) weighted string automata has already been addressed in [15,16] (also cf. [17–19]).

In this paper we combine the two concepts of fuzziness and tree series by considering fuzzy sets, whose domain is T_Σ and co-domain is a poset or a partially ordered semiring.

The motivation for the present investigation originates in the following simple facts. Tree series over an ordered set can be considered as fuzzy sets. Within this new framework, these classical objects from automata theory become poset-valued (fuzzy) structures. Consequently, these can be treated and investigated by appropriate (fuzzy) techniques. As it is usual in such investigations, the outcomes could point to two directions: there might be some new results in the classical theory, and on the other hand the fuzzy aspect may provide some unknown insight into the topic. In our case, the cut sets approach turned out to be a successful tool. We were able to show that under particular conditions the foregoing structures can be viewed as fuzzified recognizable tree languages. In addition to this new fuzzy aspect of tree series, some new properties of these objects were deduced.

In our investigation, we focus attention to cut sets of the considered tree series, and in particular, we investigate the following two questions.

Let $(A, \leq)$ be a poset, $a \in A$, and $\varphi : T_\Sigma \to A$ be a fuzzy set.

- Under which conditions is the cut φ_a a recognizable tree language (i.e., φ_a is accepted by some bottom-up finite-state tree automaton)?
- How do such conditions look like if we additionally require that φ is a recognizable tree series over some semiring $\mathcal{A}$?

We will partially answer these questions by proving sufficient conditions in Section 3.1. More precisely, we prove that φ_a is a recognizable tree language if one of the following conditions holds:

- φ is order-preserving (where T_Σ is partially ordered by the subtree relation) and compatible with top-concatenation, and $A \setminus \uparrow a$ is finite, where $\uparrow a$ is the prime filter of a (cf. Theorem 6),
- φ is order-preserving and compatible with top-concatenation, and $\varphi(T_\Sigma)$ is finite (cf. Theorem 7).

In Section 3.2 we start with the additional requirement that φ is recognizable. First we show an example of a recognizable tree series, of which the cut sets are not recognizable as tree languages (cf. Example 8). Then we prove that φ_a is a recognizable tree language if one of the following conditions holds:

- $\mathcal{A}$ is a locally finite semiring (cf. Theorem 9),
- $\mathcal{A}$ is non-decreasing with respect to $\leq$ and $A \setminus \uparrow a$ is finite (cf. Theorem 10).

The paper is organized as follows. In Section 2 we fix notions and notations, which we use in Section 3. There we provide sufficient conditions which, if imposed, guarantee that every cut set of a fuzzy set is a recognizable tree language. We conclude this paper in Section 4 by stating open problems and discussing the converse problem, which, in fuzzy terms, is known as the synthesis problem. Namely, starting with a collection of recognizable tree languages, we ask for the existence of a formal tree series whose cuts are precisely the members of the collection. Within this, further open problems are unearthed.

2 Preliminaries

2.1 Partial orders and fuzzy sets

In this section we briefly review well-known facts on posets and fuzzy sets. For more details we refer the reader to [20,2].

Given a nonempty set A, a binary relation $\leq \subseteq A \times A$ is called *partial order (on A)*, if $\leq$ is reflexive, antisymmetric, and transitive. As usual, the fact $(a, b) \in \leq$ is denoted infix, i.e, by $a \leq b$, and the relation $< \subseteq A \times A$ is defined for every $a, b \in A$ by $a < b$ if and only if $a \leq b$ and $a \neq b$. Moreover, the pair $(A, \leq)$ is called *partially ordered set* (for short: *poset*). For the rest of this section let $(A, \leq)$ be a poset. A poset $(B, \leq_B)$ is a *sub-poset of* $(A, \leq)$, if $B \subseteq A$ and $\leq_B$ is the restriction of $\leq$ to B, i.e., $\leq_B = \leq \cap (B \times B)$. The poset $(A, \leq)$ is termed finite if A is finite. Now let $S \subseteq A$. An element $m \in S$ is called *maximal element of* S, if for every $s \in S$ the fact $m \leq s$ implies $m = s$. Moreover, an element $u \in A$ is termed *upper bound of* S, if $s \leq u$ for every $s \in S$. The set of all upper bounds of S is denoted by $\uparrow S$; if $S = \{s\}$ then we write $\uparrow s$. The *smallest element of* $\uparrow S$, i.e., the element $u \in \uparrow S$ which satisfies $u \leq v$ for every $v \in \uparrow S$, is called *supremum of* S and denoted by $\sup S$, if it exists. Analogously, a *minimal element of* S, a *lower bound of* S, the *set* $\downarrow S$ *of lower bounds of* S, the *largest element of* $\downarrow S$, and the *infimum of* S, denoted by $\inf S$, are defined.

Let us now turn to particular classes of posets and thereby approach a finiteness-condition, which we assume in several of our recognizability results of Section 3. We call a poset $(A, \leq)$ *chain* (also: *linearly* or *totally ordered*) if all elements are comparable via $\leq$, i.e., $a \leq b$ or $b \leq a$ for every $a, b \in A$. Moreover, the chain $(C, \leq_C)$ is a *sub-chain* of $(A, \leq)$, if $(C, \leq_C)$ is a sub-poset of $(A, \leq)$. An *anti-chain* is a poset in which there are no comparable distinct elements, i.e., a poset (A, id_A) where $\mathrm{id}_A = \{\, (a, a) \mid a \in A \,\}$. The *width* of the poset $(A, \leq)$ is the cardinality of a maximal anti-chain in $(A, \leq)$, when such an anti-chain exists, and is ∞ otherwise. The following is known (see any extensive book on ordered sets and lattices, e.g. [21]).

Lemma 1 *A poset is finite if and only if it has finite width and no infinite chains.*

The poset $(A, \leq)$ satisfies the *descending chain condition* (for short: *DCC*) if each descending chain $c_1 > c_2 > \dots$ in $(A, \leq)$ is finite. Moreover, we call $(A, \leq)$ *up-chain connected* if it satisfies the following condition: for every $a \in A$ and every infinite sub-chain $(C, \leq_C)$ of $(A, \leq)$ the set $(\uparrow a) \cap C$ is infinite. It is easy to see that $(A, \leq)$ satisfies the DCC, whenever it is up-chain connected. Moreover, if $(A, \leq)$ is a chain, then the descending chain condition coincides with up-chain connectedness. Finally, $(A, \leq)$ is called *F-poset* if it is up-chain connected and has finite width.

Lemma 2 *The following are equivalent for every poset $(A, \leq)$.*

(i) For every $a \in A$ the set $A \setminus \uparrow a$ is finite.
(ii) $(A, \leq)$ is an F-poset.

PROOF. *(i) $\Rightarrow$ (ii): Suppose $(A, \leq)$ satisfies that for every $a \in A$ the set $A \setminus \uparrow a$ is finite. If it has an infinite anti-chain (D, id_D), then for every $a \in D$ the set $A \setminus \uparrow a$ is infinite, which is a contradiction. Further, if $(A, \leq)$ is not up-chain connected, then there is an element $a \in A$ and an infinite sub-chain $(C, \leq_C)$ of $(A, \leq)$ such that $(\uparrow a) \cap C$ is finite. Since C is infinite, it follows that $C \setminus \uparrow a$ is infinite and hence $A \setminus \uparrow a$ is also infinite, which contradicts the assumption.*

(ii) $\Rightarrow$ (i): If $(A, \leq)$ does not satisfy Property (i), then there are infinitely many elements in $A \setminus \uparrow a$ for some $a \in A$. Let $\leq'$ be the restriction of $\leq$ to $A \setminus \uparrow a$. Then, by Lemma 1, either $(A \setminus \uparrow a, \leq')$ has infinite width or there is an infinite chain in this sub-poset. In both cases $(A, \leq)$ is not an F-poset. $\square$

Let us recall fuzzy sets, cuts and fuzzy (algebraic) structures. A *fuzzy set* (also: *A-fuzzy set*, *poset-valued fuzzy set*) is a mapping $\varphi : S \to A$ such that S is a nonempty set and A is the carrier set of some poset $(A, \leq)$. A *cut set* (also: *cut*) of S is a set $\varphi_a = \{ s \in S \mid \varphi(s) \geq a \}$ for some $a \in A$. If S is the universe (underlying set) of some algebraic structure $\mathcal{S}$ (e.g., a group or a ring), then a fuzzy set $\varphi : S \to A$ is a *fuzzy subalgebra* of $\mathcal{S}$ if each cut set is a crisp subalgebra of $\mathcal{S}$. In particular, if A is a lattice, then $\varphi : S \to A$ is a fuzzy subalgebra of $\mathcal{S}$ if and only if $\varphi(f(x_1, \dots, x_n)) \geq \inf\{\varphi(x_1), \dots, \varphi(x_n)\}$ for every fundamental operation f on $\mathcal{S}$.

We conclude this section by considering mappings between posets. For this purpose let $(A, \leq_A)$ and $(B, \leq_B)$ be two posets. A mapping $f : A \to B$ is *order-preserving* (also: *isotone*) if $f(a_1) \leq_B f(a_2)$ for every $a_1, a_2 \in A$ with $a_1 \leq_A a_2$.

2.2 Semirings

We now present notions and notations concerning semirings, which we frequently use throughout this paper. For a more detailed presentation the reader may consult [22,23]. A *semiring* is an algebra $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ such that $(A, \oplus, \mathbf{0})$ is a commutative monoid, $(A, \odot, \mathbf{1})$ is a monoid, the *distributivity laws* $(a_1 \oplus a_2) \odot b = (a_1 \odot b) \oplus (a_2 \odot b)$ and $b \odot (a_1 \oplus a_2) = (b \odot a_1) \oplus (b \odot a_2)$ hold, and $\mathbf{0}$ is an *absorptive element*. By convention, we assume that multiplication $\odot$ has a higher (binding) priority than addition $\oplus$, e.g., we read $a_1 \oplus a_2 \odot a_3$ as $a_1 \oplus (a_2 \odot a_3)$. An example of a semiring is the Boolean semiring $\mathrm{Bool} = (\mathbb{B}, \vee, \wedge, 0, 1)$ where $\mathbb{B} = \{0, 1\}$ is the set of truth values and $\vee$ and $\wedge$ denote the disjunction and conjunction, respectively. Throughout this section let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring. We call $\mathcal{A}$ *locally finite*, if for every finite subset $B \subseteq A$ the smallest sub-semiring containing B, whose carrier set is denoted by $\langle B \rangle$, is still finite.

The semiring $\mathcal{A}$ is *naturally ordered*, if for every $a, b, c \in A$ the condition $a \oplus b \oplus c = a$ implies $a \oplus b = a$. Suppose that $\mathcal{A}$ is naturally ordered, then the relation $\sqsubseteq \subseteq A \times A$, which is defined for every $a, b \in A$ by

$$a \sqsubseteq b \quad \iff \quad (\exists c \in A) : \ a \oplus c = b \ ,$$

is a partial order [22]. Note that each additively idempotent semiring is naturally ordered. Now let $\leq$ be an arbitrary partial order on A. We say that $\otimes \in \{\oplus, \odot\}$ *is non-decreasing (with respect to* $\leq$*)*, if for every $a, b \in A$ it holds

(ND$\otimes$) $a \leq a \otimes b$ whenever $b \neq \mathbf{0}$.

The semiring $\mathcal{A}$ is called *non-decreasing (with respect to* $\leq$*)*, if both $\oplus$ and $\odot$ are non-decreasing with respect to $\leq$. Note that in every naturally ordered semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ the operation $\oplus$ is non-decreasing with respect to $\sqsubseteq$.

Let us conclude this section by considering mappings between semirings. For this purpose let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}_\mathcal{A}, \mathbf{1}_\mathcal{A})$ and $\mathcal{B} = (B, +, \cdot, \mathbf{0}_\mathcal{B}, \mathbf{1}_\mathcal{B})$ be two semirings and $f : A \to B$ be a mapping. We call f *semiring-homomorphism (from* $\mathcal{A}$ *to* $\mathcal{B}$*)*, if f is compatible with the semiring operations, i.e., for every $a, b \in A$ it holds that $f(a \oplus b) = f(a) + f(b)$, $f(a \odot b) = f(a) \cdot f(b)$, $f(\mathbf{0}_\mathcal{A}) = \mathbf{0}_\mathcal{B}$, and $f(\mathbf{1}_\mathcal{A}) = \mathbf{1}_\mathcal{B}$.

Now we recall notions and notations concerning tree languages and tree series as well as their relation to each other (cf. [5,14] for tree languages and [7] for tree series). Some of the basic concepts which we will use in this paper (like ranked alphabet and tree) are known from universal algebra under different names (finite, nonempty operator domain and term, respectively). However, we will use the notions which are established in the theory of formal languages and automata theory.

A *ranked alphabet* is a pair (Σ, rk), where Σ is a finite, nonempty set and $\mathrm{rk} : \Sigma \to \mathbb{N}$ associates to every symbol of Σ its *rank* (from the viewpoint of universal algebra, a ranked alphabet is a finite, nonempty operator domain, cf. e.g. page 48 of [24]). We note that $0 \in \mathbb{N}$. For every $k \in \mathbb{N}$ we define $\Sigma^{(k)} = \{\, \sigma \in \Sigma \mid \mathrm{rk}(\sigma) = k \,\}$. In the following, we will usually assume that the mapping rk is implicitly given and we denote a ranked alphabet by Σ only. Throughout the paper, we assume that $\Sigma^{(0)} \neq \emptyset$.

If Σ is a ranked alphabet, then an *algebra* $\mathcal{A}$ *of type* Σ (or equivalently: a Σ-*algebra*) is an ordered pair (A, F), where A is a nonempty set and F is a family of operations on A indexed by the ranked alphabet Σ, such that there is an n-ary operation f^A on A corresponding to each symbol from Σ of rank n.

Given a ranked alphabet Σ, we define the *set of trees over* Σ, denoted by T_Σ, inductively as follows: (i) if $\alpha \in \Sigma^{(0)}$, then $\alpha \in T_\Sigma$ and (ii) if $k \geq 1$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$, then $\sigma(t_1, \ldots, t_k) \in T_\Sigma$. (Thus, from the viewpoint of universal algebra, trees are terms over a finite, nonempty operator domain.) A *tree language* is a subset of T_Σ. For every $s, t \in T_\Sigma$ we denote the fact that s is a subtree of t by $s \sqsubseteq t$. Observe that $(T_\Sigma, \sqsubseteq)$ is a poset. Now let $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$. The *top-concatenation* with σ is the operation $\mathrm{top}_\sigma : T_\Sigma \times \cdots \times T_\Sigma \to T_\Sigma$ with k arguments defined for every $t_1, \ldots, t_k \in T_\Sigma$ by $\mathrm{top}_\sigma(t_1, \ldots, t_k) = \sigma(t_1, \ldots, t_k)$. The *term algebra* over Σ is the Σ-algebra $\mathcal{T}_\Sigma = (T_\Sigma, \{\, \mathrm{top}_\sigma \mid \sigma \in \Sigma \,\})$. We note that $\mathcal{T}_\Sigma$ is the initial object in the class of all Σ-algebras, i.e., for every other Σ-algebra $\mathcal{B} = (B, \{\, f_\sigma \mid \sigma \in \Sigma \,\})$ there is a uniquely determined homomorphism $h : T_\Sigma \to B$ from $\mathcal{T}_\Sigma$ to $\mathcal{B}$.

Let us now briefly review the notion of tree series. To this end, let $\mathcal{A}$ be a semiring. Every mapping $\varphi : T_\Sigma \to A$ is called *(formal) tree series (over* Σ *and* $\mathcal{A}$*)*. We use $A\langle\langle T_\Sigma \rangle\rangle$ to denote the set of all formal tree series over Σ and $\mathcal{A}$. Given a tree $t \in T_\Sigma$, we usually write (φ, t), termed the *coefficient* of t, instead of $\varphi(t)$ and $\bigoplus_{t \in T_\Sigma} (\varphi, t)\, t$ instead of the tree series φ. Moreover, the *support* of a tree series $\varphi \in A\langle\langle T_\Sigma \rangle\rangle$ is the set $\mathrm{supp}(\varphi) = \{\, t \in T_\Sigma \mid (\varphi, t) \neq \mathbf{0} \,\}$. We extend supp to sets of tree series in the usual manner. Observe that $\mathrm{supp}(\varphi)$ is a tree language and $\mathrm{supp}(\mathbb{B}\langle\langle T_\Sigma \rangle\rangle)$ is the set of all tree languages. Conversely, given a tree language $L \subseteq T_\Sigma$ and a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, the *characteristic mapping of* L *and* $\mathcal{A}$ is the formal tree series $\chi_L^{\mathcal{A}} = \bigoplus_{t \in L} \mathbf{1}\, t$.

Observe that, if $\leq$ is a partial order on A, then every tree series $\varphi \in A\langle\langle T_\Sigma \rangle\rangle$ is also a fuzzy set. We call a tree series φ *compatible with top-concatenation*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and k pairs $(s_i, t_i) \in T_\Sigma \times T_\Sigma$ satisfying $\varphi(s_i) = \varphi(t_i)$ for every $i \in [k]$ it holds that $\varphi(\sigma(s_1, \ldots, s_k)) = \varphi(\sigma(t_1, \ldots, t_k))$.

2.4 Recognizable tree languages and recognizable tree series

Let us now present the automata-theoretic concept which we investigate in this paper. More precisely, in this section we recall notions concerning bottom-up finite-state weighted tree automata, recognizable tree series, and recognizable tree languages. For more details on recognizable tree series, we refer the reader to [7,25,26,9,6,13] and the reader may consult [5,14] for more details on recognizable tree languages.

Let Q be a finite set (of states), Σ a ranked alphabet (of input symbols), and A a set (of weights). A *(bottom-up) tree representation (over Q, Σ, and A)* is a family $\mu = (\mu_k)_{k \in \mathbb{N}}$ of mappings $\mu_k : \Sigma^{(k)} \to A^{Q^k \times Q}$. Let $F \subseteq Q$, $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring, and μ be a tree representation over Q, Σ, and A. The tuple $M = (Q, \Sigma, F, \mathcal{A}, \mu)$ is called *bottom-up finite state weighted tree automaton (over $\mathcal{A}$)* (for short: *bu-w-fta*), where F is the set of *final states.* Let us now define the semantics of a bu-w-fta $M = (Q, \Sigma, F, \mathcal{A}, \mu)$. For this purpose let us consider the Σ-algebra

$$\mathcal{D}_M = (A^Q, \{\, \overline{\mu(\sigma)} : A^Q \times \cdots \times A^Q \to A^Q \mid \sigma \in \Sigma \,\}) \ ,$$

where for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $q \in Q$, and $v_1, \ldots, v_k \in A^Q$, the operation $\overline{\mu(\sigma)}$ is defined by

$$\overline{\mu(\sigma)}(v_1, \ldots, v_k)_q = \bigoplus_{q_1, \ldots, q_k \in Q} (v_1)_{q_1} \odot \cdots \odot (v_k)_{q_k} \odot \mu_k(\sigma)_{(q_1, \ldots, q_k), q} \ .$$

Since $\mathcal{D}_M$ is a Σ-algebra, there is a uniquely determined homomorphism h from $\mathcal{T}_\Sigma$ to $\mathcal{D}_M$; in the sequel, h will be denoted just by $h_\mu : T_\Sigma \to A^Q$. The tree series φ_M that is *accepted* or *recognized* by M is defined pointwise for every $t \in T_\Sigma$ by $(\varphi_M, t) = \bigoplus_{q \in F} h_\mu(t)_q$. Moreover, we denote by $A^{\mathrm{rec}}\langle\langle T_\Sigma \rangle\rangle$ the class of all tree series which are accepted by bu-w-fta. In particular, a tree language $L \subseteq T_\Sigma$ is *recognizable* (in the sense of [5]) if and only if $L \in \mathrm{supp}(\mathbb{B}^{\mathrm{rec}}\langle\langle T_\Sigma \rangle\rangle)$. The *class of all recognizable tree languages* is denoted by RECOG.

Let us now show that the recognizability of a tree series is preserved by applying a semiring-homomorphism to it.

Lemma 3 *Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}_{\mathcal{A}}, \mathbf{1}_{\mathcal{A}})$ and $\mathcal{B} = (B, +, \cdot, \mathbf{0}_{\mathcal{B}}, \mathbf{1}_{\mathcal{B}})$ be two semirings and $f : A \to B$ be a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. If $\varphi \in A^{\mathrm{rec}}\langle\langle T_\Sigma \rangle\rangle$, then $f(\varphi) \in B^{\mathrm{rec}}\langle\langle T_\Sigma \rangle\rangle$, where f is extended to a mapping $f : A\langle\langle T_\Sigma \rangle\rangle \to B\langle\langle T_\Sigma \rangle\rangle$ as usual by setting $(f(\varphi), t) = f(\varphi, t)$ for every $\varphi \in A\langle\langle T_\Sigma \rangle\rangle$ and $t \in T_\Sigma$.*

PROOF. *Let $M = (Q, \Sigma, F, \mathcal{A}, \mu)$ be a bu-w-fta which accepts φ. We construct a bu-w-fta $M' = (Q, \Sigma, F, \mathcal{B}, \mu')$ with $\mu'_k(\sigma)_{(q_1,\ldots,q_k),q} = f(\mu_k(\sigma)_{(q_1,\ldots,q_k),q})$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q, q_1, \ldots, q_k \in Q$. We show that M' accepts $f(\varphi)$. For this purpose, one can prove that, for every $t \in T_\Sigma$ and $q \in Q$, the equation $h_{\mu'}(t)_q = f(h_\mu(t)_q)$ holds. The proof can be done by induction on t and it is left to the reader. Using the aforementioned equation let us now show that M' accepts $f(\varphi)$. We have for every $t \in T_\Sigma$ that*

$$(\varphi_{M'}, t) = \sum_{q \in F} h_{\mu'}(t)_q = \sum_{q \in F} f(h_\mu(t)_q) = f\Big(\bigoplus_{q \in F} h_\mu(t)_q\Big) = f(\varphi, t) \ .$$

Hence M' accepts $f(\varphi)$, i.e., $f(\varphi) \in B^{rec}\langle\langle T_\Sigma \rangle\rangle$. $\square$

In the following we recall the well-known theorem of Myhill and Nerode, which characterizes the class of recognizable tree languages (cf. [27–29]). For this, let $\theta \subseteq T_\Sigma \times T_\Sigma$ be an equivalence relation, i.e., a reflexive, symmetric, and transitive relation. Then θ is a *congruence relation (with respect to the initial term algebra)*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k, s_1, \ldots, s_k \in T_\Sigma$ with $t_i \ \theta \ s_i$ for every $i \in [k]$ it holds that $\sigma(t_1, \ldots, t_k) \ \theta \ \sigma(s_1, \ldots, s_k)$. Let $B \subseteq T_\Sigma$. Then θ *saturates* B if B is the union of some equivalence classes of θ.

Theorem 4 (Myhill-Nerode theorem [27–29]) *Let $L \subseteq T_\Sigma$ be a tree language. Then the following two statements are equivalent.*

(1) L is a recognizable tree language.
(2) There exists a congruence relation θ on T_Σ which has finite index and saturates L.

3 Recognizability of cut sets

In this section, we investigate the question under which conditions a cut set of a given fuzzy set φ, which is defined on T_Σ, is a recognizable tree language. In this way we obtain a fuzzy structure — a kind of a fuzzy recognizable tree language.

We pursue two approaches. In Section 3.1 we require that the fuzzy set φ, which has domain T_Σ and a poset as co-domain, satisfies several order-theoretic assumptions, e.g., isotonicity or compatibility with top-concatenation. In Section 3.2 we additionally assume that $\varphi \in A^{\mathrm{rec}}\langle\langle T_\Sigma \rangle\rangle$ is a recognizable tree series over a semiring, which has a partially ordered carrier set, and show which of the conditions of Section 3.1 can be dropped or replaced by others.

3.1 Recognizability of cut sets of fuzzy sets

Let $\varphi : T_\Sigma \to A$ be a fuzzy set over T_Σ, and let $a \in A$. We begin this section by proving that a cut set φ_a of φ is a recognizable tree language, if φ satisfies several order-theoretic assumptions. Specifically, these assumptions are: (i) φ is isotone where the order on T_Σ is $\sqsubseteq$, and (ii) φ is compatible with top-concatenation. Recall that recognizable tree languages are recognized by bottom-up *finite-state* weighted tree automata over the Boolean semiring. The finite number of states is reflected in an additional assumption, which we require for recognizability of φ_a, viz., (i) $\varphi(T_\Sigma) \setminus {\uparrow}a$ is a finite set (cf. Lemma 5) or (ii) $(A, \leq)$ is an F-poset (cf. Theorem 6). Moreover, if the range of φ is a finite set, then the isotonicity requirement can be dropped (cf. Theorem 7).

First we prove that, given a poset $(A, \leq)$, an isotone fuzzy set $\varphi : T_\Sigma \to A$, which is compatible with top-concatenation, and $a \in A$ such that $\varphi(T_\Sigma) \setminus {\uparrow}a$ is a finite set, the cut set φ_a is recognizable. This statement is a consequence of the Theorem of Myhill and Nerode (cf. Theorem 4); we define a congruence relation θ on the initial term algebra $(T_\Sigma, \{\, \mathrm{top}_\sigma \mid \sigma \in \Sigma \,\})$ by setting $s\ \theta\ t$ if and only if either (i) s and t are mapped to the same element by φ or (ii) both s and t are mapped to elements greater or equal to a. It turns out that θ saturates φ_a and thus, by the Myhill-Nerode theorem, φ_a is a recognizable tree language.

Lemma 5 *Let $(A, \leq)$ be a poset. Moreover, let $\varphi : T_\Sigma \to A$ be an isotone fuzzy set which is compatible with top-concatenation. Then, for every $a \in A$ such that $\varphi(T_\Sigma) \setminus {\uparrow}a$ is finite, the cut set φ_a is a recognizable tree language.*

PROOF. *Let $a \in A$ such that $\varphi(T_\Sigma) \setminus {\uparrow}a$ is a finite set. Consider the relation $\theta \subseteq T_\Sigma \times T_\Sigma$, which is defined for every $s, t \in T_\Sigma$ by $s\ \theta\ t$ if and only if (i) $\varphi(s) = \varphi(t) \not\geq a$ or (ii) $\varphi(s) \geq a$ and $\varphi(t) \geq a$. Note that, for every $s, t \in T_\Sigma$, the equation $\varphi(s) = \varphi(t)$ implies $s\ \theta\ t$. Apparently, θ is reflexive and symmetric. To prove that θ is transitive, let $t_1, t_2, t_3 \in T_\Sigma$ such that $t_1\ \theta\ t_2$ and $t_2\ \theta\ t_3$. From $t_1\ \theta\ t_2$ it follows that either (i) $\varphi(t_1) = \varphi(t_2) \not\geq a$ or (ii) $\varphi(t_1) \geq a$ and $\varphi(t_2) \geq a$. In Case (i) we deduce from the facts $\varphi(t_2) \not\geq a$ and $t_2\ \theta\ t_3$ that $\varphi(t_1) = \varphi(t_2) = \varphi(t_3) \not\geq a$, i.e., $t_1\ \theta\ t_3$. In Case (ii) it follows from the facts $\varphi(t_2) \geq a$ and $t_2\ \theta\ t_3$ that $\varphi(t_3) \geq a$ and thus, also applying the assumption $\varphi(t_1) \geq a$, it holds that $t_1\ \theta\ t_3$. Hence θ is transitive, and consequently, it is an equivalence relation. Let us now show that θ is a* *congruence relation (with respect to the initial term algebra). For this purpose let $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k, t_1, \ldots, t_k \in T_\Sigma$ such that $s_1 \; \theta \; t_1, \ldots, s_k \; \theta \; t_k$. We show $\sigma(s_1, \ldots, s_k) \; \theta \; \sigma(t_1, \ldots, t_k)$ by the following case analysis.*

(α) *First assume that $\varphi(s_1) = \varphi(t_1) \not\geq a, \ldots, \varphi(s_k) = \varphi(t_k) \not\geq a$. Since φ is compatible with top-concatenation, it holds that*

$$\varphi(\sigma(s_1, \ldots, s_k)) = \varphi(\sigma(t_1, \ldots, t_k)) \ .$$

Consequently, $\sigma(s_1, \ldots, s_k) \; \theta \; \sigma(t_1, \ldots, t_k)$.

(β) *Now assume that there exists an $i \in [k]$ such that $\varphi(s_i) \geq a$ and $\varphi(t_i) \geq a$. Apparently, since φ is order-preserving, it holds that $\varphi(\sigma(s_1, \ldots, s_k)) \geq a$ and $\varphi(\sigma(t_1, \ldots, t_k)) \geq a$. Then $\sigma(s_1, \ldots, s_k) \; \theta \; \sigma(t_1, \ldots, t_k)$.*

Thus θ is a congruence relation. Next we show that θ has finite index. Consider the mapping $\pi_\theta : T_\Sigma/\theta \to (\varphi(T_\Sigma) \setminus \uparrow a) \cup \{a\}$, which is defined by $\pi_\theta([t]_\theta) = \varphi(t)$ if $\varphi(t) \not\geq a$ and $\pi_\theta([t]_\theta) = a$ if $\varphi(t) \geq a$. Clearly, π_θ is well-defined and injective. Thus,

$$\mathrm{card}(T_\Sigma/\theta) = \mathrm{card}(\pi_\theta(T_\Sigma/\theta)) \leq \mathrm{card}(\varphi(T_\Sigma) \setminus \uparrow a) + 1,$$

and since $\varphi(T_\Sigma) \setminus \uparrow a$ is a finite set by assumption, also T_Σ/θ is a finite set. Hence θ has finite index. Moreover, θ saturates φ_a, because φ_a is either the empty set or it is one equivalence class. Applying the theorem of Myhill *and* Nerode *(cf. Theorem 4) shows that φ_a is a recognizable tree language.* □

Next we replace the requirement of Lemma 5, i.e., $\varphi(T_\Sigma) \setminus \uparrow a$ is a finite set, by a restriction on the underlying poset, which ensures that for *every* $a \in A$ the set $\varphi(T_\Sigma) \setminus \uparrow a$ is finite.

Theorem 6 *Let $(A, \leq)$ be an F-poset. Moreover, let $\varphi : T_\Sigma \to A$ be an isotone fuzzy set, which is compatible with top-concatenation. Then for every $a \in A$ the cut set φ_a is a recognizable tree language.*

PROOF. *Let $a \in A$. By Lemma 2 we have that $A \setminus \uparrow a$ is a finite set. Since $\varphi(T_\Sigma) \setminus \uparrow a \subseteq A \setminus \uparrow a$, also $\varphi(T_\Sigma) \setminus \uparrow a$ is a finite set. Then, by Lemma 5, φ_a is a recognizable tree language.* □

Last in this section, we consider fuzzy sets with finite range. It turns out that if the underlying fuzzy set is compatible with top-concatenation, then every cut set is recognizable; note that in contrast to Lemma 5 and Theorem 6 we do not require the underlying fuzzy set to be isotone. The proof of the claimed statement, which is similar to the proof of Lemma 5, is based on an application of the Myhill-Nerode theorem (cf. Theorem 4); the considered congruence relation on the initial term algebra is the kernel of φ, i.e., two trees s and t are equivalent if and only if they are mapped to the same element.

Theorem 7 *Let $(A, \leq)$ be a poset. Moreover, let $\varphi : T_\Sigma \to A$ be a fuzzy set which is compatible with top-concatenation and such that $\varphi(T_\Sigma)$ is a finite set. Then, for every $a \in A$, the cut set φ_a is recognizable.*

PROOF. *Let $\theta \subseteq T_\Sigma \times T_\Sigma$ be the relation which is defined for every two trees $s, t \in T_\Sigma$ by $s\ \theta\ t$ if and only if $\varphi(s) = \varphi(t)$. Clearly, θ is an equivalence relation, and since φ is compatible with top-concatenation, θ is a congruence relation on T_Σ. Moreover, by the definition of θ, the cardinalities of the two sets T_Σ/θ and $\varphi(T_\Sigma)$ are equal. Since, by assumption, $\varphi(T_\Sigma)$ is a finite set, also T_Σ/θ is a finite set, and consequently, θ has finite index. Let us now show that θ saturates φ_a. We compute as follows:*

$$\varphi_a = \{\, t \in T_\Sigma \mid \varphi(t) \geq a \,\} = \varphi^{-1}(\uparrow a) = \bigcup_{b \in \uparrow a} \varphi^{-1}(\{b\}).$$

Observe that, for every $b \in \varphi(T_\Sigma)$, there exists a tree $t \in T_\Sigma$ such that $\varphi^{-1}(\{b\}) = [t]_\theta$, and for every $b \in A \setminus \varphi(T_\Sigma)$ it holds that $\varphi^{-1}(\{b\}) = \emptyset$. Hence φ_a is the union of some equivalence classes of θ, i.e., θ saturates φ_a. Thus, the Theorem of Myhill *and* Nerode *(cf. Theorem 4) implies that φ_a is a recognizable tree language.* $\square$

3.2 Recognizability of cut sets of recognizable tree series

In this section we start from recognizable tree series, and we investigate the question under which additional requirements we can drop, from the results of Section 3.1, the assumptions that φ is an isotone fuzzy set and that φ is compatible with top-concatenation.

First however, we show an example of a recognizable tree series, of which the cut sets are not recognizable (as tree languages).

Example 8 *Let $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}, \beta^{(0)}\}$ and $\mathrm{Trop} = (\mathbb{Z} \cup \{\infty\}, \min, +, \infty, 0)$ be the tropical semiring of integers. Consider the bu-w-fta $M = (Q, \Sigma, F, \mathrm{Trop}, \mu)$ with $Q = \{\alpha, \beta, q\}$, $F = \{q\}$, and*

$$\mu_0(\alpha)_{(),\alpha} = 1 \qquad \mu_0(\beta)_{(),\beta} = -1$$
$$\mu_0(\alpha)_{(),\beta} = 0 \qquad \mu_0(\beta)_{(),\alpha} = 0$$

$$\mu_2(\sigma)_{(\alpha,\alpha),\alpha} = \mu_2(\sigma)_{(\beta,\beta),\beta} = \mu_2(\sigma)_{(\alpha,\beta),q} = 0$$

and all remaining entries of μ are supposed to be ∞. Figure 1 shows this bu-w-fta. Clearly, for every $t \in T_\Sigma$, $h_\mu(t)_\alpha$ is the number of α-labeled nodes

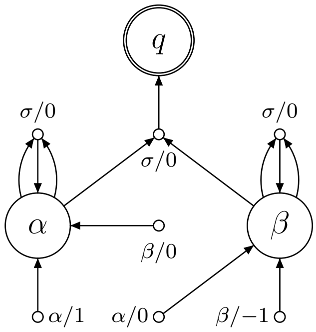

Fig. 1. Bu-w-fta M which recognizes a tree series/fuzzy set with a non-recognizable cut set.

of t, which we denote by $|t|_\alpha$. Similarly, it holds that $h_\mu(t)_\beta = -|t|_\beta$, where $|t|_\beta$ denotes the number of β-labeled nodes of t. Hence we obtain that

$$(\varphi_M, \sigma(t_1, t_2)) = h_\mu(\sigma(t_1, t_2))_q = |t_1|_\alpha - |t_2|_\beta$$

for all $t_1, t_2 \in T_\Sigma$. Now let us consider the cut set $(\varphi_M)_0$, which is defined to be the set $\{\, t \in T_\Sigma \mid (\varphi_M, t) \geq 0 \,\}$. Obviously, $\sigma(t_1, t_2) \in (\varphi_M)_0$ if and only if $(\varphi_M, \sigma(t_1, t_2)) \geq 0$. The latter holds if and only if $|t_1|_\alpha \geq |t_2|_\beta$. This is not a recognizable property, which can easily be shown using the pumping lemma for recognizable tree languages (cf., e.g., [5]). Consequently, the cut set $(\varphi_M)_0$ is not a recognizable tree language.

For the following considerations, let φ be a recognizable tree series over a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, whose carrier set A is partially ordered by $\leq$. Clearly, φ is a fuzzy set whose co-domain is an ordered semiring, and this allows us to prove similar results as in Section 3.1. Firstly we prove a result on locally finite semirings. Namely, given a recognizable tree series φ over a locally finite semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, then for every $a \in A$ the cut set φ_a is a recognizable tree language. Roughly speaking, from the bu-w-fta M recognizing φ we construct a bu-w-fta over Bool, which performs the computation of the weight in its state set. Then we set all those states to final states, where the summation of the weights yields a result greater than or equal to a. Our construction is a straightforward generalization of the construction found in the proof of Theorem 2.1 in [18]. There complete lattices $\mathcal{S}$, which fulfil the additional constraint that for each finite set S of lattice elements the sublattice generated by S is still finite, are considered. However, such lattices are locally finite semirings. It is shown in Theorem 2.1 of [18] that for each fuzzy automaton (i.e., finite-state weighted *string* automaton over $\mathcal{S}$) a deterministic fuzzy automaton recognizing the same fuzzy language can be constructed. We extend the construction to bu-w-fta over locally finite semirings in a straightforward manner and adapt the final states according to our purposes. In this way, we also obtain a deterministic device. Altogether, we obtain a result which parallels Theorem 7.

Theorem 9 *Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a locally finite semiring, $(A, \leq)$ be a poset, $\varphi \in A^{\mathrm{rec}}\langle\langle T_\Sigma \rangle\rangle$ be a recognizable tree series, and $a \in A$. Then the cut set φ_a is a recognizable tree language.*

PROOF. *Let $M = (Q, \Sigma, F, \mathcal{A}, \mu)$ be a bu-w-fta such that $\varphi_M = \varphi$. We construct a bu-w-fta M' over the Boolean semiring such that $\mathrm{supp}(\varphi_{M'}) = \varphi_a$. Therefore, let*

- $C = \{\, \mu_k(\sigma)_{(q_1,\ldots,q_k),q} \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q, q_1, \ldots, q_k \in Q \,\}$,
- $Q' = \langle C \rangle^Q$, *and*
- $F' = \{\, v \in Q' \mid a \leq \bigoplus_{q \in F} v(q) \,\}$.

Clearly, C is a finite set, and so, by local finiteness, also Q' is a finite set. Moreover, for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $v, v_1, \ldots, v_k \in Q'$ we let

$$
\begin{aligned}
& \mu'_k(\sigma)_{(v_1,\ldots,v_k),v} = 1 \\
\iff \quad & (\forall q \in Q) : \ v(q) = \bigoplus_{q_1,\ldots,q_k \in Q} v_1(q_1) \odot \cdots \odot v_k(q_k) \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q}.
\end{aligned}
$$

Then $M' = (Q', \Sigma, F', \mathrm{Bool}, \mu')$ is a bu-w-fta over the Boolean semiring Bool *such that for every $t \in T_\Sigma$ and $v \in Q'$ we have $h_{\mu'}(t)_v = 1$ if and only if $h_\mu(t) = v$. We prove this statement inductively. Let $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, symbol $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$.*

$$
\begin{aligned}
& h_\mu(\sigma(t_1, \ldots, t_k)) = v \\
\iff & (\forall q \in Q) : \bigoplus_{q_1,\ldots,q_k \in Q} h_\mu(t_1)_{q_1} \odot \cdots \odot h_\mu(t_k)_{q_k} \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} = v(q) \\
\iff & (\forall i \in [k])(\exists v_i \in Q')(\forall q \in Q) : \ h_\mu(t_i)_q = v_i(q) \quad \textit{and} \\
& \qquad \bigoplus_{q_1,\ldots,q_k \in Q} v_1(q_1) \odot \cdots \odot v_k(q_k) \odot \mu_k(\sigma)_{(q_1,\ldots,q_k),q} = v(q) \\
\iff & \quad \textit{(by the definition of } \mu' \textit{)} \\
& (\forall i \in [k])(\exists v_i \in Q')(\forall q \in Q) : \ h_\mu(t_i)_q = v_i(q) \quad \textit{and} \\
& \qquad (\mu')_k(\sigma)_{(v_1,\ldots,v_k),v} = 1 \\
\iff & \quad \textit{(by induction hypothesis)} \\
& (\forall i \in [k])(\exists v_i \in Q') : \ h_{\mu'}(t_i)_{v_i} = 1, \ (\mu')_k(\sigma)_{(v_1,\ldots,v_k),v} = 1 \\
\iff & \bigvee_{v_1,\ldots,v_k \in Q'} h_{\mu'}(t_1)_{v_1} \wedge \cdots \wedge h_{\mu'}(t_k)_{v_k} \wedge (\mu')_k(\sigma)_{(v_1,\ldots,v_k),v} = 1 \\
\iff & h_{\mu'}(\sigma(t_1, \ldots, t_k))_v = 1
\end{aligned}
$$

Hence we continue with

$$\begin{aligned}
& t \in \operatorname{supp}(\varphi_{M'}) \\
\Longleftrightarrow\ & \bigvee_{v \in F'} h_{\mu'}(t)_v = 1 \\
\Longleftrightarrow\ & (\exists v \in F') : \ h_{\mu'}(t)_v = 1 \\
\Longleftrightarrow\ & (\exists v \in F') : \ h_{\mu}(t) = v \\
\Longleftrightarrow\ & (\exists v \in Q') : \ h_{\mu}(t) = v \quad \textit{and} \quad a \leq \bigoplus_{q \in F} v(q) \\
\Longleftrightarrow\ & a \leq \bigoplus_{q \in F} h_{\mu}(t)_q \\
\Longleftrightarrow\ & a \leq \varphi_M(t) \\
\Longleftrightarrow\ & t \in \varphi_a \ ,
\end{aligned}$$

which proves that φ_a is a recognizable tree language. $\square$

>From the previous theorem we now derive statements similar to Lemma 5 and Theorem 6. Therefore, we observe that, given a non-decreasing semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ with respect to $\leq$ and an element $a \in A$, then the mapping $h : A \to A$, defined for every $a' \in A$ by (i) $h(a') = a$, if $a \leq a'$, and (ii) $h(a') = a'$ otherwise, is a homomorphism to a semiring with carrier set $(A \setminus \uparrow a) \cup \{a\}$. Consequently, if we demand that $A \setminus \uparrow a$ is a finite set, then this semiring is finite and Theorem 9 is applicable.

Theorem 10 *Let $(A, \leq)$ be a poset and $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring which is non-decreasing with respect to $\leq$. Moreover, let $\varphi \in A^{\mathrm{rec}}\langle\langle T_\Sigma \rangle\rangle$ be a recognizable tree series and $a \in A$. If $A \setminus \uparrow a$ is finite, then the cut set φ_a is a recognizable tree language.*

PROOF. *Let $D = (A \setminus \uparrow a) \cup \{a\}$ and define the operations $+, \cdot : D^2 \to D$ for every $d_1, d_2 \in D$ as follows.*

$$d_1 + d_2 = \begin{cases} a & , \textit{ if } a \leq d_1 \oplus d_2 \\ d_1 \oplus d_2 & , \textit{ otherwise} \end{cases}$$

$$d_1 \cdot d_2 = \begin{cases} a & , \textit{ if } a \leq d_1 \odot d_2 \\ d_1 \odot d_2 & , \textit{ otherwise} \end{cases}$$

Further, let $h : A \to D$ be the mapping defined for every $a' \in A$ by

$$h(a') = \begin{cases} a & , \textit{ if } a \leq a' \\ a' & , \textit{ otherwise} \end{cases} \quad .$$

Using the non-decreasing property of $\mathcal{A}$, we can easily prove that h is a semiring homomorphism. Here we only show $h(a_1 \oplus a_2) = h(a_1) + h(a_2)$ for every $a_1, a_2 \in A$. The proof for the multiplication is similar.

$$\begin{aligned}
h(a_1 \oplus a_2) &= \begin{cases} a & \text{, if } a \leq a_1 \oplus a_2 \\ a_1 \oplus a_2 & \text{, otherwise} \end{cases} \\
&= \begin{cases} a & \text{, if } a \leq a_1 \text{ or } a \leq a_2 \\ a & \text{, if } a \leq a_1 \oplus a_2, a \not\leq a_1, \text{ and } a \not\leq a_2 \\ a_1 \oplus a_2 & \text{, otherwise} \end{cases} \\
&\quad \text{(because } \mathcal{A} \text{ is non-decreasing)} \\
&= \begin{cases} a & \text{, if } a = h(a_1) \text{ or } a = h(a_2) \\ a & \text{, if } a = h(a_1) + h(a_2), a \not\leq a_1, \text{ and } a \not\leq a_2 \\ h(a_1) + h(a_2) & \text{, otherwise} \end{cases} \\
&= h(a_1) + h(a_2)
\end{aligned}$$

Hence $\mathcal{D} = (D, +, \cdot, h(\mathbf{0}), h(\mathbf{1}))$ is a finite semiring. Let $\varphi' = h(\varphi)$, i.e, $\varphi' = \sum_{t \in T_\Sigma} h((\varphi, t))\, t$. Since recognizable tree series are closed under semiring homomorphisms (cf. Lemma 3), we conclude that $\varphi' \in D^{\text{rec}}\langle\langle T_\Sigma \rangle\rangle$. Moreover, we note that $\varphi_a = \varphi'_a$, because for every $t \in T_\Sigma$

$$t \in \varphi_a \iff a \leq (\varphi, t) \iff a = h((\varphi, t)) \iff a = (\varphi', t) \iff t \in \varphi'_a \ .$$

Hence by Theorem 9 it follows that φ_a is a recognizable tree language. □

Finally, we consider non-decreasing semirings $\mathcal{A}$ such that the finiteness condition is fulfilled. Then it follows from Theorem 10 that every cut set of a recognizable tree series $\varphi \in A^{\text{rec}}\langle\langle T_\Sigma \rangle\rangle$ is recognizable.

Corollary 11 *Let $(A, \leq)$ be an F-poset and $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring which is non-decreasing with respect to $\leq$. Moreover, let $\varphi \in A^{\text{rec}}\langle\langle T_\Sigma \rangle\rangle$ be a recognizable tree series. Then for every $a \in A$ the cut set φ_a is a recognizable tree language.*

PROOF. *Since $(A, \leq)$ is an F-poset, it holds for every $a \in A$ that $A \setminus \uparrow a$ is a finite set. The claim now follows from Theorem 10.* □

4 Conclusion and open questions

We have presented a connection between the theory of fuzzy sets and structures on one side and automata theory (in particular: recognizable tree series) on the other side. For this purpose we considered a class of fuzzy sets, namely the fuzzy sets $\varphi : T_\Sigma \to A$ whose domain is the set of all trees over a given ranked alphabet Σ and whose co-domain is the carrier set of some poset $(A, \leq)$ satisfying various finiteness conditions. Our aim was to obtain a fuzzy structure which fuzzifies the notion of a recognizable tree language, in the sense that its cut sets are crisp recognizable tree languages. We have shown that every cut set of φ is a recognizable tree language provided that (i) φ is isotone and compatible with top-concatenation (cf. Subsection 3.1) or (ii) A is the carrier set of some non-decreasing semiring and φ is a recognizable tree series (cf. Subsection 3.2). Clearly, these are sufficient conditions, so we also consider it interesting to find out necessary and sufficient conditions.

Moreover, it is an interesting question under which conditions an inverse theory can be established. More precisely, given a collection $\mathcal{S}$ of recognizable tree languages, does there exist a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, a partial order $\leq$ on A, and a recognizable tree series φ over $\mathcal{A}$, i.e., a fuzzy set, such that the collection φ_S of cut sets equals the given collection $\mathcal{S}$ of tree languages. One could also investigate an extended version of the aforementioned problem and ask for conditions under which φ is a fuzzy sub-algebra of the initial term algebra $(T_\Sigma, \{\, \mathrm{top}_\sigma \mid \sigma \in \Sigma \,\})$ such that $\varphi_S = \mathcal{S}$ where a fuzzy sub-algebra of the initial term algebra is a fuzzy set $\varphi : T_\Sigma \to A$ fulfilling $\varphi(\sigma(t_1, \ldots, t_k)) \geq \inf\{\varphi(t_1,), \ldots, \varphi(t_k)\}$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$. We leave these questions open, but refer the reader to Proposition 2 of [4]. There it is proved in a more general framework that, given a collection $\mathcal{S}$ of subsets (ordered by inclusion) of some nonempty set D which is closed under centralized intersection, i.e., $\bigcap\{\, S \in \mathcal{S} \mid d \in S \,\} \in \mathcal{S}$ for every $d \in D$, and which covers D, i.e., $\bigcup \mathcal{S} = D$, the fuzzy set $\varphi : D \to \mathcal{S}$ sending every $d \in D$ to the set $\bigcap\{\, S \in \mathcal{S} \mid d \in S \,\}$ induces the family of cut sets $\varphi_{\mathcal{S}}$ being equal to the given collection $\mathcal{S}$ of subsets of D. In order to approach an answer to the question under which conditions φ is a fuzzy sub-algebra, let us now additionally assume that D is the carrier set of some algebra $\mathcal{D} = (D, F)$ and that $\leq$ is a partial order on D, which is order-preserving with respect to all operations of F. From Proposition 2 of [4] it straightforwardly follows that, if every $S \in \mathcal{S}$ is an up-set, i.e., $S = {\uparrow}S$, then the above specified mapping φ is a fuzzy sub-algebra of $\mathcal{D}$. Apparently, these two general statements can be instantiated to $\mathcal{S}$ being a collection of tree languages and hence φ being a tree series. It is still open which requirements have to be made to ensure that φ is recognizable.

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