

# The Substitution Vanishes

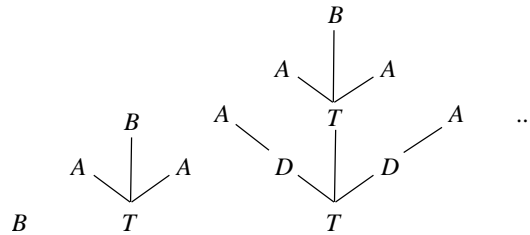
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**Abstract.** Accumulation techniques were invented to transform functional programs, which intensively use append functions (like inefficient list reversal), into more efficient programs, which use accumulating parameters instead (like efficient list reversal). In this paper we present a generalized and automatic accumulation technique that also handles programs operating with unary functions on arbitrary tree structures and employing substitution functions on trees which may replace different designated symbols by different trees. We show that this transformation does not deteriorate the efficiency with respect to call-by-need reduction.

## 1 Introduction

The sequence of trees in Figure 1 illustrates the stepwise growth of a tree, where in every step in parallel every occurrence of a symbol  $A$  (and  $B$ , respectively) is substituted by a tree  $(D A)$  (and  $(T A B A)$ , respectively).



**Fig. 1.** Stepwise growth of a tree.

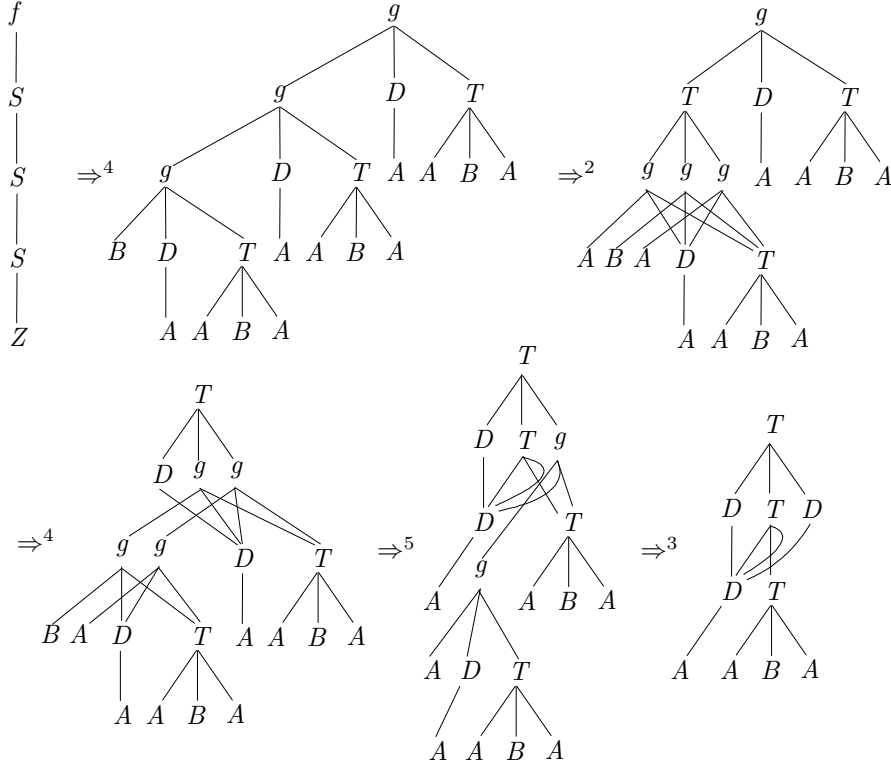
In a functional program  $p_{non}$  (formulated e.g. in Haskell), this substitution can be defined by a ternary function  $g$  that takes the “previous tree” and the two kinds of “fresh branches”  $(D A)$  and  $(T A B A)$  as arguments. Additionally, a unary function  $f$  generates as many nested calls of  $g$  as the argument of  $f$  indicates, where natural numbers are represented by a nullary  $Z$  and a unary  $S$ , i.e. the initial expression  $(f (S^n Z))$  generates  $n$  nested calls of  $g$ .<sup>1</sup>

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<sup>1</sup> Since there is only one unary input symbol  $S$  for  $f$ , the actual parameters  $(D A)$  and  $(T A B A)$  of  $g$  are unique. Hence, an alternative version of  $g$  could avoid its formal parameters  $y_1$  and  $y_2$  and directly use  $(D A)$  and  $(T A B A)$  in its  $A$ - and  $B$ -rules, respectively. In a more elaborate example with different unary input symbols for  $f$  the actual parameters of  $g$  may be different and hence the formal parameters are essential. For convenience we avoided to blow up our example into this direction.

$$\begin{aligned}
f Z &= B \\
f (S x_1) &= g (f x_1) (D A) (T A B A) \\
g A y_1 y_2 &= y_1 \\
g B y_1 y_2 &= y_2 \\
g (D x_1) y_1 y_2 &= D (g x_1 y_1 y_2) \\
g (T x_1 x_2 x_3) y_1 y_2 &= T (g x_1 y_1 y_2) (g x_2 y_1 y_2) (g x_3 y_1 y_2)
\end{aligned}$$

Unfortunately,  $p_{non}$  has cubic time-complexity in the input number  $n$ , since  $g$  has to process  $n$  intermediate results with sizes  $1^2, 2^2, \dots, n^2$ , respectively (though they are not explicitly constructed in a call-by-need evaluation). Figure 2 shows an evaluation of  $(f (S^3 Z))$  obeying a call-by-need strategy.



**Fig. 2.** Evaluation using  $p_{non}$  where  $\Rightarrow$  is the call-by-need reduction relation.

Therefore we would prefer the following program  $p_{acc}$  which has linear time-complexity in  $n$  to evaluate the (modified) initial expression  $(f (S^n Z) A B)$ :

$$\begin{aligned}
f Z y_1 y_2 &= y_2 \\
f (S x_1) y_1 y_2 &= f x_1 (D y_1) (T y_1 y_2 y_1)
\end{aligned}$$

Since  $p_{acc}$  uses its second and third argument to accumulate step by step a “ $D$ -branch” and an output tree, respectively, we call  $p_{acc}$  an *accumulative* program, whereas we call  $p_{non}$  *non-accumulative*. Techniques which transform non-accumulative into accumulative programs are called *accumulation*.

In the case that substitutions on tree structures are restricted to append functions on list structures, there is a long history of research on accumulation: Already in [4] it is shown in the context of transforming programs into iterative (tail-recursive) form, how non-accumulative programs can be transformed (non-automatically) into their accumulative versions. In [1, 14] a similar technique for linear recursive functions is presented. Other non-automatic realizations of accumulation are given e.g. in [2, 10]. Finally, the transformation of [20] is fully automatic and is accompanied by an efficiency analysis. The crucial laws, on which the transformation of [20] is based, can be found in our paper in a similar form. All mentioned techniques essentially rely on the properties of the monoid of lists with append. This fact is detailed in [3].

Our automatic transformation technique is more general in two aspects: we consider (i) arbitrary tree structures (instead of lists) as input and output, and (ii) substitutions on trees (instead of append) which additionally may replace different designated symbols by different trees. On the other hand, our technique is restricted to unary functions (apart from substitutions), though also in [20] the only example program involving a non-unary function could not be optimized. Hence the scope of our technique includes the unary functions in the examples of [20] (in particular, inefficient list reversal).

In this paper we view functional programs like  $p_{non}$  as special *2-modular tree transducers* [7]. Every function in module 1 (like  $f$  in  $p_{non}$ ) is unary and is defined by a case analysis on the root symbol  $c$  of its argument  $t$ . The right-hand side of the equation for  $f$  and  $c$  may contain recursive calls of functions in module 1 on subtrees of  $t$  and arbitrary calls of the function in module 2. The function in module 2 (like  $g$  in  $p_{non}$ ) is a *substitution function*, i.e. designated nullary *substitution constructors* are replaced by parameters and other constructors are left unchanged. In [12] it was shown, how such programs can be transformed into *macro tree transducers* [5, 6], in which every function (like  $f$  in  $p_{acc}$ ) may have arbitrary rank and is defined by a case analysis on the root symbol  $c$  of its first argument  $t$ . The right-hand side of the equation for  $f$  and  $c$  may contain (nested) recursive function calls on subtrees of  $t$ .

The accumulation technique of [12] is divided into three indirect transformation steps which are mainly based on a composition result [6, 11] for *top-down tree transducers* [15, 17] with macro tree transducers and on the associativity of substitution functions. Although the resulting programs avoid substitution functions, they are not always more efficient than the original programs. There and in the present paper the efficiency is measured in the number of performed call-by-need reduction steps. This point of view, which neglects the actual complexity of rule applications, is also taken in, e.g., [16, 19]. In [12] also the reverse transformation is presented. Both transformations together induce that the classes of macro tree transducers and of the special 2-modular tree transducers have the same computational power. Although the reverse transformation deteriorates in general the efficiency, it also has practical relevance: In [9] it is extended to a *deaccumulation* technique which is useful to improve the automatic verification of functional (and even imperative) programs.

In [13] the deficiencies of the accumulation technique in [12] were solved by presenting a direct transformation which additionally employs let-expressions to avoid causes of inefficiency. Moreover, it was shown in [13] that the transformation does not deteriorate the efficiency. To this end, a call-by-need reduction on term graphs was defined and compared for the original and resulting program. The efficiency result is based on the fact that the number of applications of functions in module 1 of the original program equals the number of function applications in the resulting program. Hence, the applications of the substitution function in module 2 of the original program are saved!

We simplify the presentation of [13] by avoiding an explicit call-by-need reduction and by adopting a technique presented in [16, 18, 19], where function applications (in [18, 19] for (compositions of) macro tree transducers) additionally produce special “ticking symbols” in order to make the number of performed (call-by-name) reduction steps visible in the output. Instead of a call-by-need reduction relation on term graphs which (implicitly) uses sharing to avoid that unevaluated function arguments are copied, we use a nondeterministic reduction relation on expressions with an explicit denotation for sharing. Unfortunately, this *explicit sharing* does not prevent that our nondeterministic reduction relation creates a shared subexpression  $e$  such that  $e$  contains function applications and  $e$  is not relevant in the overall expression (in the sense that a call-by-need reduction would delete all references to  $e$ ). To avoid that the ticking symbols which are generated by  $e$  are counted (and thus the reduction steps needed to evaluate  $e$ ), we additionally use a *counting function* which takes care of such non-relevant subexpressions. Hence the concepts of *explicit sharing* and the *counting function* provide a new technique to count the number of steps performed by a call-by-need reduction.

## 2 Preliminaries

We denote the set of natural numbers including 0 by  $\mathbb{N}$  and the set  $\mathbb{N} - \{0\}$  by  $\mathbb{N}_+$ . For every  $m \in \mathbb{N}$ , the set  $\{1, \dots, m\}$  is denoted by  $[m]$ . The cardinality of a set  $K$  is denoted by  $\text{card}(K)$ . We use the sets  $X = \{x_0, x_1, x_2, x_3, \dots\}$ ,  $Y = \{y_1, y_2, y_3, \dots\}$ , and  $Z = \{z_1, z_2, z_3, \dots\}$  of *variables*. For every  $n \in \mathbb{N}$ , let  $X_n = \{x_1, \dots, x_n\}$ ,  $Y_n = \{y_1, \dots, y_n\}$ , and  $Z_n = \{z_1, \dots, z_n\}$ . Let  $\Rightarrow$  be a binary relation on a set  $K$  and  $n \in \mathbb{N}$ . Then,  $\Rightarrow^n$  denotes the  $n$ -fold composition and  $\Rightarrow^*$  the transitive, reflexive closure of  $\Rightarrow$ . If  $k \Rightarrow^* k'$  for  $k, k' \in K$  and there is no  $k'' \in K$  such that  $k' \Rightarrow k''$ , then  $k'$  is called a *normal form of  $k$  with respect to  $\Rightarrow$* , which is denoted by  $\text{nf}(\Rightarrow, k)$ , if it exists and if it is unique.

A *ranked alphabet* is a pair  $(C, \text{rank})$  where  $C$  is a finite set and  $\text{rank}$  is a mapping which associates with every symbol  $c \in C$  a natural number called the *rank* of  $c$ . We simply write  $C$  instead of  $(C, \text{rank})$  and assume  $\text{rank}$  as implicitly given. The set of symbols of  $C$  with rank  $n$  is denoted by  $C^{(n)}$  and if  $c \in C^{(n)}$ , we also use the notation  $c^{(n)}$ . The set of *trees* (or *terms*) over  $C$  indexed by (a set of variables)  $U$ , denoted by  $T_C(U)$ , is the smallest subset  $T \subseteq (C \cup U \cup \{(\cdot)\})^*$  such that  $U \subseteq T$  and for every  $c \in C^{(n)}$  with  $n \in \mathbb{N}$  and  $t_1, \dots, t_n \in T$ :

$(c \ t_1 \dots t_n) \in T$ . If  $c \in C^{(0)}$ , we write just  $c$  instead of  $(c)$ . The set  $T_C(\emptyset)$  is abbreviated by  $T_C$ . If  $R$  is the set of rules of a term rewriting system, then  $\Rightarrow_R$  denotes the (nondeterministic) reduction relation induced by  $R$ . If there is at most one occurrence of a variable  $v$  in a term  $t$ , then we call  $t$  *linear* in  $v$ .

For a term  $t$ , pairwise distinct variables  $v_1, \dots, v_n$ , and terms  $t_1, \dots, t_n$ , we denote by  $t[v_1/t_1, \dots, v_n/t_n]$  the term that is obtained from  $t$  by substituting for every  $i \in [n]$  every occurrence of  $v_i$  in  $t$  by  $t_i$ . We abbreviate  $[v_1/t_1, \dots, v_n/t_n]$  by  $[v_i/t_i]$ , if the involved variables and terms are clear from the context. We use a linear, “substitution-like” notation for term graphs to express the sharing of subgraphs:  $e[z_1 \rightsquigarrow e_1, \dots, z_n \rightsquigarrow e_n]$  denotes a term graph, in which for every occurrence of  $z_i$  in the subgraph denoted by  $e$  there is a directed edge from the direct ancestor node of  $z_i$  to the root node of the subgraph denoted by  $e_i$ .

For the rest of the paper, let  $n \in \mathbb{N}_+$ .

**Definition 1** Let  $C$  be a ranked alphabet and  $U \in \{Z_n, \emptyset\}$ . The set  $E_{C,n}(U)$  of  $C$ -expressions with sharing (and free variables of  $U$ ) is defined by:

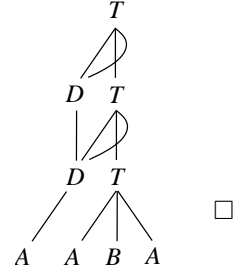
- For every  $z_j \in U$ :  $z_j \in E_{C,n}(U)$ .
- For every  $c \in C^{(k)}$  and  $e_1, \dots, e_k \in E_{C,n}(U)$ :  $(c \ e_1 \dots e_k) \in E_{C,n}(U)$ .
- For every  $e_1, \dots, e_n \in E_{C,n}(U)$  and  $e \in E_{C,n}(Z_n)$ :  
 $e[z_1 \rightsquigarrow e_1, \dots, z_n \rightsquigarrow e_n] \in E_{C,n}(U)$ .

The set  $E_{C,n}$  of  $C$ -expressions with sharing is defined as  $E_{C,n}(\emptyset)$ . □

Note that a  $C$ -expression with sharing  $e \in E_{C,n}(U)$  can be considered as a tree  $e \in T_{C'}(U)$  where  $C' = C \cup \{(\cdot[z_1 \rightsquigarrow \cdot, \dots, z_n \rightsquigarrow \cdot])^{(n+1)}\}$  is the ranked alphabet obtained from  $C$  by adding a new  $(n+1)$ -ary symbol. Thus we can employ the notions and notations, which we introduced for trees, also for  $C$ -expressions.

**Example 2** Let  $C = \{A^{(0)}, B^{(0)}, D^{(1)}, T^{(3)}\}$ . Then,

$$\begin{aligned} (T \ z_1 \ z_2 \ z_1) &\in E_{C,2}(Z_2), \\ (T \ z_1 \ z_2 \ z_1)[z_1 \rightsquigarrow (D \ z_1), z_2 \rightsquigarrow (T \ z_1 \ z_2 \ z_1)] &\in E_{C,2}(Z_2), \\ (T \ z_1 \ z_2 \ z_1)[z_1 \rightsquigarrow (D \ z_1), z_2 \rightsquigarrow (T \ z_1 \ z_2 \ z_1)] \\ &\quad [z_1 \rightsquigarrow (D \ A), z_2 \rightsquigarrow (T \ A \ B \ A)] \in E_{C,2}, \end{aligned}$$



and the latter represents the depicted term graph. □

If  $e_1, \dots, e_n$  are clear from the context, then we abbreviate an expression of the form  $e[z_1 \rightsquigarrow e_1, \dots, z_n \rightsquigarrow e_n]$  by  $e[z_i \rightsquigarrow e_i]$ .

In the following we define a function on  $C$ -expressions with sharing, which constructs trees by unfolding all sharings.

**Definition 3** Let  $C$  be a ranked alphabet and  $U \in \{Z_n, \emptyset\}$ . The function tree :  $E_{C,n}(U) \rightarrow T_C(U)$  is defined as follows:

- For every  $z_j \in U$ : tree( $z_j$ ) =  $z_j$ .
- For every  $c \in C^{(k)}$  and  $e_1, \dots, e_k \in E_{C,n}(U)$ :  
 $\underline{\text{tree}}(c \ e_1 \dots e_k) = (c \ \underline{\text{tree}}(e_1) \dots \underline{\text{tree}}(e_k))$ .
- For every  $e_1, \dots, e_n \in E_{C,n}(U)$  and  $e \in E_{C,n}(Z_n)$ :  
 $\underline{\text{tree}}(e[z_i \rightsquigarrow e_i]) = \underline{\text{tree}}(e)[z_i/\underline{\text{tree}}(e_i)]$ . □

We call  $z_i \in Z_n$  a *free occurrence* in  $e \in E_{C,n}(Z_n)$ , if  $z_i$  occurs in  $\text{tree}(e)$ . Note that this clarifies the scope of a sharing. The scope of  $z_1, \dots, z_n$  in an expression  $e[z_i \rightsquigarrow e_i]$  is limited to the free occurrences of  $z_1, \dots, z_n$  in  $e$ .

### 3 Nonaccumulating and Accumulating Tree Transducers

First we define nonaccumulating tree transducers as functional source language. Nonaccumulating tree transducers are special 2-modular tree transducers [7, 12].

**Definition 4** An *n-nonaccumulating tree transducer* (for short *n-ntt*) is a tuple  $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$ , where

- $F$  is a ranked alphabet (of *function symbols*) with  $F = F^{(1)}$ ,
- $Sub = \{sub^{(n+1)}\}$  (*substitution function*),
- $C$  is a ranked alphabet (of *constructors*),
- $\Pi = \{\Pi_1, \dots, \Pi_n\} \subseteq C^{(0)}$  with  $\text{card}(\Pi) = n$  (the *substitution constructors*)

such that  $F$ ,  $Sub$ , and  $C$  are pairwise disjoint,

- $R_1$  is a set of *rules* such that for every  $f \in F$  and  $c \in C^{(k)}$  there is exactly one rule  $f(c x_1 \dots x_k) = rhs_{R_1, f, c}$  with  $rhs_{R_1, f, c} \in RHS(F, Sub, C, X_k)$ , where for every  $k \in \mathbb{N}$ ,  $RHS(F, Sub, C, X_k)$  is the smallest set  $RHS$  such that
  - for every  $f \in F$  and  $i \in [k]$ :  $(f x_i) \in RHS$ ,
  - for every  $r_0, \dots, r_n \in RHS$ :  $(sub r_0 \dots r_n) \in RHS$ , and
  - for every  $c \in C^{(l)}$  and  $r_1, \dots, r_l \in RHS$ :  $(c r_1 \dots r_l) \in RHS$ ,
- $R_2$  is a set of *rules* such that
  - for every  $j \in [n]$  there is the rule  $sub \Pi_j y_1 \dots y_n = y_j$
  - and for every  $c \in (C - \Pi)^{(k)}$  there is the rule  $sub(c x_1 \dots x_k) y_1 \dots y_n = c(sub x_1 y_1 \dots y_n) \dots (sub x_k y_1 \dots y_n)$ ,
- $r_{in} \in RHS(F, Sub, C, X_1)$  is the *initial right-hand side*.  $\square$

Since every function is defined by recursion on its first argument, this argument is called *recursion argument*. The other arguments are called *context arguments*. The set  $RHS$  formalizes the description of right hand sides found in the introduction. The initial right hand side  $r_{in}$  serves as *call pattern* for the *n-ntt*, where  $x_1$  acts as a placeholder for the actual input tree. Note that the concept of *n-ntts* (with one substitution function of rank  $n + 1$ ) can easily be generalized to a model with several substitution functions. This, however, does not increase the computational power.

In the following examples we will avoid rules, which are never used.

**Example 5**  $M_{non} = (F, Sub, C, \Pi, R_1, R_2, r_{in})$  is a 2-ntt, where  $F = \{f\}$ ,  $sub = g$ ,  $C = \{S^{(1)}, Z^{(0)}, A^{(0)}, B^{(0)}, D^{(1)}, T^{(3)}\}$ ,  $\Pi_1 = A$ ,  $\Pi_2 = B$ ,  $R_1$  and  $R_2$  contain the *f*-rules and *g*-rules, respectively, of  $p_{non}$ , and  $r_{in} = (f x_1)$ .  $\square$

Now we present *n-ntts* with sharings as abstractions for our functional source programs. In contrast to functional programs, where in a call-by-need reduction the sharing of expressions which are bound to variables of rules is performed

implicitly, in  $n$ -ntts with sharings the sharing is performed explicitly, whenever there is the risk to copy unevaluated expressions. This concerns only the context arguments of substitution functions (since other arguments are not copied or are constructor trees). To denote an explicit sharing in a right-hand side of a rule or in a sentential form, we also use expressions with sharing. Thus, because of possibly nested substitution functions during an evaluation, additionally expressions with sharing may occur in the recursion argument of substitution functions. Hence they must be handled by a special rule. Actually,  $n$ -ntts with sharings could be considered as special “2-modular tree-to-graph transducers”. See [7, 8] for the concepts of modular tree transducers and top-down tree-to-graph transducers, respectively. Note that the additional sharing mechanism does not change the computational power of  $n$ -ntts, but may improve the efficiency.

**Definition 6** An  $n$ -nonaccumulating tree transducer with sharings (for short  $n$ -sntt) is a tuple  $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$ , where

- $F, C, \Pi, R_1$ , and  $r_{in}$  are defined as in Definition 4,
- $Sub = \{sub^{(n+1)}\}$ ,
- $R_2$  is a set of rules such that
  - for every  $j \in [n]$  there is the rule  $sub \Pi_j y_1 \dots y_n = y_j$ ,
  - for every  $c \in (C - \Pi)^{(k)}$  there is the rule<sup>2</sup>

$$\begin{aligned} &sub (c x_1 \dots x_k) y_1 \dots y_n \\ &= (c (sub x_1 z_1 \dots z_n) \dots (sub x_k z_1 \dots z_n))[z_i \rightsquigarrow y_i], \end{aligned}$$
  - and there is the rule<sup>3</sup>

$$sub x_0[z_i \rightsquigarrow x_i] y_1 \dots y_n = x_0[z_i \rightsquigarrow (sub x_i z_1 \dots z_n)][z_i \rightsquigarrow y_i]. \quad \square$$

Note that in the last rule in the previous definition  $sub$  walks into  $x_1, \dots, x_n$ , but not into  $x_0$ . This is due to the fact that every instantiation of  $x_0[z_i \rightsquigarrow x_i]$  was generated by an inner occurrence of  $sub$  which already handled the substitution constructors in  $x_0$ . Moreover, it is easily seen that the inner occurrence of  $sub$  does not introduce substitution constructors in  $x_0$  because only calls of the form  $(sub x_i z_1 \dots z_n)$  can occur in  $x_0$ . We further note that an application of the last rule represents a short cut, since a call-by-need reduction on term graphs would (i) walk stepwise through the expression bound to  $x_0$  and would (ii) end up with different occurrences of  $sub$  at different occurrences of a  $z_i$  (thus performing several runs on the expression bound to  $x_i$ ).

**Example 7**  $\tilde{M}_{non} = (F, Sub, C, \Pi, R_1, R_2, r_{in})$  is a 2-sntt, where  $F = \{f\}$ ,  $Sub = \{g^{(3)}\}$ ,  $C = \{S^{(1)}, Z^{(0)}, A^{(0)}, B^{(0)}, D^{(1)}, T^{(3)}\}$ ,  $\Pi_1 = A$ ,  $\Pi_2 = B$ ,  $R_1$  contains the  $f$ -rules of  $p_{non}$  and  $R_2$  contains rules

$$\begin{aligned} g A y_1 y_2 &= y_1 \\ g B y_1 y_2 &= y_2 \\ g (D x_1) y_1 y_2 &= D (g x_1 y_1 y_2) \\ g (T x_1 x_2 x_3) y_1 y_2 &= (T (g x_1 z_1 z_2) (g x_2 z_1 z_2) (g x_3 z_1 z_2))[z_i \rightsquigarrow y_i] \\ g x_0[z_i \rightsquigarrow x_i] y_1 y_2 &= x_0[z_i \rightsquigarrow (g x_i z_1 z_2)][z_i \rightsquigarrow y_i], \end{aligned}$$

and  $r_{in} = (f x_1)$ . Let  $R = R_1 \cup R_2$ . Then,

<sup>2</sup> If  $c$  is nullary or unary, then the explicit sharing will be avoided in examples.

<sup>3</sup> If  $n = 1$ , then the explicit sharing  $[z_i \rightsquigarrow y_i]$  could be avoided.

$$\begin{aligned}
f(S^3 Z) &\Rightarrow_R^5 g(g(T A B A)(D A)(T A B A))(D A)(T A B A) \\
&\Rightarrow_R^4 g(T z_1 z_2 z_1)[z_1 \rightsquigarrow (D A), z_2 \rightsquigarrow (T A B A)](D A)(T A B A) \\
&\Rightarrow_R (T z_1 z_2 z_1)[z_1 \rightsquigarrow (g(D A) z_1 z_2), z_2 \rightsquigarrow (g(T A B A) z_1 z_2)] \\
&\quad [z_1 \rightsquigarrow (D A), z_2 \rightsquigarrow (T A B A)] \\
&\Rightarrow_R^6 (T z_1 z_2 z_1)[z_1 \rightsquigarrow (D z_1), z_2 \rightsquigarrow (T z_1 z_2 z_1)[z_1 \rightsquigarrow z_1, z_2 \rightsquigarrow z_2]] \\
&\quad [z_1 \rightsquigarrow (D A), z_2 \rightsquigarrow (T A B A)]. \quad \square
\end{aligned}$$

Our main transformation will deliver accumulating tree transducers with sharings, which could be considered as special “macro tree-to-graph transducers”. See [6, 8] for the concepts of macro tree transducers and top-down tree-to-graph transducers, respectively.

**Definition 8** An  $n$ -accumulating tree transducer with sharings (for short  $n$ -satt) is a tuple  $M = (F, C, R, r_{in})$ , where

- $F$  is a ranked alphabet (of *function symbols*) with  $F = F^{(n+1)}$ ,
- $C$  is a ranked alphabet (of *constructors*), such that  $F$  and  $C$  are disjoint,
- $R$  is a set of *rules* such that for every  $f \in F$  and  $c \in C^{(k)}$  there is exactly one rule  $f(c x_1 \dots x_k) y_1 \dots y_n = rhs_{R,f,c}$  with  $rhs_{R,f,c} \in RHS'(F, C, X_k, Y_n)$ , where for every  $j \in [n]$ , the right-hand side  $rhs_{R,f,c}$  is linear in  $y_j$ , and for every  $k \in \mathbb{N}$  and  $U \in \{Y_n, Z_n, \emptyset\}$ , the set  $RHS'(F, C, X_k, U)$  is the smallest set such that
  - for every  $f \in F$ ,  $i \in [k]$ , and  $r_1, \dots, r_n \in RHS'(F, C, X_k, U)$ :  
 $(f x_i r_1 \dots r_n) \in RHS'(F, C, X_k, U)$ ,
  - for every  $c \in C^{(l)}$  and  $r_1, \dots, r_l \in RHS'(F, C, X_k, U)$ :  
 $(c r_1 \dots r_l) \in RHS'(F, C, X_k, U)$ ,
  - for every  $r_1, \dots, r_n \in RHS'(F, C, X_k, U)$  and  $r_0 \in RHS'(F, C, X_k, Z_n)$ :  
 $r_0[z_i \rightsquigarrow r_i] \in RHS'(F, C, X_k, U)$ , and
  - for every  $u \in U$ :  $u \in RHS'(F, C, X_k, U)$ ,
- $r_{in} \in RHS'(F, C, X_1, \emptyset)$  is the *initial right-hand side*. □

The linearity condition in the previous definition will be called *context-linearity*. Note that it guarantees that no unevaluated subexpressions are copied in a nondeterministic reduction relation. This fact will be needed in Subsections 4.3 and 5.3, where  $n$ -satts are realized by functional programs under call-by-need evaluation, such that the number of performed reduction steps is equal.

**Example 9**  $M_{acc} = (F', C, R, r'_{in})$  is a 2-satt, where  $F' = \{f^{(3)}\}$ ,  $C = \{S^{(1)}, Z^{(0)}, A^{(0)}, B^{(0)}, D^{(1)}, T^{(3)}\}$ ,  $R$  contains the rules

$$\begin{aligned}
f Z y_1 y_2 &= y_2 \\
f(S x_1) y_1 y_2 &= (f x_1 (D z_1) (T z_1 z_2 z_1))[z_i \rightsquigarrow y_i]
\end{aligned}$$

and  $r'_{in} = (f x_1 A B)$ . Then,

$$\begin{aligned}
f(S^3 Z) A B &\Rightarrow_R (f(S^2 Z)(D z_1)(T z_1 z_2 z_1))[z_1 \rightsquigarrow A, z_2 \rightsquigarrow B] \\
&\Rightarrow_R (f(S Z)(D z_1)(T z_1 z_2 z_1)) \\
&\quad [z_1 \rightsquigarrow (D z_1), z_2 \rightsquigarrow (T z_1 z_2 z_1)][z_1 \rightsquigarrow A, z_2 \rightsquigarrow B] \\
&\Rightarrow_R^2 (T z_1 z_2 z_1)[z_1 \rightsquigarrow (D z_1), z_2 \rightsquigarrow (T z_1 z_2 z_1)] \\
&\quad [z_1 \rightsquigarrow (D z_1), z_2 \rightsquigarrow (T z_1 z_2 z_1)][z_1 \rightsquigarrow A, z_2 \rightsquigarrow B]. \quad \square
\end{aligned}$$



For every  $n$ -sntt  $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$  with  $R = R_1 \cup R_2$  and for every  $n$ -satt  $M = (F, C, R, r_{in})$ ,  $\Rightarrow_R$  is locally confluent, because there are no critical pairs. Similarly to modular tree transducers [7] and macro tree transducers [6],  $\Rightarrow_R$  is also terminating, since every rule application to a function symbol with its recursion argument  $t$  delivers only (i) new function symbols with subtrees of  $t$  as recursion arguments or (in the case of an  $n$ -sntt) (ii) occurrences of the substitution function which does not call any other function and also “strictly walks down” on its recursion argument. Thus, for every  $t \in T_C$ ,  $nf(\Rightarrow_R, r_{in}[x_1/t])$  exists. Moreover, there are no function symbols in this normal form, because all functions are exhaustively defined on their possible recursion arguments (in particular on all outputs of functions which are nested in their recursion arguments). Hence, the normal form is a  $C$ -expression with sharing.

**Definition 10** Let  $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$  be an  $n$ -sntt with  $R = R_1 \cup R_2$  or let  $M = (F, C, R, r_{in})$  be an  $n$ -satt. The *tree transformation computed by  $M$*  is the function  $\tau_M : T_C \rightarrow T_C$ , which is for every  $t \in T_C$  defined by  $\tau_M(t) = \underline{tree}(nf(\Rightarrow_R, r_{in}[x_1/t]))$ .  $\square$

## 4 Accumulation Technique

Our transformation technique consists of three steps: (i) a preprocessing step which abstracts  $n$ -ntts into  $n$ -sntts, (ii) the main transformation on the level of tree transducers with sharings (transforming  $n$ -sntts into  $n$ -satts), and (iii) a postprocessing step which realizes  $n$ -satts as functional programs. Since the pre- and postprocessing steps are relatively simple compared to the main transformation, they will be presented only informally.

### 4.1 Preprocessing

Explicit sharings which were introduced in the previous section, are added. More exactly, the *sub*-rules of Definition 4 are replaced by those of Definition 6.

### 4.2 Main Transformation

The main transformation processes an  $n$ -sntt  $M$  and yields an  $n$ -satt  $M'$ . The construction introduces a new  $(n + 1)$ -ary function symbol  $f$  for every function symbol  $f$  of  $M$ . The context arguments of the new  $f$  shall store replacements for the substitution constructors. Intuitively speaking, a call like  $(f\ t\ e_1 \dots e_n)$  should evaluate (using  $M'$ ) to the result of  $(sub\ (f\ t)\ e_1 \dots e_n)$  (evaluated using  $M$ ). Thereby the intermediate result that is produced by the call  $(f\ t)$  is avoided. The formalization of this intuitive relation can be found in Lemma 13. The construction uses an auxiliary function *sub* to transform right hand sides of rules thereby evaluating substitutions at compile time.

**Definition 11** Let  $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$  be an  $n$ -sntt. First, we define the set  $\overline{R_2}$  of *transformation rules* which contains

- for every  $j \in [n]$  a rule  $\underline{sub} \Pi_j y_1 \dots y_n = y_j$ ,
- for every  $c \in (C - \Pi)^{(k)}$  a rule<sup>4</sup>

$$\underline{sub} (c x_1 \dots x_k) y_1 \dots y_n$$

$$= (c (\underline{sub} x_1 z_1 \dots z_n) \dots (\underline{sub} x_k z_1 \dots z_n)) [z_i \rightsquigarrow y_i],$$
- for every  $f \in F$  a rule  $\underline{sub} (f x_0) y_1 \dots y_n = f x_0 y_1 \dots y_n$ ,
- and the rule<sup>5</sup>

$$\underline{sub} (sub x_0 x_1 \dots x_n) y_1 \dots y_n$$

$$= (\underline{sub} x_0 (\underline{sub} x_1 z_1 \dots z_n) \dots (\underline{sub} x_n z_1 \dots z_n)) [z_i \rightsquigarrow y_i].$$

Then, the  $n$ -satt constructed from  $M$  by accumulation is defined as  $acc(M) = (acc(F), C, acc(R_1), acc(r_{in}))$ , where

- $acc(F) = \{f^{(n+1)} \mid f \in F\}$ ,
- $acc(R_1)$  contains for every  $f \in acc(F)$  and  $c \in C^{(k)}$  the rule
$$f (c x_1 \dots x_k) y_1 \dots y_n = nf(\Rightarrow_{R_2}, \underline{sub} rhs_{R_1, f, c} y_1 \dots y_n),$$
- $acc(r_{in}) = nf(\Rightarrow_{R_2}, \underline{sub} r_{in} \Pi_1 \dots \Pi_n)$ . □

It can be shown easily that  $acc(M)$  is in fact a well-defined  $n$ -satt. In particular, the context-linearity is induced by the fact that the  $\underline{sub}$ -rules do not copy variables. Given the above intuition, the rules for  $\underline{sub}$  should be straightforward: The first rule avoids the explicit construction of  $\Pi_j$ -symbols. The second rule is standard and the third rule encodes our intuition. Finally, the fourth rule represents the “associativity” of substitutions. Note the similarity of these rules to the laws (1), (2), (\*), and (3), respectively, of [20]. In the second and fourth rule we use explicit sharings in order to avoid that occurrences of  $F$ -functions are copied and thus are executed more than once in the transformed program.

**Example 12** Let  $\tilde{M}_{non} = (F, Sub, C, \Pi, R_1, R_2, r_{in})$  be the 2-sntt from Example 7. Then, the set  $R_2$  contains the rules

$$\begin{aligned} \underline{g} A y_1 y_2 &= y_1 \\ \underline{g} B y_1 y_2 &= y_2 \\ \underline{g} (D x_1) y_1 y_2 &= D (\underline{g} x_1 y_1 y_2) \\ \underline{g} (T x_1 x_2 x_3) y_1 y_2 &= (T (\underline{g} x_1 z_1 z_2) (\underline{g} x_2 z_1 z_2) (\underline{g} x_3 z_1 z_2)) [z_i \rightsquigarrow y_i] \\ \underline{g} (f x_0) y_1 y_2 &= f x_0 y_1 y_2 \\ \underline{g} (g x_0 x_1 x_2) y_1 y_2 &= (\underline{g} x_0 (\underline{g} x_1 z_1 z_2) (\underline{g} x_2 z_1 z_2)) [z_i \rightsquigarrow y_i] \end{aligned}$$

and the 2-satt constructed from  $\tilde{M}_{non}$  by accumulation is defined as  $\tilde{M}_{acc} = acc(\tilde{M}_{non}) = (acc(F), C, acc(R_1), acc(r_{in}))$ , where  $acc(F) = \{f^{(3)}\}$ ,  $acc(R_1)$  contains the following rules with underlined left- and right-hand sides

$$\begin{aligned} \underline{f} Z y_1 y_2 &= nf(\Rightarrow_{R_2}, \underline{g} B y_1 y_2) = \underline{y_2}, \\ \underline{f} (S x_1) y_1 y_2 &= nf(\Rightarrow_{R_2}, \underline{g} (g (f x_1) (D A) (T A B A)) y_1 y_2) \\ &= nf(\Rightarrow_{R_2}, (\underline{g} (f x_1) (\underline{g} (D A) z_1 z_2) (\underline{g} (T A B A) z_1 z_2))) [z_i \rightsquigarrow y_i] \\ &= \underline{(f x_1 (D z_1) (T z_1 z_2 z_1)) [z_i \rightsquigarrow z_i]} [z_i \rightsquigarrow y_i] \end{aligned}$$

and  $acc(r_{in}) = nf(\Rightarrow_{R_2}, \underline{g} (f x_1) A B) = (f x_1 A B)$ . □

<sup>4</sup> If  $c$  is nullary or unary, then the explicit sharing will be avoided in examples.

<sup>5</sup> If  $n = 1$ , then the explicit sharing could be avoided.

The correctness proof<sup>6</sup> of the main transformation is based on the following lemma which formalizes our intuition from the beginning of this subsection.

**Lemma 13** Let  $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$  be an  $n$ -sntt and  $acc(M) = (acc(F), C, acc(R_1), acc(r_{in}))$ . For every  $f \in F$  and  $t \in T_C$ :

$$\underline{tree}(nf(\Rightarrow_{R_1 \cup R_2}, sub(f t) z_1 \dots z_n)) = \underline{tree}(nf(\Rightarrow_{acc(R_1)}, f t z_1 \dots z_n)). \quad \square$$

**Theorem 14** Let  $M$  be an  $n$ -sntt. Then,  $\tau_M = \tau_{acc(M)}$ .  $\square$

### 4.3 Postprocessing

Finally, an  $n$ -satt resulting from the main transformation is translated into a functional program by replacing in the right-hand sides of rules and in the initial right-hand side the explicit sharings with let-expressions. More exactly, an expression of the form

$$r[z_1 \rightsquigarrow r_1, \dots, z_n \rightsquigarrow r_n] \quad \text{is replaced by} \quad \text{let } \{v_1 = \bar{r}_1; \dots; v_n = \bar{r}_n\} \text{ in } \bar{r},$$

where  $v_1, \dots, v_n$  are fresh variables (which can be obtained using tree-structured addresses in the translation process) and  $\bar{r}_1, \dots, \bar{r}_n, \bar{r}$  result from recursively replacing explicit sharings in  $r_1, \dots, r_n, r$ , respectively, and additionally using  $v_1, \dots, v_n$  instead of the free occurrences of  $z_1, \dots, z_n$  in  $r$ .

**Example 15** Let  $\tilde{M}_{acc}$  be the 2-satt of Example 12. Postprocessing translates  $f(S x_1) y_1 y_2 = (f x_1 (D z_1) (T z_1 z_2 z_1)[z_i \rightsquigarrow z_i])[z_i \rightsquigarrow y_i]$  into the rule

$$f(S x_1) y_1 y_2 = \text{let } \{v_1 = y_1; v_2 = y_2\} \\ \text{in } f x_1 (D v_1) (\text{let } \{v_{11} = v_1; v_{12} = v_2\} \text{ in } (T v_{11} v_{12} v_{11})). \quad \square$$

A more elaborate translation could simplify (or even avoid) some let-expressions, e.g. if  $z_j$  does not occur or occurs only once freely in  $r$  or if  $r_j = z_j$  or  $r_j = y_j$  (i.e. we have  $z_j \rightsquigarrow z_j$  or  $z_j \rightsquigarrow y_j$ ). For the case  $r_j = y_j$  note that the resulting program will be again treated call-by-need, and hence  $y_j$  is shared implicitly.

**Example 16** Instead of constructing the rule as in Example 15, the following rule can be used (cf. also the introduction):

$$f(S x_1) y_1 y_2 = f x_1 (D y_1) (T y_1 y_2 y_1) \quad \square$$

## 5 Efficiency Non-deterioration by Accumulation

Our aim is to show the efficiency non-deterioration for call-by-need reduction. Unfortunately, it is technically difficult to formally compare the number of steps caused by deterministic reduction relations (cf. e.g. [13]). Hence we will base our comparison on the nondeterministic reduction relations for  $n$ -sntts and  $n$ -satts.

<sup>6</sup> Available online at <http://wwwtcs.inf.tu-dresden.de/~maletti/proof>

Therefore we first present a mechanism such that the number of call-by-need reduction steps caused by the  $R_1$ -rules of an  $n$ -ntt  $M$  equals the number of “relevant” nondeterministic reduction steps caused by the  $R_1$ -rules of the corresponding  $n$ -sntt  $\tilde{M}$ : In both reduction relations the copying of unevaluated applications of  $F$ -functions is avoided (by implicit and explicit sharing, respectively). But, whereas the deletion of a useless unevaluated application of an  $F$ -function is performed automatically in a call-by-need reduction, the nondeterministic reduction relation for  $\tilde{M}$  either simply evaluates such an application and later moves the result into a subexpression  $e_i$  of an expression of the form  $e[z_i \rightsquigarrow e_i]$  or, vice versa, the reduction relation for  $\tilde{M}$  first moves it into an  $e_i$  of an expression  $e[z_i \rightsquigarrow e_i]$ , where it is evaluated later. In both situations the normal form of  $e$  will not contain a free occurrence of  $z_i$ , but nevertheless the useless evaluation is performed! In order to consider only the relevant  $R_1$ -reduction steps, in our mechanism (i) every application of an  $R_1$ -rule will additionally generate a special symbol  $\circ$  and (ii) in the normal form only those  $\circ$ -symbols are counted by a function step, which do not occur in a subexpression  $e_i$  of an expression  $e[z_i \rightsquigarrow e_i]$ , where  $z_i$  does not occur freely in  $e$ .

Then we use the same counting mechanism for the  $n$ -satt  $acc(\tilde{M})$  in order to prove that the number of relevant  $R_1$ -reduction steps of  $\tilde{M}$  equals the number of relevant reduction steps of  $acc(\tilde{M})$ . Together with a final argumentation that the postprocessing phase does not change the number of reduction steps, we obtain the desired efficiency result. Note that our comparison procedure does not consider the  $R_2$ -reduction steps of  $M$  or  $\tilde{M}$ , which do not occur in  $acc(\tilde{M})$  and hence are saved by accumulation!

In the following we assume that  $\circ$  is a new unary symbol and for every ranked alphabet  $C$  we define  $C^\circ = C \cup \{\circ\}$ .

**Definition 17** Let  $M = (F, Sub, C, II, R_1, R_2, r_{in})$  be an  $n$ -sntt. The  $n$ -sntt  $M^\circ = (F, Sub, C^\circ, II, R_1^\circ, R_2^\circ, r_{in})$  is defined by

- if  $R_1$  contains a rule  $f(c x_1 \dots x_k) = rhs_{R_1, f, c}$ ,  
then  $R_1^\circ$  contains a rule  $f(c x_1 \dots x_k) = \circ rhs_{R_1, f, c}$ , and
- $R_1^\circ$  contains for every  $f \in F$  a (dummy; never applied) rule  $f(\circ x_1) = \dots$ ,
- $R_2^\circ$  contains the rules of  $R_2$ , and
- $R_2^\circ$  contains the rule  $sub(\circ x_1) y_1 \dots y_n = \circ(sub x_1 y_1 \dots y_n)$ .

Let  $M = (F, C, R, r_{in})$  be an  $n$ -satt. The  $n$ -satt  $M^\circ = (F, C^\circ, R^\circ, r_{in})$  is defined by

- if  $R$  contains a rule  $f(c x_1 \dots x_k) y_1 \dots y_n = rhs_{R, f, c}$ ,  
then  $R^\circ$  contains a rule  $f(c x_1 \dots x_k) y_1 \dots y_n = \circ rhs_{R, f, c}$ , and
- $R^\circ$  contains for every  $f \in F$  a (dummy) rule  $f(\circ x_1) y_1 \dots y_n = \dots$  □

Note that by the additional *sub*-rule of  $R_2^\circ$  in the previous definition the  $\circ$ -symbols produced by  $R_1^\circ$ -rules are retained.

**Definition 18** Let  $C$  be a ranked alphabet.

The function  $\underline{step} : E_{C^\circ, n}(Z_n) \rightarrow \mathbb{N}$  is defined as follows:

- For every  $e \in E_{C^\circ, n}(Z_n)$ :  $\underline{step}(\circ e) = 1 + \underline{step}(e)$ .
- For every  $c \in C^{(k)}$  and  $e_1, \dots, e_k \in E_{C^\circ, n}(Z_n)$ :  
 $\underline{step}(c e_1 \dots e_k) = \sum_{i=1}^k \underline{step}(e_i)$ .
- For every  $e_1, \dots, e_n, e \in E_{C^\circ, n}(Z_n)$ :  
 $\underline{step}(e[z_i \rightsquigarrow e_i]) = \underline{step}(e) + \sum_{i=1}^n (\underline{step}(e_i) * \underline{rel}(z_i, e))$ .
- For every  $i \in [n]$ :  $\underline{step}(z_i) = 0$ .

The function  $\underline{rel} : Z_n \times E_{C^\circ, n}(Z_n) \rightarrow \{0, 1\}$  is for every  $i \in [n]$  and  $e \in E_{C^\circ, n}(Z_n)$  defined by  $\underline{rel}(z_i, e) = 1$  iff  $z_i$  occurs in  $\underline{tree}(e)$ .  $\square$

**Example 19** Since the phenomenon of non-relevant subexpressions does not occur in our running example, we choose an artificial example here:

$$\begin{aligned} & \underline{step}((\circ z_1)[z_1 \rightsquigarrow (\circ A)][z_1 \rightsquigarrow (\circ A)]) \\ &= \underline{step}((\circ z_1)[z_1 \rightsquigarrow (\circ A)]) + \underline{step}(\circ A) * \underline{rel}(z_1, (\circ z_1)[z_1 \rightsquigarrow (\circ A)]) \\ &= \underline{step}(\circ z_1) + \underline{step}(\circ A) * \underline{rel}(z_1, (\circ z_1)) + 1 * 0 = 1 + 1 * 1 + 1 * 0 = 2 \quad \square \end{aligned}$$

Now we have to consider again our three transformation steps, where the second step involves a formal proof and the first and last step are argumentations concerning call-by-need reduction, which we avoided to define formally.

## 5.1 Preprocessing

For an  $n$ -ntt  $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$  and a term  $t \in T_C$  we will denote by  $cbn_{R_1}(t)$  the number of  $R_1$ -reduction steps which are used to reduce  $r_{in}[x_1/t]$  to a term graph corresponding to  $nf(\Rightarrow_{R_1 \cup R_2}, r_{in}[x_1/t])$  with a call-by-need reduction. Let  $\tilde{M} = (F, \{sub\}, C, \Pi, \tilde{R}_1, \tilde{R}_2, \tilde{r}_{in})$  result from  $M$  by preprocessing. Then we have to argue that

$$cbn_{R_1}(t) = \underline{step}(nf(\Rightarrow_{\tilde{R}_1^\circ \cup \tilde{R}_2^\circ}, \tilde{r}_{in}[x_1/t])).$$

First we only consider the  $F$ -functions: Since every application of a rule in  $\tilde{R}_1^\circ$  delivers exactly one occurrence of  $\circ$ , the number of occurrences of  $\circ$  in  $nf(\Rightarrow_{\tilde{R}_1^\circ}, \tilde{r}_{in}[x_1/t])$  equals the number of applied  $\tilde{R}_1$ -steps to calculate  $nf(\Rightarrow_{\tilde{R}_1}, \tilde{r}_{in}[x_1/t])$ . This number is in turn equal to the number of applied  $R_1$ -steps to calculate a term graph corresponding to  $nf(\Rightarrow_{R_1}, r_{in}[x_1/t])$  with call-by-need, because occurrences of  $F$ -functions are not nested and hence are neither copied nor deleted.

Now we additionally consider the substitution function  $sub$ : (i) Occurrences of  $F$ -functions in the recursion argument of  $sub$  are neither copied nor deleted in a reduction by  $R_1 \cup R_2$ . Correspondingly, in a reduction by  $\tilde{R}_1^\circ \cup \tilde{R}_2^\circ$  occurrences of  $\circ$  in the recursion argument of  $sub$  are exactly once reproduced and counted by  $\underline{step}$ . (ii) In a call-by-need reduction by  $R_1 \cup R_2$  no application of an  $F$ -function inside a context argument of  $sub$  is copied and also in a reduction by  $\tilde{R}_1^\circ \cup \tilde{R}_2^\circ$  (with explicit sharing) no corresponding occurrence of  $\circ$  is copied. (iii) But, in a call-by-need reduction by  $R_1 \cup R_2$ , every  $R_1$ -step which constitutes a subgraph of the term graph corresponding to  $nf(\Rightarrow_{R_1}, r_{in}[x_1/t])$ , such that the subgraph occurs in a deleted context argument position  $j$  of an occurrence of  $sub$  will not

be executed, whereas a reduction by  $\tilde{R}_1^\circ \cup \tilde{R}_2^\circ$  may behave differently: either the occurrence of  $\circ$  in  $nf(\Rightarrow_{\tilde{R}_1^\circ}, \tilde{r}_{in}[x_1/t])$  that corresponds to the above  $R_1$ -step is also deleted (by a *sub*-rule on a  $\Pi_i$  with  $i \neq j$ ) or it is shifted into a subexpression  $e_j$  of an expression of the form  $e[z_i \rightsquigarrow e_i]$  in which  $z_j$  does not occur freely in  $e$  and hence it is not counted by *step*.

## 5.2 Main Transformation

The proof<sup>7</sup> of efficiency non-deterioration of the main transformation is based on the following lemma. Note the similarity of this lemma to Lemma 13: Instead of the reduction relations of  $M$  and  $acc(M)$ , their “ $\circ$ -generating versions” are used here. Moreover, instead of calculating the output tree by *tree*, the number of  $\circ$ -symbols is counted by *step*.

**Lemma 20** Let  $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$  be an  $n$ -sntt and  $acc(M) = (acc(F), C, acc(R_1), acc(r_{in}))$ . For every  $f \in F$  and  $t \in T_C$ :

$$\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, sub(f\ t\ z_1 \dots z_n))) = \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, f\ t\ z_1 \dots z_n)). \quad \square$$

**Theorem 21** Let  $M = (F, Sub, C, \Pi, R_1, R_2, r_{in})$  be an  $n$ -sntt and  $acc(M) = (acc(F), C, acc(R_1), acc(r_{in}))$ . Then, for every  $t \in T_C$ :

$$\underline{step}(nf(\Rightarrow_{R_1^\circ \cup R_2^\circ}, r_{in}[x_1/t])) = \underline{step}(nf(\Rightarrow_{acc(R_1)^\circ}, acc(r_{in})[x_1/t])). \quad \square$$

## 5.3 Postprocessing

Let-expressions do not cause additional reduction steps, rather they denote in functional languages explicit sharings. Thus, if we denote for an  $n$ -satt  $M = (F, C, R, r_{in})$ , for  $\tilde{R}$  and  $\tilde{r}_{in}$  obtained from  $R$  and  $r_{in}$ , respectively, by introducing let-expressions, and for a term  $t \in T_C$ , by  $cbn_{\tilde{R}}(t)$  the number of call-by-need reduction steps which are used to reduce  $\tilde{r}_{in}[x_1/t]$  to a term graph corresponding to  $nf(\Rightarrow_{\tilde{R}}, r_{in}[x_1/t])$  with  $\tilde{R}$ , then it suffices to argue that

$$cbn_{\tilde{R}}(t) = \underline{step}(nf(\Rightarrow_{R^\circ}, r_{in}[x_1/t])).$$

The introduction of let-expressions does not change the copying or deletion properties of the rules in  $R$ . Moreover, in a call-by-need reduction by  $\tilde{R}$  no function application inside some context argument is copied, and because of the context-linearity of  $R$  and hence also of  $R^\circ$ , no corresponding occurrence of  $\circ$  is copied in a reduction by  $R^\circ$ . Hence there is only one main difference in the calculations of  $\tilde{R}$  and  $R^\circ$ : On the one hand, in a call-by-need reduction by  $\tilde{R}$  a function application is not evaluated, if it occurs in a subexpression  $e_j$  of an expression of the form *let*  $\{v_1 = e_1; \dots; v_n = e_n\}$  *in*  $e$ , in which  $e$  is evaluated and  $v_j$  does not occur in  $e$ . On the other hand, the corresponding function application in a subexpression  $e'_j$  of an expression of the form  $e'[z_i \rightsquigarrow e'_i]$  (where  $v_1, \dots, v_n$  correspond to  $z_1, \dots, z_n$ , respectively) is evaluated by  $R^\circ$  and a symbol  $\circ$  is produced. But since  $z_j$  does not occur freely in  $e'$ , the produced  $\circ$  is not counted by *step*.

<sup>7</sup> Available online at <http://wwwtcs.inf.tu-dresden.de/~maletti/proof>

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