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Incomparability Results for Classes of Polynomial Tree Series Transformations

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ABSTRACT

Polynomial bottom-up and top-down tree series transducers over partially ordered semirings are considered, and the classes of ε-tree-to-tree-series (for short: ε-t-ts) and o-tree-to-tree-series (for short: o-t-ts) transformations computed by such transducers are compared. The main result is the following. Let $\mathcal{A}$ be a weakly growing semiring and $x, y \in \{\text{deterministic}, \text{homomorphism}\}$. The class of o-t-ts transformations computed by x bottom-up tree series transducers over $\mathcal{A}$ is incomparable (with respect to set inclusion) with the class of ε-t-ts transformations computed by y bottom-up tree series transducers over $\mathcal{A}$. Moreover, the latter class is incomparable with the class of ε-t-ts transformations computed by x top-down tree series transducers over $\mathcal{A}$. If additionally $\mathcal{A}$ is additively idempotent, then the above statements even hold for every $x, y \in \{\text{polynomial}, \text{deterministic}, \text{homomorphism}\}$.

1. Introduction

Tree series transducers [40, 19, 25, 27] were introduced as a joint generalization of tree transducers [45, 47, 16] and weighted tree automata [46, 38, 7]. Both historical predecessors of tree series transducers have successfully been motivated from and applied in practice. Specifically, tree transducers are motivated from syntax-directed translations in compilers [33, 17, 26], and they are applied in, e.g., functional program analysis and transformation [37, 30, 34, 48], computational linguistics [43, 36, 42, 35], generation of pictures [11, 12], and query languages of XML databases [3, 20]. Weighted tree automata have been applied to code selection in compilers [24, 5] and tree pattern matching [46]. Moreover, a rich theory of tree transducers was developed (see [16, 1, 18] as seminal papers and [28, 44, 9, 29, 26] as survey papers and monographs) during the seventies, whereas weighted tree automata just recently received some attention (e.g., [46, 38, 4, 6, 13, 14, 23, 15]).

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Roughly speaking, tree series transducers capture both (a) the way of translating input trees into output trees, as it is inherent in bottom-up and top-down tree transducers, and (b) the computation of a weight (or cost) in a semiring, as it is inherent in weighted tree automata. More formally, a (bottom-up or top-down) tree series transducer is a tuple $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$, where Q is a finite set of states, Σ and Δ are ranked alphabets of input and output symbols, respectively, $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring, $D \subseteq Q$ is a set of designated states (also called final states if M is bottom-up, or initial states if M is top-down), and $\mu = (\, \mu_k \mid k \in \mathbb{N} \,)$ is a (bottom-up or top-down) tree representation. The tree representation consists of mappings μ_k that map the set of k-ary symbols of Σ, denoted by $\Sigma^{(k)}$, into $(Q \times Q(X_k)^*)$-matrices over $A\langle\langle T_\Delta(X)\rangle\rangle$, where $Q(X_k) = \{\, q(x_i) \mid q \in Q, 1 \leq i \leq k \,\}$, $T_\Delta(X)$ denotes the set of Δ-trees indexed by variables of $X = \{x_1, x_2, \ldots\}$, and $A\langle\langle T_\Delta(X)\rangle\rangle$ denotes the set of mappings $\varphi : T_\Delta(X) \longrightarrow A$ (called tree series). Using m-substitution of tree series (with $m \in \{\varepsilon, \mathrm{o}\}$; see [27]) in order to substitute tree series into tree series, we can impose a Σ-algebraic structure on $A\langle\langle T_\Delta\rangle\rangle^Q$ and thereby obtain the unique Σ-homomorphism $h^m_\mu : T_\Sigma \longrightarrow A\langle\langle T_\Delta\rangle\rangle^Q$. Then the m-tree-to-tree-series (for short: m-t-ts) transformation computed by M is the mapping $\tau^m_M : T_\Sigma \longrightarrow A\langle\langle T_\Delta\rangle\rangle$ defined by $\tau^m_M(s) = \sum_{q\in D} h^m_\mu(s)_q$. Thus, for a given input tree $s \in T_\Sigma$, M computes a (potentially infinite) set $\mathrm{supp}(\tau^m_M(s)) = \{\, t \in T_\Delta \mid (\tau^m_M(s), t) \neq \mathbf{0} \,\}$ of output trees and associates a coefficient $(\tau^m_M(s), t) \in A$ to every output tree $t \in T_\Delta$. Note that $(\tau^m_M(s), t)$ denotes the application $\varphi(t)$ with $\varphi = \tau^m_M(s)$. For every so-called polynomial tree series transducer M and input tree $s \in T_\Sigma$, the set $\mathrm{supp}(\tau^m_M(s))$ of computed and relevant output trees is finite. Polynomial bottom-up and top-down tree series transducers over the boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$ essentially are bottom-up and top-down tree transducers, respectively (see Section 4 of [19]).

In the same way as tree transducers, also tree series transducers can have particular properties, e.g., they can be deterministic or they are homomorphisms (see, e.g., [16]). Note that homomorphism tree series transducers are deterministic and deterministic tree series transducers are polynomial. The classes of m-t-ts transformations computed by bottom-up and top-down tree series transducers having the property x (e.g., being deterministic) over $\mathcal{A}$ are denoted by x–$\mathrm{BOT}^m(\mathcal{A})$ and x–$\mathrm{TOP}^m(\mathcal{A})$, respectively.

In [27] several classes of the form x–$\mathrm{BOT}^m(\mathcal{A})$ and x–$\mathrm{TOP}^m(\mathcal{A})$ have been compared with respect to set inclusion. For instance, it was proved that:

- x–$\mathrm{TOP}^\varepsilon(\mathcal{A}) = x$–$\mathrm{TOP}^\mathrm{o}(\mathcal{A})$, see Theorem 5.2 of [27], for every semiring $\mathcal{A}$ and $x \in \{\text{polynomial}, \text{deterministic}, \text{homomorphism}\}$;
- p–$\mathrm{BOT}^\varepsilon(\mathbb{N}_\infty) \bowtie$ p–$\mathrm{BOT}^\mathrm{o}(\mathbb{N}_\infty)$, where p stands for polynomial, the semiring $\mathbb{N}_\infty$ of non-negative integers (with infinity) is $(\mathbb{N}\cup\{\infty\}, +, \cdot, 0, 1)$, and $\bowtie$ denotes incomparability with respect to set inclusion (see Corollary 5.18 of [27]); and
- p–$\mathrm{BOT}^\varepsilon(\mathbb{T}) \bowtie$ p–$\mathrm{BOT}^\mathrm{o}(\mathbb{T})$, where $\mathbb{T} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ is the tropical semiring (see Corollary 5.23 of [27]).

The latter two incomparability results motivated us to investigate the question whether this incomparability also holds for semirings different from $\mathbb{N}_\infty$ and $\mathbb{T}$. In this paper we answer this question in the affirmative. Additionally, we compare classes of ε-t-ts transformations that are computed by different types of tree series transducers; i.e., bottom-up and top-down tree series transducers. Our main result is Theorem 5.10, which states the following:

> We have that x–$\mathrm{BOT}^{\mathrm{o}}(\mathcal{A}) \bowtie y$–$\mathrm{BOT}^{\varepsilon}(\mathcal{A})$ and y–$\mathrm{BOT}^{\varepsilon}(\mathcal{A}) \bowtie x$–$\mathrm{TOP}^{\varepsilon}(\mathcal{A})$ for every $x, y \in \{\text{deterministic}, \text{homomorphism}\}$ and weakly growing semiring $\mathcal{A}$. Provided that $\mathcal{A}$ is additively idempotent, x and y may even be polynomial in this statement.

Let us add some details and then briefly discuss the way how to prove this theorem. A partially ordered semiring $(A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ is a semiring $(A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ with a partial order $\preceq$ on A such that the order is preserved by both semiring operations. The partially ordered semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ is called weakly growing, if:

(i) there is an $a \in A$ such that $a^i \prec a^j$ for all non-negative integers $i < j$; and

(ii) for every $a_1, a_2, b \in A \setminus \{\mathbf{0}\}$, $d \in A$, and $n \in \mathbb{N}$, if $a^n = a_1 \odot b \odot a_2 \oplus d$, then there is an $m \in \mathbb{N}$ such that $b \preceq a^m$.

Roughly speaking, Condition (ii) requires that every element b that occurs in a decomposition of a power of a can be bounded (from above) by another power of a. In particular, the following semirings are weakly growing:

- $\mathbb{N}_\infty$ with $\preceq \,=\, \leq$, $a = 2$, and $m = n$;
- $\mathbb{T}$ with $\preceq \,=\, \leq$, $a = 1$, and $m = \max(n, d)$;
- the arctic semiring $\mathbb{A} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ with $\preceq \,=\, \leq$, $a = 1$, and $m = n$; and
- the formal language semiring $(\mathcal{P}(S^*), \cup, \circ, \emptyset, \{\varepsilon\})$ over the alphabet S with $\preceq \,=\, \subseteq$, $a = \{\varepsilon, s\}$ for some $s \in S$, and $m = n$.

In order to prove the non-inclusion results of the main theorem, we use the partial order on the semiring and establish a framework of mappings called coefficient majorizations. For a given m-t-ts transformation $\tau : T_\Sigma \longrightarrow A\langle\langle T_\Delta \rangle\rangle$, a coefficient majorization $f : \mathbb{N}_+ \longrightarrow A$ is a mapping such that $f(n)$ is an upper bound of the set $C_\tau(n)$, which is the set of all coefficients generated from input trees of height at most n; i.e.,

$$C_\tau(n) = \{\, (\tau(s), t) \mid s \in T_\Sigma, \mathrm{height}(s) \leq n, t \in \mathrm{supp}(\tau(s)) \,\}\,.$$

Given two classes $\mathfrak{T}_1$ and $\mathfrak{T}_2$ of transformations, we can prove $\mathfrak{T}_1 \not\subseteq \mathfrak{T}_2$ by exhibiting (i) a mapping f that is a coefficient majorization for the class $\mathfrak{T}_2$ (i.e., a coefficient majorization for every $\tau \in \mathfrak{T}_2$) and (ii) a transformation $\tau \in \mathfrak{T}_1$ for which f is no coefficient majorization. For particular classes, this is achieved in Lemma 5.9.

Coefficient majorizations have been investigated for the specific case in which the coefficient $(\tau(s), t)$ is the height or size of the output tree t, e.g., for top-down tree transducers (see Lemma 3.27 of [26]), for attributed tree transducers (see Lemmata 3.3 of [17] and 5.40 of [26]), for macro tree transducers (see Lemmata 3.3 of [17] and 4.22 of [26]), and for bottom-up tree transducers (which have the same coefficient majorization as top-down tree transducers, which follows in a straightforward manner from Theorem 3.15 in [16]).

This paper is structured as follows. Section 2 recalls the relevant basic mathematical notions and notations, in particular partially ordered semirings, tree series, and substitution of tree series. Section 3 presents the definition of tree series transducers from [19] in some detail along with the definition of several subclasses of tree series transducers. Section 4 establishes a coefficient majorization for the m-t-ts transformation computed by a polynomial tree series transducer that is bottom-up or top-down. Finally, in Section 5 the incomparability results outlined above are derived.

2. Preliminaries

In this section we present some basic notions and notations required in the sequel. Section 2.1 recalls partial orders [10] and associated notions. Words [41] and trees [28, 29] are considered in Section 2.2, whereas Section 2.3 is dedicated to algebraic structures and, in particular, (partially ordered) semirings [39, 32, 31]. Finally, this section is concluded by the presentation of formal tree series [2, 39] and two definitions of substitution for tree series [19, 27].

2.1. Partial Orders

The set $\{0, 1, 2, \dots\}$ of all non-negative integers is denoted by $\mathbb{N}$, and we let $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. For every $i, j \in \mathbb{N}$ we denote the interval $\{\, n \in \mathbb{N} \mid i \leq n \leq j \,\}$ by $[i, j]$, and we use $[j]$ to abbreviate $[1, j]$. The *cardinality* of a set S (i.e., the number of elements of S) is denoted by $\mathrm{card}(S)$, and for every set S the *power set of* S, comprising all subsets of S, is denoted by $\mathcal{P}(S)$.

Given a non-empty set A, a binary relation $\preceq \subseteq A \times A$ is called *partial order (on* A*)*, if $\preceq$ is reflexive, antisymmetric, and transitive. The pair $(A, \preceq)$ is termed *partially ordered set*, and we represent the pair by A alone whenever $\preceq$ is understood from the context. Let $a_1, a_2 \in A$. The fact that neither $a_1 \preceq a_2$ nor $a_2 \preceq a_1$ (or equivalently: a_1 and a_2 are *incomparable*) is expressed as $a_1 \bowtie a_2$. In case there are no incomparable elements, the partial order $\preceq$ is said to be a *total order*. As usual, the *strict order* $\prec \subseteq A \times A$ corresponding to $\preceq$ is defined by $a_1 \prec a_2$, if and only if $a_1 \preceq a_2$ and $a_1 \neq a_2$.

Let $S \subseteq A$. An element $u \in A$ is called *upper bound of* S, if $s \preceq u$ for every $s \in S$. The set of all upper bounds of S is denoted by $\uparrow S$, and if $\uparrow S$ contains a least element, then it is called *supremum of* S and denoted by $\sup S$. If, for every $a_1, a_2 \in A$ there is an upper bound of $\{a_1, a_2\}$ (i.e., $\uparrow\{a_1, a_2\} \neq \emptyset$), then A is called *directed*.

2.2. Words and trees

By a *word of length* $n \in \mathbb{N}$ we mean an element of the n-fold Cartesian product $S^n = S \times \cdots \times S$ of a set S. The set of all words over S is denoted by S^*, where the particular element $() \in S^0$, called the *empty word*, is displayed as ε. The *length of a word* $w \in S^*$ is denoted by $|w|$; thus $|\varepsilon| = 0$.

Every non-empty and finite set is called *alphabet*; its elements are termed *symbols*. A *ranked alphabet* is defined to be a pair (Σ, rk), where Σ is an alphabet and the mapping $\mathrm{rk}: \Sigma \longrightarrow \mathbb{N}$ associates to every symbol a *rank*. For every $k \in \mathbb{N}$ we let $\Sigma^{(k)} = \{\, \sigma \in \Sigma \mid \mathrm{rk}(\sigma) = k \,\}$ be the set of symbols having rank k. In the following, we usually assume that rk is implicitly given. Hence we identify (Σ, rk) with Σ and specify the ranked alphabet by listing the symbols with their rank put in parentheses as superscripts as in $\{\sigma^{(2)}, \alpha^{(0)}\}$. The *maximal rank* of the ranked alphabet Σ, denoted by mx_Σ, is the maximal $n \in \mathbb{N}$ such that $\Sigma^{(n)} \neq \emptyset$.

Let Σ be a ranked alphabet and $X = \{\, x_i \mid i \in \mathbb{N}_+ \,\}$ be a set of (formal) variables. The set of (finite, labeled, and ordered) Σ-*trees indexed by* $V \subseteq X$, denoted by $T_\Sigma(V)$, is inductively defined to be the smallest set T such that (i) $V \subseteq T$ and (ii) for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T$ also $\sigma(t_1, \ldots, t_k) \in T$. We use T_Σ as an abbreviation for $T_\Sigma(\emptyset)$ and usually write α instead of $\alpha()$ whenever $\alpha \in \Sigma^{(0)}$. The number of occurrences of a given $x \in X$ in $t \in T_\Sigma(X)$ is denoted by $|t|_x$. Since we often deal with finite subsets of X, we let $X_n = \{\, x_i \mid i \in [n] \,\}$ for every $n \in \mathbb{N}$ (note that $X_0 = \emptyset$).

We distinguish a subset $\widehat{T_\Sigma}(X_n)$ of $T_\Sigma(X_n)$ as follows. Let $t \in T_\Sigma(X_n)$ be in $\widehat{T_\Sigma}(X_n)$, if and only if for every $i \in [n]$ the variable x_i occurs exactly once in t and the variables occur in the order $x_1, \ldots, x_n$ when reading t from left to right. Moreover, we recursively define the mapping $\mathrm{height}: T_\Sigma(X) \longrightarrow \mathbb{N}_+$ by the following equalities:

- $\mathrm{height}(x) = 1$ for every $x \in X$; and
- $\mathrm{height}(\sigma(t_1, \ldots, t_k)) = 1 + \max\{\, \mathrm{height}(t_i) \mid i \in [k] \,\}$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma(X)$.

Given $n \in \mathbb{N}$, $t \in T_\Sigma(X_n)$, and $s_1, \ldots, s_n \in T_\Sigma(X)$, the expression $t[s_1, \ldots, s_n]$ denotes the result of replacing (in parallel) every occurrence of x_i in t with s_i for every $i \in [n]$; i.e., $x_i[s_1, \ldots, s_n] = s_i$ for every $i \in [n]$ and

$$\sigma(t_1, \ldots, t_k)[s_1, \ldots, s_n] = \sigma(t_1[s_1, \ldots, s_n], \ldots, t_k[s_1, \ldots, s_n])$$

for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma(X_n)$. Let Σ be a ranked alphabet with exactly one non-nullary symbol, more precisely $\bigcup_{k \in \mathbb{N}_+} \Sigma^{(k)} = \{\sigma\}$. The set of *fully balanced trees (over* Σ*)* is defined to be the smallest $T \subseteq T_\Sigma$ such that $\alpha \in T$ for every $\alpha \in \Sigma^{(0)}$, and whenever $t \in T$ also $\sigma(\underbrace{t, \ldots, t}_{\mathrm{rk}(\sigma)}) \in T$.

2.3. Monoids and (partially ordered) semirings

A *monoid* is an algebraic structure $\mathcal{A} = (A, \otimes, \mathbf{1})$ consisting of a *carrier set* A with an associative binary operation $\otimes: A \times A \longrightarrow A$ and a unit $\mathbf{1} \in A$. By a *semiring (with one and absorbing zero)* we mean an algebraic structure $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ such that $(A, \oplus, \mathbf{0})$ is a commutative monoid, $(A, \odot, \mathbf{1})$ is a monoid, multiplication distributes over addition, and $\mathbf{0}$ acts as a zero for multiplication; i.e., $a \odot \mathbf{0} = \mathbf{0} = \mathbf{0} \odot a$ for every $a \in A$. By convention, we assume that multiplication has a higher (binding) priority than addition; e.g., we read $a_1 \oplus a_2 \odot a_3$ as $a_1 \oplus (a_2 \odot a_3)$.

For every $a \in A$ and $n \in \mathbb{N}$ we denote by a^n the product $a \odot \cdots \odot a$ containing n times the factor a and set $a^0 = \mathbf{1}$. Moreover, given $a_i \in A$ for $i \in [n]$, we also let $\sum_{i \in [n]} a_i = a_1 \oplus \cdots \oplus a_n$ and $\prod_{i \in [n]} a_i = a_1 \odot \cdots \odot a_n$. Note that $\sum_{i \in \emptyset} a_i = \mathbf{0}$ and $\prod_{i \in \emptyset} a_i = \mathbf{1}$. Finally, we also use the sum over arbitrary index sets, given that only finitely many summands are non-zero, exploiting that $\oplus$ is commutative (i.e., the order in which elements are summed up is irrelevant). Important semirings are, for example:

- the *non-negative integers (with infinity)* $\mathbb{N}_\infty = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ with the common operations of addition and multiplication extended to ∞ by $a + \infty = \infty$ and $a \cdot \infty = \infty = \infty \cdot a$ for every element $a \in \mathbb{N}_+ \cup \{\infty\}$;
- the *tropical semiring* $\mathbb{T} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ with minimum and addition extended to ∞ such that ∞ is the unit of min and $+$ is the addition of $\mathbb{N}_\infty$;
- the *arctic semiring* $\mathbb{A} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ with maximum and addition extended to $-\infty$ such that $-\infty$ is the unit of max and acts as a zero for $+$;
- the *boolean semiring* $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ with the usual operations of disjunction and conjunction;
- the semiring $\mathbb{Z}/4\mathbb{Z} = (\{0, 1, 2, 3\}, +, \cdot, 0, 1)$ with the usual addition and multiplication modulo 4;
- the *min-max semiring* $\mathbb{R}_{\min,\max} = (\mathbb{R} \cup \{\infty, -\infty\}, \min, \max, \infty, -\infty)$ with the common minimum and maximum operations; and
- the *language semiring* $\mathbb{L}_S = (\mathcal{P}(S^*), \cup, \circ, \emptyset, \{\varepsilon\})$ for some alphabet S with union and concatenation of words lifted to sets of words.

Several more examples of semirings can be found, e.g., in [32, 31]. For the sake of simplicity, we assume $\mathbf{0} \neq \mathbf{1}$ for all semirings we consider; i.e., we ignore the trivial semiring with the singleton carrier set.

A semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ with finite carrier set A is called *finite*, and $\mathcal{A}$ is called *additively idempotent*, if it fulfils $\mathbf{1} \oplus \mathbf{1} = \mathbf{1}$. Clearly, this immediately yields $a \oplus a = a$ for every $a \in A$ by distributivity. Finally, $\mathcal{A}$ is called *multiplicatively periodic*, if for every $a \in A$ there exist $i, j \in \mathbb{N}$ such that $i < j$ and $a^i = a^j$. For example, the semirings $\mathbb{B}$, $\mathbb{Z}/4\mathbb{Z}$, and $\mathbb{R}_{\min,\max}$ are multiplicatively periodic, whereas $\mathbb{N}_\infty$, $\mathbb{A}$, $\mathbb{T}$, and $\mathbb{L}_S$ are not. Obviously, every finite semiring is multiplicatively periodic.

Now we consider semirings endowed with a partial order. Given a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ and a partial order $\preceq \subseteq A \times A$, we say that $\preceq$ *partially orders* $\mathcal{A}$, or equivalently, $\mathcal{A}$ is *partially ordered (by $\preceq$)*, if for every $a_1, a_2, b_1, b_2 \in A$:

(PO$\oplus$) if $a_1 \preceq a_2$ and $b_1 \preceq b_2$, then $a_1 \oplus b_1 \preceq a_2 \oplus b_2$; and

(PO$\odot$) if $a_1 \preceq a_2$ and $b_1 \preceq b_2$, then $a_1 \odot b_1 \preceq a_2 \odot b_2$.

The literature contains several different notions of partially ordered semirings. For example, in [32, 31] the axiom (PO$\odot$) is only demanded for multiplication with *positive* elements $a \in A$ (i.e., $\mathbf{0} \preceq a$). The positive elements of such a partially ordered semiring form a partially ordered subsemiring according to our definition. In contrast, the definition of partially ordered semirings in [39] requires every element to be positive.

In the sequel we denote a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ partially ordered by $\preceq$ simply by $(A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ and call it *totally ordered* if $\preceq$ is a total order. The set $P_\mathcal{A} = \{ a \in A \mid \mathbf{0} \preceq a \}$ is called the *positive cone* of $\mathcal{A}$ (via $\preceq$) and $N_\mathcal{A} = \{ a \in A \mid a \preceq \mathbf{0} \}$ is called the *negative cone* of $\mathcal{A}$ (note that $P_\mathcal{A} \cap N_\mathcal{A} = \{\mathbf{0}\}$). Moreover, we say that $\mathcal{A}$ has the growth property (G$\oplus$) whenever $\mathbf{1} \preceq \mathbf{1} \oplus \mathbf{1}$. Note that (G$\oplus$) implies that $a \preceq a \oplus a$ for every $a \in A$. Moreover, every additively idempotent and partially ordered semiring trivially satisfies (G$\oplus$). Finally, if every element of a partially ordered semiring is comparable to $\mathbf{0}$, then Observation 2.1(i,ii) characterizes (G$\oplus$).

2.1 Observation Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ be a partially ordered semiring.

(i) If $\mathbf{0} \prec \mathbf{1}$, then $A = P_\mathcal{A}$ and (G$\oplus$) is satisfied.

(ii) If $\mathbf{1} \prec \mathbf{0}$, then $A = N_\mathcal{A}$ and (G$\oplus$) implies that $\mathcal{A}$ is additively idempotent.

(iii) If $\mathbf{0} \prec \mathbf{1}$ or $\mathbf{1} \prec \mathbf{0}$, then A is directed.

Proof. The first parts of Statements (i) and (ii) are immediate by (PO$\odot$). If $\mathbf{0} \prec \mathbf{1}$ then $\mathbf{1} \preceq \mathbf{1} \oplus \mathbf{1}$ by (PO$\oplus$). If $\mathbf{1} \prec \mathbf{0}$ then $\mathbf{1} \oplus \mathbf{1} \preceq \mathbf{1}$ by (PO$\oplus$). Together with $\mathbf{1} \preceq \mathbf{1} \oplus \mathbf{1}$ from (G$\oplus$) we obtain $\mathbf{1} = \mathbf{1} \oplus \mathbf{1}$ and thus additive idempotency. Finally, let us prove Statement (iii). Let $a, b \in A$. We claim that there is a $c \in A$ with $a \preceq c$ and $b \preceq c$. Since $A = P_\mathcal{A} \cup N_\mathcal{A}$, we may assume that $a, b \in P_\mathcal{A}$. Now put $c = a \oplus b$ to obtain the claim and the result. ■

We conclude that a totally ordered semiring $\mathcal{A}$ possesses property (G$\oplus$) if and only if $\mathcal{A}$ is partially ordered in the sense of [39] or additively idempotent. The next observation groups together some simple statements concerning partially ordered semirings. The statements lift several conditions like (PO$\oplus$), (PO$\odot$), and (G$\oplus$) from exactly two elements to several elements. The proofs are straightforward and therefore left to the reader.

2.2 Observation Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ be a partially ordered semiring.

(i) Let $n \in \mathbb{N}$, and let $a_i, b_i \in A$ for $i \in [n]$. If $a_i \preceq b_i$ for every $i \in [n]$, then $\sum_{i \in [n]} a_i \preceq \sum_{i \in [n]} b_i$ and $\prod_{i \in [n]} a_i \preceq \prod_{i \in [n]} b_i$.

(ii) Assume that $\mathcal{A}$ has property (G$\oplus$), and let $a \in A$. For all $m, n \in \mathbb{N}_+$, if $m \leq n$ then $\sum_{i \in [m]} a \preceq \sum_{i \in [n]} a$. Note that $m = 0$ is excluded, because there may be $a \in A$ with $\mathbf{0} \not\preceq a$.

(iii) Let $b \in A$ with $\mathbf{1} \preceq b$. For all $m, n \in \mathbb{N}$, if $m \leq n$ then $b^m \preceq b^n$. □

Generalizing the usual total order $\leq$ on $\mathbb{N}$, some semirings are partially ordered by a partial order defined in terms of the semiring addition. We consider the relation $\sqsubseteq \subseteq A \times A$ that is defined for every $a_1, a_2 \in A$ by $a_1 \sqsubseteq a_2$, if and only if there exists an $a \in A$ such that $a_1 \oplus a = a_2$. The semiring $\mathcal{A}$ is said to be *naturally ordered* if $\sqsubseteq$ is a partial order (for this it suffices to show that $\sqsubseteq$ is antisymmetric). We always write $\sqsubseteq$ for the natural order. Note that every additively idempotent semiring is naturally ordered (e.g., [49]).

Theorem 2.1 of [39] establishes that naturally ordered semirings are partially ordered by $\sqsubseteq$. In addition, they always fulfil (G$\oplus$) and have a directed carrier set by Observation 2.1(iii).

2.3 Proposition Let $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a naturally ordered semiring. Then $(A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \sqsubseteq)$ is a partially ordered semiring and $\mathbf{0} \sqsubseteq a$ for every $a \in A$. $\square$

Let us consider some examples. The semiring $\mathbb{Z}/4\mathbb{Z}$ is partially ordered only by the trivial order $=$. All the semirings $\mathbb{N}_\infty$, $\mathbb{A}$, $\mathbb{T}$, $\mathbb{B}$, $\mathbb{R}_{\min,\max}$, and $\mathbb{L}_S$ are naturally ordered, and among these only $\mathbb{L}_S$ is not totally ordered. In each of these cases the positive cone is the whole carrier set (by Proposition 2.3 and Observation 2.1(i)). Note that $\mathbb{T}$ is also partially ordered by the total order $\leq$; this is the converse of the natural order $\sqsubseteq$ induced by the minimum operation.

In fact, if $\mathcal{A}$ is totally ordered by $\preceq$, then the carrier set is directed, while, in general, it cannot be concluded that $\mathcal{A}$ has property (G$\oplus$). For example, the semiring $(\{0,1,2\}, +, \cdot, 0, 1, \preceq)$ completely determined by $1+1 = 2+1 = 2+2 = 2$ and $2 \cdot 2 = 2$ (note that the remaining cases are such that 0 and 1 are unit elements with respect to addition and multiplication, respectively, and 0 acts as zero), is totally ordered by $2 \prec 1 \prec 0$, but $1 + 1 \prec 1$. Also note that, e.g., $\mathbb{B}$ is additively idempotent and partially ordered by $=$, but the carrier set is not directed because $\uparrow\{0,1\} = \emptyset$.

2.4. Formal tree series

Let Δ be a ranked alphabet, $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a semiring, and $V \subseteq X$. Every mapping $\varphi : T_\Delta(V) \longrightarrow A$ is called *(formal) tree series (over Δ, V, and A)*. We use $A\langle\!\langle T_\Delta(V)\rangle\!\rangle$ to denote the set of all tree series over Δ, V, and A. Given $t \in T_\Delta(V)$, we usually write (φ, t), termed *coefficient* of t, instead of $\varphi(t)$ and $\sum_{t \in T_\Delta(V)} (\varphi, t)\, t$ instead of φ. For example, $\sum_{t \in T_\Delta(V)} \mathrm{height}(t)\, t$ is the tree series that associates to every tree its height. We add tree series pointwise; i.e., $(\varphi \oplus \psi, t) = (\varphi, t) \oplus (\psi, t)$ for every $\varphi, \psi \in A\langle\!\langle T_\Delta(V)\rangle\!\rangle$ and $t \in T_\Delta(V)$. As usual, we use the $\sum$-notation to abbreviate sums over finite index sets.

Given $\varphi \in A\langle\!\langle T_\Delta(V)\rangle\!\rangle$, the set $\mathrm{supp}(\varphi) = \{\, t \in T_\Delta(V) \mid (\varphi, t) \neq \mathbf{0} \,\}$ is called the *support* of φ. Whenever $\mathrm{supp}(\varphi)$ is finite, φ is said to be *polynomial*, and φ is *monomial* if $\mathrm{supp}(\varphi)$ contains at most one element. The set of all polynomial tree series (over Δ, V, and A) is denoted by $A\langle T_\Delta(V)\rangle$. Moreover, if there is an $a \in A$ such that $(\varphi, t) = a$ for every $t \in T_\Delta(V)$, then φ is said to be *constant* and we use $\widetilde{a}$ to abbreviate such φ.

Tree substitution can be generalized to tree languages [21, 22] as well as tree series. We follow the IO-substitution approach [8, 19]. Since we will only need tree series substitution for polynomial tree series, we restrict the definitions appropriately. Then, in particular, the summations over tree series are finite. Let $n \in \mathbb{N}$, $\varphi \in A\langle T_\Delta(X_n)\rangle$, and $\psi_1, \ldots, \psi_n \in A\langle T_\Delta(V)\rangle$. *(Pure) substitution of* $(\psi_1, \ldots, \psi_n)$ *into* φ, denoted by $\varphi \xleftarrow{\varepsilon} (\psi_1, \ldots, \psi_n)$, is defined by

$$\varphi \xleftarrow{\varepsilon} (\psi_1, \ldots, \psi_n) = \sum_{\substack{t \in \mathrm{supp}(\varphi),\\ (\forall i \in [n]):\, t_i \in \mathrm{supp}(\psi_i)}} \Big((\varphi, t) \odot \prod_{i \in [n]} (\psi_i, t_i)\Big)\, t[t_1, \ldots, t_n].$$

Thus $\varphi \xleftarrow{\varepsilon} (\psi_1, \ldots, \psi_n)$ is in $A\langle T_\Delta(V)\rangle$, and irrespective of the number of occurrences of x_i for some $i \in [n]$, the coefficient (ψ_i, t_i) is taken into account exactly once (even if x_i does not appear at all in t). This particularity led to the introduction of a different notion of substitution defined in [27] as follows.

$$\varphi \xleftarrow{\mathrm{o}} (\psi_1, \ldots, \psi_n) = \sum_{\substack{t \in \mathrm{supp}(\varphi), \\ (\forall i \in [n]) \colon t_i \in \mathrm{supp}(\psi_i)}} \Big((\varphi, t) \odot \prod_{i \in [n]} (\psi_i, t_i)^{|t|_{x_i}} \Big)\, t[t_1, \ldots, t_n].$$

This notion of substitution, called *o-substitution*, takes (ψ_i, t_i) into account as often as x_i appears in t. Proposition 3.4 of [27] lists some properties common to both types of substitution. In particular, note the third property in case $m = \mathrm{o}$.

2.4 Proposition Let $n \in \mathbb{N}$, $\varphi \in A\langle T_\Delta(X_n)\rangle$, and $\psi_1, \ldots, \psi_n \in A\langle T_\Delta(X)\rangle$. Then for every $m \in \{\varepsilon, \mathrm{o}\}$ and $i \in [n]$:

(i) $\varphi \xleftarrow{m} () = \varphi$;

(ii) $\widetilde{\mathbf{0}} \xleftarrow{m} (\psi_1, \ldots, \psi_n) = \widetilde{\mathbf{0}}$; and

(iii) $\varphi \xleftarrow{m} (\psi_1, \ldots, \psi_{i-1}, \widetilde{\mathbf{0}}, \psi_{i+1}, \ldots, \psi_n) = \widetilde{\mathbf{0}}$. □

Finally, in [40] a notion of substitution based on the OI-substitution approach [21, 22] is introduced. There the number of occurrences of a certain variable is taken into account as well. However, in this paper we exclusively deal with the IO-substitution approach.

3. Tree series transducers

In this section we recall the notions of bottom-up and top-down tree series transducers from [19]. Figure 1 attempts to display the automata and transducer concepts subsumed by tree series transducers. Roughly speaking, moving upwards-left adds weights (costs or multiplicity), moving upwards performs the generalization from strings to trees, and finally, moving upwards-right adds an output component.

Before we proceed with the definition of tree series transducers, we recall some basic notions concerning matrices. Let I and S be sets. The set of all mappings $f : I \longrightarrow S$ is denoted by S^I, and we occasionally write f_i instead of $f(i)$ with $i \in I$. Now let I and J be sets. An *$(I \times J)$-matrix over S* is a mapping $M : I \times J \longrightarrow S$. The element $M(i,j)$, usually written $M_{i,j}$, is called the *(i,j)-entry* of M.

Next we define tree representations, which encode the transitions and output trees of tree series transducers. Let Q be a finite set representing the state set of a tree series transducer. For every $V \subseteq X$ we abbreviate $\{\, q(v) \mid q \in Q, v \in V \,\}$ by $Q(V)$. Roughly speaking, a tree representation is a family of mappings, each of which maps an input symbol to a matrix indexed by sequences of (annotated) states (more formally, by a state and an element of $Q(X)^*$). The entries of those matrices are tree series over Δ, X, and A, where Δ is an output ranked alphabet and A is the carrier set of a semiring.

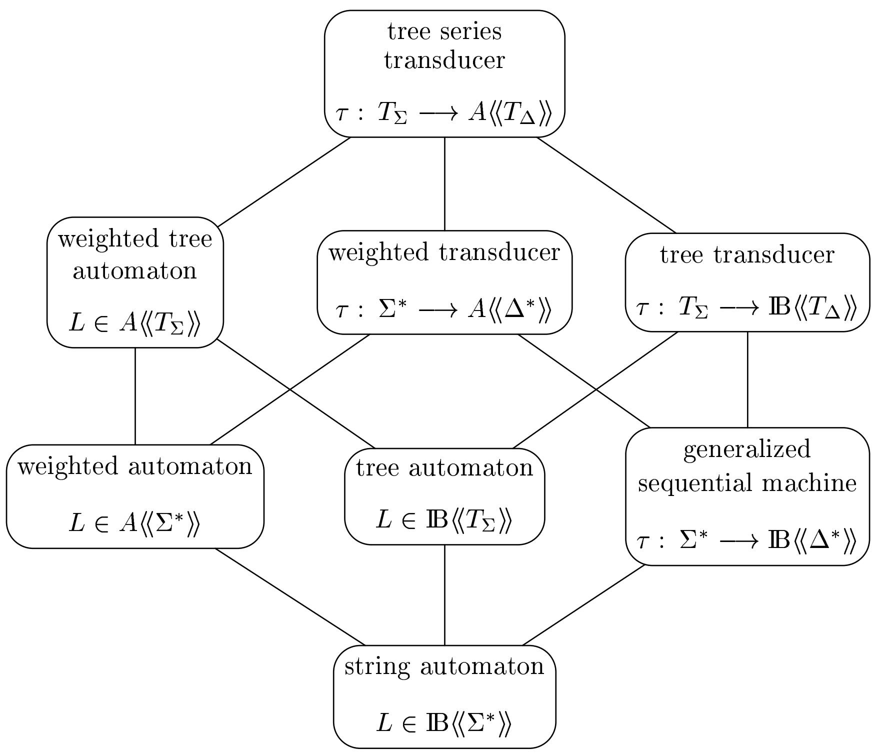

Figure 1: Generalization hierarchy.

3.1 Definition Given a finite set Q, ranked alphabets Σ and Δ, and a semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$, a *tree representation (over Q, Σ, Δ, and A)* is a family $(\mu_k \mid k \in \mathbb{N})$ of mappings

$$\mu_k : \Sigma^{(k)} \longrightarrow A\langle\!\langle T_\Delta(X)\rangle\!\rangle^{Q \times Q(X_k)^*}$$

such that for every $\sigma \in \Sigma^{(k)}$ there exist only finitely many $(q, w) \in Q \times Q(X_k)^*$ with $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$, and $\mu_k(\sigma)_{q,w} \in A\langle\!\langle T_\Delta(X_{|w|})\rangle\!\rangle$ for every such (q, w).

- The tree representation μ is called *bottom-up*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(q, w) \in Q \times Q(X_k)^*$ with $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$ we have $w = (q_1(x_1), \ldots, q_k(x_k))$ for some $q_1, \ldots, q_k \in Q$.
- The tree representation μ is called *top-down*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(q, w) \in Q \times Q(X_k)^*$ we have $\operatorname{supp}(\mu_k(\sigma)_{q,w}) \subseteq \widehat{T_\Delta}(X_{|w|})$.

Finally, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $(q, w) \in Q \times Q(X_k)^*$ the tree series $\mu_k(\sigma)_{q,w}$ is polynomial, then μ is called *polynomial*. □

Note that polynomial tree representations are finitely representable due to the finiteness of the input ranked alphabet Σ and the fact that for every $k \in \mathbb{N}$ almost all entries in the matrices in the range of μ_k are $\widetilde{\mathbf{0}}$. A tree series transducer is basically a tree representation together with supportive information about the state set Q, the input and output ranked alphabets Σ and Δ, respectively, and the semiring $\mathcal{A}$. Additionally, we distinguish certain states, which are called designated states. Depending on the mode of traversing the input, these might be initial or final states.

3.2 Definition A *tree series transducer* is a sextuple $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$, where:

- Q and $D \subseteq Q$ are alphabets of *states* and *designated states*, respectively;
- Σ and Δ are ranked alphabets (both disjoint to Q) of *input* and *output symbols*, respectively;
- $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is a semiring; and
- μ is a tree representation over Q, Σ, Δ, and A.

Tree series transducers inherit the properties bottom-up, top-down, and polynomial from their tree representation; i.e., M is called *bottom-up* (*top-down* and *polynomial*, respectively), if μ is bottom-up (top-down and polynomial, respectively). The elements of D are also called *final states* (respectively, *initial states*), if M is bottom-up (respectively, top-down). □

For the rest of the paper we only consider polynomial tree series transducers that are bottom-up or top-down. For an investigation of general tree series transducers, we refer the reader to [19, 27, 25]. To be concise, we drop the variables from the second index of the matrices in the range of bottom-up tree representations; e.g., we write $\mu_k(\sigma)_{q,(q_1,\ldots,q_k)}$ instead of $\mu_k(\sigma)_{q,(q_1(x_1),\ldots,q_k(x_k))}$.

3.3 Definition Let $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ be a bottom-up tree series transducer. We say that M is:

- *deterministic*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k \in Q$ there exists at most one $(q,t) \in Q \times T_\Delta(X)$ such that $t \in \mathrm{supp}(\mu_k(\sigma)_{q,(q_1,\ldots,q_k)})$;
- *total*, if $D = Q$ and for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k \in Q$ there exists at least one $(q,t) \in Q \times T_\Delta(X)$ such that $t \in \mathrm{supp}(\mu_k(\sigma)_{q,(q_1,\ldots,q_k)})$; and
- a *homomorphism*, if M is total and deterministic, and Q is a singleton. □

Similarly these concepts (of determinism, totality, and homomorphism) can be defined for top-down tree series transducers.

3.4 Definition Let $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ be a top-down tree series transducer. We say that M is:

- *deterministic*, if D is a singleton and for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q \in Q$ there exists at most one $(w,t) \in Q(X_k)^* \times T_\Delta(X)$ such that $t \in \mathrm{supp}(\mu_k(\sigma)_{q,w})$;
- *total*, if for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q \in Q$ there exists at least one $(w,t) \in Q(X_k)^* \times T_\Delta(X)$ such that $t \in \mathrm{supp}(\mu_k(\sigma)_{q,w})$; and
- a *homomorphism*, if M is total and deterministic, and Q is a singleton. □

Note that a deterministic (top-down or bottom-up) tree series transducer is polynomial. Finally, we assign a formal semantics to polynomial tree series transducers. We define two different semantics; namely, the ε-tree-to-tree-series transformation and the o-tree-to-tree-series transformation computed by a polynomial tree series transducer. Both are defined in the same manner; the only difference is the type of substitution used.

3.5 Definition Let $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ be a polynomial tree series transducer and $m \in \{\varepsilon, \mathrm{o}\}$.

(i) For every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$ the tree representation μ induces a mapping $\overline{\mu_k(\sigma)}^m : \left(A\langle T_\Delta\rangle^Q\right)^k \longrightarrow A\langle T_\Delta\rangle^Q$ defined for every $q \in Q$ and $R_1, \ldots, R_k \in A\langle T_\Delta\rangle^Q$ by

$$\overline{\mu_k(\sigma)}^m(R_1, \ldots, R_k)_q = \sum_{\substack{w \in Q(X_k)^*,\\ w = (q_1(x_{i_1}),\ldots,q_l(x_{i_l}))}} \mu_k(\sigma)_{q,w} \overset{m}{\longleftarrow} \big((R_{i_1})_{q_1}, \ldots, (R_{i_l})_{q_l}\big).$$

The above sum is essentially finite because there are only finitely many $w \in Q(X_k)^*$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$ (if $\mu_k(\sigma)_{q,w} = \widetilde{\mathbf{0}}$ then the summand is $\widetilde{\mathbf{0}}$ by Proposition 2.4(ii)). Note that

$$\Big(A\langle T_\Delta\rangle^Q, \big(\overline{\mu_k(\sigma)}^m \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}\big)\Big)$$

defines a Σ-algebra, and T_Σ is the free Σ-algebra. Thus there exists a unique homomorphism $h_\mu^m : T_\Sigma \longrightarrow A\langle T_\Delta\rangle^Q$, which is inductively defined for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma$ by

$$h_\mu^m(\sigma(s_1, \ldots, s_k)) = \overline{\mu_k(\sigma)}^m(h_\mu^m(s_1), \ldots, h_\mu^m(s_k)).$$

(ii) The *m-tree-to-tree-series transformation*, abbreviated m-t-ts transformation, computed by M is the mapping $\tau_M^m : T_\Sigma \longrightarrow A\langle T_\Delta\rangle$ specified for every $s \in T_\Sigma$ by $\tau_M^m(s) = \sum_{q\in D} h_\mu^m(s)_q$. □

3.6 Example The bottom-up tree series transducer $M = (\{*\}, \Sigma, \Delta, \mathbb{A}, \{*\}, \mu)$ over the arctic semiring $\mathbb{A}$ with $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$, $\Delta = \{\alpha^{(0)}\}$, and μ defined by

$$\mu_2(\sigma)_{*,(*,*)} = \max(1\,x_1,\ 1\,x_2) \qquad \text{and} \qquad \mu_0(\alpha)_{*,\varepsilon} = 1\,\alpha$$

is total and polynomial, but not deterministic, and consequently, not a homomorphism. The o-t-ts transformation computed by M is $\tau_M^{\mathrm{o}}(s) = \mathrm{height}(s)\,\alpha$ for every $s \in T_\Sigma$. To illustrate the previous definition, we prove this property by structural induction over s.

Induction base: Let $s = \alpha$.

$$\begin{aligned}\tau_M^{\mathrm{o}}(\alpha) &\overset{\text{Def. 3.5(ii)}}{=} \max_{q\in\{*\}} h_\mu^{\mathrm{o}}(\alpha)_q \ = \ h_\mu^{\mathrm{o}}(\alpha)_* \overset{\text{Def. 3.5(i)}}{=} \overline{\mu_0(\alpha)}^{\mathrm{o}}()_* \\ &\overset{\text{Def. 3.5(i)}}{=} \mu_0(\alpha)_{*,\varepsilon} \overset{\mathrm{o}}{\longleftarrow} () \overset{\text{Prop. 2.4(i)}}{=} \mu_0(\alpha)_{*,\varepsilon} \ = \ 1\,\alpha \ = \ \mathrm{height}(\alpha)\,\alpha\end{aligned}$$

Induction step: Let $s = \sigma(s_1, s_2)$ for some $s_1, s_2 \in T_\Sigma$. Note that $a^0 = 0$.

$$\begin{aligned}\tau_M^{\mathrm{o}}\big(\sigma(s_1,s_2)\big) &\overset{\text{Def. 3.5(ii)}}{=} \max_{q\in\{*\}} h_\mu^{\mathrm{o}}\big(\sigma(s_1,s_2)\big)_q \ = \ h_\mu^{\mathrm{o}}\big(\sigma(s_1,s_2)\big)_* \\ &\overset{\text{Def. 3.5(i)}}{=} \overline{\mu_2(\sigma)}^{\mathrm{o}}\big(h_\mu^{\mathrm{o}}(s_1), h_\mu^{\mathrm{o}}(s_2)\big)_* \\ &\overset{\text{Def. 3.5(i)}}{=} \max_{(q_1,q_2)\in\{*\}^2} \mu_2(\sigma)_{*,(q_1,q_2)} \overset{\mathrm{o}}{\longleftarrow} \big(h_\mu^{\mathrm{o}}(s_1)_{q_1}, h_\mu^{\mathrm{o}}(s_2)_{q_2}\big) \\ &= \mu_2(\sigma)_{*,(*,*)} \overset{\mathrm{o}}{\longleftarrow} \big(h_\mu^{\mathrm{o}}(s_1)_*, h_\mu^{\mathrm{o}}(s_2)_*\big) \\ &= \max(1\,x_1, 1\,x_2) \overset{\mathrm{o}}{\longleftarrow} \big(\max_{q\in\{*\}} h_\mu^{\mathrm{o}}(s_1)_q, \max_{q\in\{*\}} h_\mu^{\mathrm{o}}(s_2)_q\big) \\ &\overset{\text{Def. 3.5(ii)}}{=} \max(1\,x_1, 1\,x_2) \overset{\mathrm{o}}{\longleftarrow} \big(\tau_M^{\mathrm{o}}(s_1), \tau_M^{\mathrm{o}}(s_2)\big) \\ &\overset{\text{I.H.}}{=} \max(1\,x_1, 1\,x_2) \overset{\mathrm{o}}{\longleftarrow} \big(\mathrm{height}(s_1)\,\alpha,\ \mathrm{height}(s_2)\,\alpha\big) \\ &\overset{\dagger}{=} \max\big(1+\mathrm{height}(s_1), 1+\mathrm{height}(s_2)\big)\,\alpha \\ &= 1+\max\big(\mathrm{height}(s_1), \mathrm{height}(s_2)\big)\,\alpha \\ &= \mathrm{height}\big(\sigma(s_1,s_2)\big)\,\alpha\end{aligned}$$

Note that $\mathrm{supp}\big(\max(1x_1,\ 1x_2)\big) = \{x_1, x_2\}$ and $\mathrm{supp}\big(\tau_M^{\mathrm{o}}(s_1)\big) = \mathrm{supp}\big(\tau_M^{\mathrm{o}}(s_2)\big) = \{\alpha\}$ are used at †. □

In the sequel we are interested in the computational power of subclasses of bottom-up and top-down tree series transducers. More precisely, to every class of restricted bottom-up or top-down tree series transducers (see the properties in Definitions 3.3 and 3.4) we associate the class of all m-t-ts transformations computed by them. Then we compare such classes of m-t-ts transformations by means of set inclusion. The next definition establishes shorthands for classes of m-t-ts transformations.

3.7 Definition Let $m \in \{\varepsilon, \mathrm{o}\}$. The class of all m-t-ts transformations computed by polynomial, deterministic, and homomorphism bottom-up tree series transducers over the semiring $\mathcal{A}$ is denoted by p–BOT$^m(\mathcal{A})$, d–BOT$^m(\mathcal{A})$, and h–BOT$^m(\mathcal{A})$, respectively. Likewise, we use p–TOP$^m(\mathcal{A})$, d–TOP$^m(\mathcal{A})$, and h–TOP$^m(\mathcal{A})$ to stand for the corresponding classes of m-t-ts transformations computed by top-down tree series transducers over $\mathcal{A}$. □

We note that there are two differences concerning the denotation of classes of transformations in comparison to [27]. One difference is that our ε disappears in the denotation of the same class in [27], and the second difference is that here we drop the index $t - ts$ (standing for tree-to-tree-series) from the denotation of our classes. Thus, e.g., d–BOT$^\varepsilon(\mathcal{A})$ is denoted by $d\text{-}BOT_{\text{t-ts}}(A)$ in [27].

3.8 Theorem Let $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ be a polynomial top-down tree series transducer.

(i) $h_\mu^\varepsilon(s)_q = h_\mu^{\mathrm{o}}(s)_q$ for every $s \in T_\Sigma$ and $q \in Q$.

(ii) x–TOP$^\varepsilon(\mathcal{A}) = x$–TOP$^{\mathrm{o}}(\mathcal{A})$ for every $x \in \{\mathrm{p}, \mathrm{d}, \mathrm{h}\}$.

Proof. See Lemma 5.1 and Theorem 5.2 of [27]. ■

Next we recall a property of deterministic tree series transducers that are bottom-up or top-down. Roughly speaking, the addition operation of the underlying semiring is irrelevant concerning computations of a deterministic tree series transducer; i.e., all computations are performed in the multiplicative monoid of the semiring.

3.9 Proposition Let $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ be a deterministic (bottom-up or top-down) tree series transducer and $m \in \{\varepsilon, \mathrm{o}\}$. Then $h_\mu^m(s)_q$ and $\tau_M^m(s)$ are monomial for every $s \in T_\Sigma$ and $q \in Q$. Moreover, if M is bottom-up then for every $s \in T_\Sigma$ there exists at most one $q \in Q$ such that $h_\mu^m(s)_q \neq \widetilde{\mathbf{0}}$.

Proof. If $m = \varepsilon$ then the proof of the statement is in Proposition 3.12 of [19]. The proof of the statement with $m = \mathrm{o}$ uses exactly the same argumentation. ■

Before we proceed to the next section, we explicitly exclude certain non-interesting tree series transducers. We call a tree series transducer $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ *non-trivial*, if $\Sigma^{(0)} \neq \emptyset$ and there exist $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q \in Q$ such that $\mu_k(\sigma)_{q,\varepsilon} \neq \widetilde{\mathbf{0}}$. Hence, in particular, for bottom-up tree series transducers, non-triviality implies that there exists a $\sigma \in \Sigma^{(0)}$ satisfying the condition above. Moreover, non-triviality implies $\Delta^{(0)} \neq \emptyset$. Let M be a trivial tree series transducer. Then $\tau_M^m(s) = \widetilde{\mathbf{0}}$ for every $s \in T_\Sigma$ and $m \in \{\varepsilon, \mathrm{o}\}$. Since this particular case is not interesting, we assume that all tree series transducers that are considered in the rest of the paper are non-trivial.

Moreover, we henceforth assume that $\mathrm{mx}_\Sigma \geq 1$ for all considered ranked alphabets Σ of input symbols. This is justified because if we restrict ourselves to input alphabets with only nullary symbols then

$$x\text{–BOT}^\varepsilon(\mathcal{A}) = x\text{–BOT}^{\mathrm{o}}(\mathcal{A}) = x\text{–TOP}^\varepsilon(\mathcal{A}) = x\text{–TOP}^{\mathrm{o}}(\mathcal{A}).$$

4. Coefficient majorization

Throughout the rest of the section, $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ is a partially ordered semiring. Thus, e.g., the semirings $\mathbb{N}_\infty$, $\mathbb{A}$, $\mathbb{T}$, $\mathbb{B}$, $\mathbb{R}_{\min,\max}$, and $\mathbb{L}_S$ (for an alphabet S) are suitable. Moreover, we let $m \in \{\varepsilon, \mathrm{o}\}$ and $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ be a non-trivial polynomial tree series transducer with $\mathrm{mx}_\Sigma \geq 1$. More specifically, M is bottom-up in Section 4.2 and top-down in Section 4.3.

4.1. The general approach

We approximate the coefficient of an output tree that is in the support of a tree series in the range of τ_M^m. More precisely, we define coefficient majorizations $f : \mathbb{N}_+ \longrightarrow A$, which fulfil $f(n) \in {\uparrow} C_M^m(n)$ for every $n \in \mathbb{N}_+$, where $C_M^m(n) \subseteq A$, the set of *coefficients generated by M on input trees of height at most n*, is

$$C_M^m(n) = \left\{ \, (h_\mu^m(s)_q, t) \;\middle|\; q \in Q, s \in T_\Sigma, \mathrm{height}(s) \leq n, t \in \mathrm{supp}(h_\mu^m(s)_q) \, \right\}.$$

The existence of such mappings gives rise to a property of polynomial tree series transducers. We exploit this property in Section 5 to reprove some recent results and to provide some insight into the relation between the two modes of traversing the input tree (i.e., bottom-up and top-down) and the two types of substitution (i.e., pure and o-substitution).

We start by defining some constants associated with the polynomial tree series transducer M. They provide the abstraction from the concrete tree series transducer used in our majorizations.

4.1 Definition We define the following constants representing basic facts of M:

- the *maximal rank* $r_M \in \mathbb{N}_+$ of the input symbols: $r_M = \mathrm{mx}_\Sigma$;
- the number $d_M \in \mathbb{N}_+$ of *follow-up states* (or *successor states*):

$$d_M = \begin{cases} 1 & \text{if } M \text{ is deterministic,} \\ \mathrm{card}(Q) & \text{if } M \text{ is bottom-up and not deterministic,} \\ \max(2, \mathrm{card}(Q) \cdot r_M) & \text{otherwise;} \end{cases}$$

- the *maximal support cardinality* $e_M \in \mathbb{N}_+$:

$$e_M = \max \left\{ \mathrm{card}\big(\mathrm{supp}(\mu_k(\sigma)_{q,w})\big) \;\middle|\; \begin{array}{l} k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, \\ q \in Q, w \in Q(X_k)^* \end{array} \right\};$$

- the *maximal variable degree* $u_{M,m} \in \mathbb{N}$:

$$u_{M,m} = \begin{cases} r_M & \text{if } m = \varepsilon, \\ \displaystyle\max_{\substack{k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q, \\ w \in Q(X_k)^*, t \in \mathrm{supp}(\mu_k(\sigma)_{q,w})}} \sum_{x \in X_{|w|}} |t|_x & \text{if } m = \mathrm{o}; \end{cases}$$

- and the *maximal length* $v_M \in \mathbb{N}$ of the second index in any matrix in the range of μ_k:

$$v_M = \max\left\{ |w| \;\middle|\; k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q, w \in Q(X_k)^*, \mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}} \right\}.$$

□

Let us discuss those constants in more detail. The constant r_M represents the maximal number of direct subtrees of any input tree. This number coincides with the maximal rank of the input ranked alphabet. Next we consider a state $q \in Q$ and a word $w \in Q(X_k)^*$ such that $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$. The constant d_M represents the number of possible combinations for a single symbol of the word w. For a deterministic tree series transducer d_M is apparently 1 by Proposition 3.9. Given that M is bottom-up, we have only $\operatorname{card}(Q)$ choices for the state because the variable of X_k is uniquely determined by the position in the word w. Finally, for polynomial top-down tree series transducers we have $\operatorname{card}(Q) \cdot k$ choices, but for technical reasons we take $\max(2, \operatorname{card}(Q) \cdot r_M)$.

The intention of the constant e_M, which is well-defined because M is polynomial, is obvious. Lastly, the constants $u_{M,m}$ and v_M fulfil a similar purpose. They both limit the number of factors representing subtree coefficients in any multiplication (see the definition of pure and o-substitution). The bottom-up case, in which at most r_M factors (if $m = \varepsilon$) or at most as many factors as there are variables in the tree selected from the tree representation (if $m = \mathrm{o}$) occur, is handled by the constant $u_{M,m}$. In the top-down case, which is handled by v_M, there is no difference between pure and o-substitution. Here there are at most as many factors as the length of the longest word $w \in Q(X_k)^*$ with $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$.

Note that $u_{M,m}$ and v_M are well-defined; for the former we need that M is polynomial. Additionally, $v_M = r_M$ if M is bottom-up, and $v_M = u_{M,\mathrm{o}}$ if M is top-down.

4.2 Definition An element $c \in A$ is an *upper bound of the coefficients of* μ, if

$$c \in \uparrow\left(\{\mathbf{1}\} \cup \left\{ (\mu_k(\sigma)_{q,w}, t) \;\middle|\; \begin{array}{l} k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, q \in Q, \\ w \in Q(X_k)^*, t \in \operatorname{supp}(\mu_k(\sigma)_{q,w}) \end{array} \right\}\right).$$

□

Note that such an element need not exist in general. However, the existence can be guaranteed, e.g., by demanding that A is directed. In the following, we often assume an upper bound c of the coefficients of μ, and apparently, to obtain the best results, it should be chosen as small as possible; hence it should be the supremum of the non-zero coefficients occurring in μ, if it exists.

Next we introduce particular mappings, namely cardinality, sum, and coefficient majorizations. Given an $n \in \mathbb{N}_+$, a cardinality majorization is supposed to limit the cardinality of the support of $h_\mu^m(s)_q$ for every $q \in Q$ and $s \in T_\Sigma$ of height at most n. A sum majorization shall provide an upper bound of the n-fold sum of an $a \in A$. This mapping represents internal knowledge of the semiring and is externally provided. For example, the mapping $g(n, a) = n \cdot a$ for every $n \in \mathbb{N}_+$ and $a \in \mathbb{N} \cup \{\infty\}$ is suitable for $\mathbb{N}_\infty$. The sum majorization allows us to omit unnecessary detail, for the mapping is only required to approximate the sum; it need not return the precise sum. Finally, given $n \in \mathbb{N}_+$, a coefficient majorization f is supposed to limit all non-zero coefficients generated by M on input trees of height at most n; i.e., it must fulfil $f(n) \in \uparrow C_M^m(n)$.

4.3 Definition The following notions are defined.

- A mapping $l : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ such that $\mathrm{card}\big(\mathrm{supp}(h_\mu^m(s)_q)\big) \leq l(n)$ for every $n \in \mathbb{N}_+$, $s \in T_\Sigma$ of height at most n, and $q \in Q$ is called *cardinality majorization (with respect to M and m).*
- A mapping $g : \mathbb{N}_+ \times A \longrightarrow A$ such that $\sum_{i\in[n]} a \preceq g(n,a)$ for every $n \in \mathbb{N}_+$ and $a \in A$ is called *sum majorization (with respect to $\mathcal{A}$).*
- A mapping $f : \mathbb{N}_+ \longrightarrow A$ such that $f(n) \in \uparrow C_M^m(n)$ for every $n \in \mathbb{N}_+$ is called *coefficient majorization (with respect to M and m).* □

We note that throughout Section 5, we will use $g(n,a) = \sum_{i\in[n]} a$ as sum majorization. Now we continue by providing an example for each of the above defined majorizations using our running example bottom-up tree series transducer of Example 3.6.

4.4 Example Let M be the tree series transducer of Example 3.6, and let $m = \mathrm{o}$. Recall that $\mathbb{A}$ (the semiring used by M) fulfils the general conditions of this section.

- The mapping $l : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ defined by $l(n) = 1$ for every $n \in \mathbb{N}_+$ is a cardinality majorization because $\mathrm{card}(T_\Delta) = 1$.
- The mapping $g : \mathbb{N}_+ \times (\mathbb{N} \cup \{-\infty\}) \longrightarrow \mathbb{N} \cup \{-\infty\}$ defined by $g(n,a) = a$ for every $n \in \mathbb{N}_+$ and $a \in \mathbb{N} \cup \{-\infty\}$ is a sum majorization because $\mathbb{A}$ is additively idempotent.
- The mapping $f : \mathbb{N}_+ \longrightarrow \mathbb{N} \cup \{-\infty\}$ defined by $f(n) = n$ is a coefficient majorization, which is immediate from Example 3.6. □

Next we discuss the general approach used to derive a coefficient majorization. Let $s \in T_\Sigma$ and let c be an upper bound of the coefficients of μ (see Definition 4.2). Using a cardinality majorization l and a sum majorization g, we can introduce a so-called ample coefficient majorization associated with l, g, and c (see Definitions 4.5 and 4.11). The different modifiers (i.e., $m = \varepsilon$ or $m = \mathrm{o}$) are taken care of by the maximal variable degree $u_{M,m}$ (see Definition 4.1) in case M is bottom-up, while the m-t-ts transformations computed by top-down tree series transducers using on the one hand $m = \varepsilon$ and on the other hand $m = \mathrm{o}$ do not differ (i.e., $\tau_M^\varepsilon = \tau_M^\mathrm{o}$; see Lemma 5.1 of [27]).

Roughly speaking, if s has height 1, then every support element of $h_\mu^m(s)_q$ has a coefficient that is at most c. Given s of height $n+1$, we first compute an upper bound of the coefficients of all subtrees of height at most n. Since those weights are multiplied in the definition of substitution, we take the result of the recursive call to the $u_{M,m}$-th power, if M is bottom-up, and to the v_M-th power, if M is top-down. Recall that $u_{M,m}$ and v_M are defined such that they hold the maximal number of multiplications in any product generated by one substitution. The other factor is provided by the tree representation, and thus c provides a suitable upper bound of this factor.

Finally, by substitution, equal trees might arise such that the coefficients of those are going to be summed up. The cardinality majorization l with the help of the sum majorization g is going to provide an upper bound of this sum as we will see in Theorems 4.7 and 4.13.

In the sections to follow we distinguish the two modes of traversing the input tree, namely bottom-up and top-down. In particular, in the top-down section we casually refer to the bottom-up section because the derived majorizations generally have the same structure and so properties only depending on the structure carry over to the top-down case.

4.2. The bottom-up case

Recall that in this section M is always a (non-trivial) polynomial bottom-up tree series transducer with $\mathrm{mx}_\Sigma \geq 1$. Moreover, let l and g be a cardinality majorization and a sum majorization, respectively. Lastly, let c be an upper bound of the coefficients of μ (see Definition 4.2). According to the outline just presented, we define the following coefficient majorization. Recall the constants r_M, d_M, e_M, and $u_{M,m}$ from Definition 4.1.

4.5 Definition The *ample coefficient majorization* $f^{\mathrm{bot}}_{M,m,l,g,c} : \mathbb{N}_+ \longrightarrow A$ *(associated with l, g, and c)* is defined recursively by

$$f^{\mathrm{bot}}_{M,m,l,g,c}(1) = c$$
$$f^{\mathrm{bot}}_{M,m,l,g,c}(n) = g\big((d_M)^{r_M} \cdot e_M \cdot l(n-1)^{r_M}, c \odot f^{\mathrm{bot}}_{M,m,l,g,c}(n-1)^{u_{M,m}}\big)$$

for every $n \geq 2$. □

Thus the ample coefficient majorization depends on the polynomial bottom-up tree series transducer M (or more specifically: a few characteristics of M), the modifier m, the cardinality majorization l, the sum majorization g, and the upper bound c. Next we prove that the ample coefficient majorization is indeed a coefficient majorization.

This result is proved for two cases:

(C1) $\mathcal{A}$ satisfies (G$\oplus$); or

(C2) M is deterministic and $l(n) = 1$ for every $n \in \mathbb{N}_+$, which is a cardinality majorization by Proposition 3.9.

In fact, (PO$\oplus$) is not even needed in Case (C2).

4.6 Observation Given (C1) or (C2), we have $\mathbf{1} \preceq f^{\mathrm{bot}}_{M,m,l,g,c}(n)$ for every $n \in \mathbb{N}_+$.

Proof. The proof is by induction on n. The induction base is immediate by Definition 4.2, so we proceed with the induction step. Clearly, $a = c \odot f^{\mathrm{bot}}_{M,m,l,g,c}(n)^{u_{M,m}}$ obeys $\mathbf{1} \preceq a$ by induction hypothesis, Observation 2.2(iii), and (PO$\odot$). In Case (C1) we have $a \preceq f^{\mathrm{bot}}_{M,m,l,g,c}(n+1)$ by Observation 2.2(ii). Thus $\mathbf{1} \preceq a \preceq f^{\mathrm{bot}}_{M,m,l,g,c}(n+1)$. Now let us consider Case (C2). If M is deterministic, then $d_M = 1$ and $e_M = 1$. Moreover, $l(n) = 1$ by assumption. Hence we obtain that $f^{\mathrm{bot}}_{M,m,l,g,c}(n+1) = g(1,a)$ and $a \preceq g(1,a)$ because g is a sum majorization. Thus again $\mathbf{1} \preceq a \preceq f^{\mathrm{bot}}_{M,m,l,g,c}(n+1)$. ∎

4.7 Theorem Given (C1) or (C2), the ample coefficient majorization $f^{\mathrm{bot}}_{M,m,l,g,c}$ is a coefficient majorization; i.e., $f^{\mathrm{bot}}_{M,m,l,g,c}(n) \in \uparrow C^m_M(n)$ for every $n \in \mathbb{N}_+$. Moreover,

- in Case (C1): $(\tau^m_M(s), t) \preceq g\big(\mathrm{card}(D), f^{\mathrm{bot}}_{M,m,l,g,c}(n)\big)$; and
- in Case (C2): $(\tau^m_M(s), t) \preceq g\big(1, f^{\mathrm{bot}}_{M,m,l,g,c}(n)\big)$

for every $n \in \mathbb{N}_+$, $s \in T_\Sigma$, and $t \in \mathrm{supp}(\tau^m_M(s))$ such that $\mathrm{height}(s) \le n$.

Proof. Obviously we have to prove $(h^m_\mu(s)_q, t) \preceq f^{\mathrm{bot}}_{M,m,l,g,c}(n)$ for every $n \in \mathbb{N}_+$, $q \in Q$, $s \in T_\Sigma$, and $t \in \mathrm{supp}(h^m_\mu(s)_q)$ such that $\mathrm{height}(s) \le n$. We proceed by structural induction over s.

Induction base: Let $s = \alpha$ with $\alpha \in \Sigma^{(0)}$. Since $t \in \mathrm{supp}(h^m_\mu(\alpha)_q)$, we have

$$(h^m_\mu(\alpha)_q, t) \overset{\text{Def. 3.5(i)}}{=} (\mu_0(\alpha)_{q,\varepsilon}, t) \preceq c = f^{\mathrm{bot}}_{M,m,l,g,c}(1).$$

Induction step: Let $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, $s_1, \ldots, s_k \in T_\Sigma$, and $s = \sigma(s_1, \ldots, s_k)$ be an input tree of height at most n. Note that throughout the proof we use the statements of Observation 2.2 without explicit reference. First we prove the induction step for M being not deterministic; i.e., Case (C1).

$$\begin{aligned}
&(h^m_\mu(\sigma(s_1,\ldots,s_k))_q, t)\\
\overset{\text{Def. 3.5(i)}}{=}\ & \Big(\sum_{q_1,\ldots,q_k \in Q} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \xleftarrow{m} (h^m_\mu(s_1)_{q_1}, \ldots, h^m_\mu(s_k)_{q_k}), t\Big)\\
\overset{\star}{=}\ & \sum_{\substack{w=(q_1,\ldots,q_k)\in Q^k,\\ t=t'[t_1,\ldots,t_k], t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall i\in[k]):\ t_i\in\mathrm{supp}(h^m_\mu(s_i)_{q_i})}} (\mu_k(\sigma)_{q,w}, t') \odot \prod_{i\in[k]} (h^m_\mu(s_i)_{q_i}, t_i)^{n_i}\\
& \text{where for every } i \in [k]\colon n_i = \begin{cases} |t'|_{x_i} & \text{if } m = \mathrm{o},\\ 1 & \text{if } m = \varepsilon \end{cases}\\
\overset{\text{I.H.}}{\preceq}\ & \sum_{\ldots} c \odot \prod_{i\in[k]} f^{\mathrm{bot}}_{M,m,l,g,c}(n-1)^{n_i} = \sum_{\ldots} c \odot f^{\mathrm{bot}}_{M,m,l,g,c}(n-1)^{n_1+\cdots+n_k}\\
& \text{where } \sum_{\ldots} \text{ abbreviates the sum of } (\star)\\
\overset{\text{Obs. 4.6}}{\preceq}\ & \sum_{\ldots} c \odot f^{\mathrm{bot}}_{M,m,l,g,c}(n-1)^{u_{M,m}} \overset{\dagger}{\preceq} \sum_{\substack{w=(q_1,\ldots,q_k)\in Q^k,\\ t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall i\in[k]):\ t_i\in\mathrm{supp}(h^m_\mu(s_i)_{q_i})}} a\\
& \text{where } a = c \odot f^{\mathrm{bot}}_{M,m,l,g,c}(n-1)^{u_{M,m}}
\end{aligned}$$

$$\preceq \sum_{j\in[(d_M)^k\cdot e_M\cdot l(n-1)^k]} a \preceq \sum_{j\in[(d_M)^{r_M}\cdot e_M\cdot l(n-1)^{r_M}]} a$$
$$\preceq g\big((d_M)^{r_M}\cdot e_M\cdot l(n-1)^{r_M}, a\big) = f^{\mathrm{bot}}_{M,m,l,g,c}(n)$$

The step at † is governed by $t \in \mathrm{supp}(h^m_\mu(s)_q)$, which implies that there exists at least one non-zero summand of the sum, needed for the application of Observation 2.2(ii). For the next step, note that $e_M \geq \mathrm{card}(\mathrm{supp}(\mu_k(\sigma)_{q,w}))$ and $l\big(n-1\big) \geq \mathrm{card}(\mathrm{supp}(h^m_\mu(s_i)_{q_i}))$ for every $i \in [k]$. This concludes the induction step for M being not deterministic.

Now let M be deterministic.

$$(h^m_\mu(\sigma(s_1,\ldots,s_k))_q, t)$$
$$\overset{\text{Def. 3.5(i)}}{=} \Big(\sum_{q_1,\ldots,q_k\in Q} \mu_k(\sigma)_{q,(q_1,\ldots,q_k)} \xleftarrow{m} (h^m_\mu(s_1)_{q_1},\ldots,h^m_\mu(s_k)_{q_k}), t\Big)$$
$$\overset{\star}{=} \sum_{\substack{w=(q_1,\ldots,q_k)\in Q^k,\\ t=t'[t_1,\ldots,t_k], t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall i\in[k]):\ t_i\in\mathrm{supp}(h^m_\mu(s_i)_{q_i})}} (\mu_k(\sigma)_{q,w}, t') \odot \prod_{i\in[k]} (h^m_\mu(s_i)_{q_i}, t_i)^{n_i}$$
$$\text{where for every } i\in[k]\text{: } n_i = \begin{cases} |t'|_{x_i} & \text{if } m = \mathrm{o},\\ 1 & \text{if } m = \varepsilon \end{cases}$$
$$\overset{\text{I.H.}}{\preceq} \sum_{j\in[(d_M)^{r_M}\cdot e_M\cdot l(n-1)^{r_M}]} c \odot f^{\mathrm{bot}}_{M,m,l,g,c}(n-1)^{n_1+\cdots+n_k}$$
$$\overset{\text{Obs 4.6}}{\preceq} \sum_{j\in[(d_M)^{r_M}\cdot e_M\cdot l(n-1)^{r_M}]} c \odot f^{\mathrm{bot}}_{M,m,l,g,c}(n-1)^{u_{M,m}}$$
$$\preceq g((d_M)^{r_M}\cdot e_M\cdot l(n-1)^{r_M}, a) = f^{\mathrm{bot}}_{M,m,l,g,c}(n)$$

Since M is deterministic, the states q_i are fixed by Proposition 3.9 because $t_i \in \mathrm{supp}(h^m_\mu(s_i)_{q_i})$. Moreover, by determinism and $t \in \mathrm{supp}(h^m_\mu(s)_q)$ we have that $\mathrm{supp}(\mu_k(\sigma)_{q,w})$ is singleton. Thus the index set in $\star$ is singleton. This makes the step marked I.H. possible. It uses the induction hypothesis and the facts that $d_M = 1$, $e_M = 1$, and that (G⊕) holds if $l(n-1) \neq 1$.

Thus we have proved the first statement of the theorem for both cases (C1) and (C2). This statement easily allows us to derive the latter statement of the theorem as follows. First we again consider the case that M is not deterministic; i.e., Case (C1).

$$(\tau^m_M(s), t)$$
$$\overset{\text{Def. 3.5(ii)}}{=} \sum_{q\in D} (h^m_\mu(s)_q, t) = \sum_{q\in D \text{ s.th. } t\in\mathrm{supp}(h^m_\mu(s)_q)} (h^m_\mu(s)_q, t)$$
$$\overset{\text{Obs. 2.2(i)}}{\preceq} \sum_{q\in D \text{ s.th. } t\in\mathrm{supp}(h^m_\mu(s)_q)} f^{\mathrm{bot}}_{M,m,l,g,c}(n) \overset{\text{Obs. 2.2(ii)}}{\preceq} \sum_{q\in D} f^{\mathrm{bot}}_{M,m,l,g,c}(n)$$

$$\preceq \quad g\big(\mathrm{card}(D), f^{\mathrm{bot}}_{M,m,l,g,c}(n)\big)$$

Recall that $t \in \mathrm{supp}(\tau^m_M(s))$. Thus the step labeled Obs. 2.2(ii) is possible because there exists a $q \in D$ such that $t \in \mathrm{supp}(h^m_\mu(s)_q)$.

Finally, let M be deterministic. Recall that by Proposition 3.9, there exists a state $p \in Q$ such that $h^m_\mu(s)_q = \widetilde{\mathbf{0}}$ for all $q \in Q$ with $q \neq p$. Moreover, since $t \in \mathrm{supp}(\tau^m_M(s))$ we also have $h^m_\mu(s)_p \neq \widetilde{\mathbf{0}}$.

$$(\tau^m_M(s), t) \overset{\text{Def. 3.5(ii)}}{=} \sum_{q \in D} (h^m_\mu(s)_q, t) \;=\; (h^m_\mu(s)_p, t) \preceq g\big(1, f^{\mathrm{bot}}_{M,m,l,g,c}(n)\big)$$

This completes the proof. ∎

Continuing with the running example, we present the ample coefficient majorization for the tree series transducer M of Example 3.6.

4.8 Example Let $M = (\{*\}, \Sigma, \Delta, \mathbb{A}, \{*\}, \mu)$ be the tree series transducer of Example 3.6. The constants of Definition 4.1 are $r_M = 2$, $d_M = 1$, $e_M = 2$, and $u_{M,\mathrm{o}} = 1$, and we let l and g be the cardinality and sum majorization presented in Example 4.4, respectively (i.e., $l(n) = 1$ and $g(n, a) = a$ for every $n \in \mathbb{N}_+$ and $a \in \mathbb{N} \cup \{-\infty\}$).

Finally, we let $c = 1$, which is an upper bound of the coefficients of μ according to Definition 4.2. We obtain the ample coefficient majorization $f^{\mathrm{bot}}_{M,\mathrm{o},l,g,c}$ with $f^{\mathrm{bot}}_{M,\mathrm{o},l,g,c}(1) = 1$ and for every $n \geq 2$

$$\begin{aligned} f^{\mathrm{bot}}_{M,\mathrm{o},l,g,c}(n) &= g\big(2 \cdot l(n-1)^2, 1 + f^{\mathrm{bot}}_{M,\mathrm{o},l,g,c}(n-1)\big) \\ &= 1 + f^{\mathrm{bot}}_{M,\mathrm{o},l,g,c}(n-1) \;=\; n. \end{aligned}$$

Theorem 4.7 applied to this example yields that $(\tau^{\mathrm{o}}_M(s), t) \leq n$ for every $n \in \mathbb{N}_+$, $s \in T_\Sigma$ of height at most n, and $t \in \mathrm{supp}(\tau^{\mathrm{o}}_M(s))$. Note, furthermore, that $f^{\mathrm{bot}}_{M,\mathrm{o},l,g,c}$ coincides with the coefficient majorization presented in Example 4.4. □

We have derived a mapping $f^{\mathrm{bot}}_{M,m,l,g,c}$ that limits the coefficients of output subtrees generated by M. By definition, $f^{\mathrm{bot}}_{M,m,l,g,c}$ depends on a cardinality majorization $l : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ and a sum majorization $g : \mathbb{N}_+ \times A \longrightarrow A$. The sum majorization g is specific for $\mathcal{A}$ and needs to be provided from the outside; i.e., it cannot be deduced from properties of M. In Section 5 we will see how restrictions on $\mathcal{A}$ allow an easy definition of g. The cardinality majorization l limits the support cardinality of the computed tree series. This mapping was also supplied from the outside, but now we derive an easy cardinality majorization l^{bot}_M.

Given $n \in \mathbb{N}_+$, we have to limit the cardinality of the support of $h^m_\mu(s)_q$ for every $s \in T_\Sigma$ of height at most n and $q \in Q$. The idea is to pessimistically assume that given $k \in \mathbb{N}$, pairs of different trees $(t, t') \in T_\Delta(X_k)^2$, and $(t_1, t'_1), \ldots, (t_k, t'_k) \in (T_\Delta)^2$, the trees $t[t_1, \ldots, t_k]$ and $t'[t'_1, \ldots, t'_k]$ are different. This is — of course — not true in general, but it is appropriate for our cardinality majorization because the number of different trees in the support might only be overestimated.

4.9 Definition The *ample cardinality majorization associated with* M is the mapping $l_M^{\mathrm{bot}} : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ defined for every $n \in \mathbb{N}_+$ by

$$\begin{aligned} l_M^{\mathrm{bot}}(n) &= (d_M)^{\sum_{i\in[1,n-1]} r_M^i} \cdot (e_M)^{\sum_{i\in[0,n-1]} r_M^i} \\ &= \begin{cases} e_M & \text{if } n = 1, \\ (d_M)^{r_M} \cdot e_M \cdot l_M^{\mathrm{bot}}(n-1)^{r_M} & \text{if } n > 1. \end{cases} \end{aligned}$$ □

4.10 Lemma The ample cardinality majorization associated with M is a cardinality majorization; i.e., $\mathrm{card}(\mathrm{supp}(h_\mu^m(s)_q)) \leq l_M^{\mathrm{bot}}(n)$ for every $n \in \mathbb{N}_+$, $q \in Q$, and $s \in T_\Sigma$ of height at most n.

Proof. If M is deterministic, then $l_M^{\mathrm{bot}}(n) = 1$, which is a cardinality majorization by Proposition 3.9. Now assume that M is not deterministic. We prove the statement by structural induction over s.

Induction base: Let $s = \alpha$ with $\alpha \in \Sigma^{(0)}$.

$$\mathrm{card}\big(\mathrm{supp}(h_\mu^m(\alpha)_q)\big) \overset{\text{Def. 3.5(i)}}{=} \mathrm{card}\big(\mathrm{supp}(\mu_0(\alpha)_{q,\varepsilon})\big) \;\leq\; e_M \;=\; l_M^{\mathrm{bot}}(1)$$

Induction step: Let $s = \sigma(s_1, \ldots, s_k)$ for some $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma$. Recall that $\mathrm{height}(s) \leq n$.

$$\begin{aligned} &\mathrm{card}\big(\mathrm{supp}(h_\mu^m(\sigma(s_1,\ldots,s_k))_q)\big) \\ \overset{\text{Def. 3.5(i)}}{=}\;& \mathrm{card}\Big(\mathrm{supp}\Big(\sum_{w=(q_1,\ldots,q_k)\in Q^k} \mu_k(\sigma)_{q,w} \overset{m}{\longleftarrow} (h_\mu^m(s_1)_{q_1}, \ldots, h_\mu^m(s_k)_{q_k})\Big)\Big) \\ =\;& \mathrm{card}\Big(\mathrm{supp}\Big(\sum_{\substack{w=(q_1,\ldots,q_k)\in Q^k,\\ t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall i\in[k]):\, t_i\in\mathrm{supp}(h_\mu^m(s_i)_{q_i})}} (\mu_k(\sigma)_{q,w}, t') \odot \prod_{i\in[k]} (h_\mu^m(s_i)_{q_i}, t_i)^{n_i}\; t'[t_1,\ldots,t_k]\Big)\Big) \\ & \text{where for every } i \in [k]\text{: } n_i = \begin{cases} |t'|_{x_i} & \text{if } m = \mathrm{o}, \\ 1 & \text{if } m = \varepsilon \end{cases} \\ \overset{\dagger}{\leq}\;& (d_M)^k \cdot e_M \cdot l_M^{\mathrm{bot}}(n-1)^k \;\leq\; (d_M)^{r_M} \cdot e_M \cdot l_M^{\mathrm{bot}}(n-1)^{r_M} \\ =\;& l_M^{\mathrm{bot}}(n) \end{aligned}$$

At † we used the induction hypothesis and $e_M \geq \mathrm{card}(\mathrm{supp}(\mu_k(\sigma)_{q,w}))$. ■

Thus we obtain an ample coefficient majorization $f^{\mathrm{bot}}_{M,m,l_M^{\mathrm{bot}},g,c}$ that only depends on the constants, a sum majorization g, and c. If we use $g(n,a) = \sum_{i\in[n]} a$ as sum majorization, we obtain a coefficient majorization that only depends on the constants and c.

4.3. The top-down case

The results of this section are also proved for two cases: (C1) $\mathcal{A}$ satisfies (G⊕), or (C2) M is deterministic and $l(n) = 1$ for every $n \in \mathbb{N}_+$, which is a cardinality majorization by Proposition 3.9. As in the bottom-up case, (PO$\oplus$) is not even needed in Case (C2). In this section we consider polynomial top-down tree series transducers and derive similar majorizations for them. Thus, M always denotes a (non-trivial) polynomial top-down tree series transducer with $\mathrm{mx}_\Sigma \geq 1$ in this section. Moreover, we let l and g be a cardinality and a sum majorization, respectively, and c be an upper bound of the coefficients of μ. Recall the constants d_M, e_M, and v_M of Definition 4.1.

4.11 Definition The *ample coefficient majorization* $f^{\mathrm{top}}_{M,l,g,c} : \mathbb{N}_+ \longrightarrow A$ *(associated with l, g, and c)* is defined recursively by

$$f^{\mathrm{top}}_{M,l,g,c}(1) = c$$
$$f^{\mathrm{top}}_{M,l,g,c}(n) = g\big((d_M)^{1+v_M} \cdot e_M \cdot l(n-1)^{v_M}, c \odot f^{\mathrm{top}}_{M,l,g,c}(n-1)^{v_M}\big)$$

for every $n \geq 2$. □

Note the structural similarity of $f^{\mathrm{top}}_{M,l,g,c}$ and the ample coefficient majorization of a polynomial bottom-up tree series transducer. Also note that $f^{\mathrm{top}}_{M,l,g,c}$ does not depend on m. Theorem 4.7, which states that the ample coefficient majorization of a polynomial bottom-up tree series transducer is indeed a coefficient majorization, and its proof can be translated in a straightforward manner to the top-down case. The general approach remains the same, though there are some notational changes, so we resupply the proof. Due to Theorem 3.8(i) it suffices to consider $m = \varepsilon$.

4.12 Observation Given (C1) or (C2), we have $\mathbf{1} \preceq f^{\mathrm{top}}_{M,l,g,c}(n)$ for every $n \in \mathbb{N}_+$.

Proof. The proof is literally the same as the proof of Observation 4.6 except that $f^{\mathrm{bot}}_{M,m,l,g,c}$ and $u_{M,m}$ have to be replaced by $f^{\mathrm{top}}_{M,l,g,c}$ and v_M, respectively. ■

4.13 Theorem Given (C1) or (C2), the ample coefficient majorization $f^{\mathrm{top}}_{M,l,g,c}$ is a coefficient majorization; i.e., $f^{\mathrm{top}}_{M,l,g,c}(n) \in {\uparrow} C^{\varepsilon}_{M}(n)$ for every $n \in \mathbb{N}_+$. Moreover,

- in Case (C1): $(\tau^{\varepsilon}_M(s), t) \preceq g\big(\mathrm{card}(D), f^{\mathrm{top}}_{M,l,g,c}(n)\big)$; and
- in Case (C2): $(\tau^{\varepsilon}_M(s), t) \preceq g\big(1, f^{\mathrm{top}}_{M,l,g,c}(n)\big)$

for every $n \in \mathbb{N}_+$, $s \in T_\Sigma$, and $t \in \mathrm{supp}(\tau^{\varepsilon}_M(s))$ such that $\mathrm{height}(s) \leq n$.

Proof. The proof of the latter statement is identical to the proof of the corresponding statement of Theorem 4.7. So it remains to prove $(h^{\varepsilon}_{\mu}(s)_q, t) \preceq f^{\mathrm{top}}_{M,l,g,c}(n)$ for every $n \in \mathbb{N}_+$, $q \in Q$, $s \in T_\Sigma$, and $t \in \mathrm{supp}(h^{\varepsilon}_{\mu}(s)_q)$ such that $\mathrm{height}(s) \leq n$. We prove this statement by structural induction over s.

Induction base: Let $s = \alpha$ with $\alpha \in \Sigma^{(0)}$. Since $t \in \mathrm{supp}(h^{\varepsilon}_{\mu}(\alpha)_q)$, we have

$$(h^{\varepsilon}_{\mu}(\alpha)_q, t) \overset{\text{Def. 3.5(i)}}{=} (\mu_0(\alpha)_{q,\varepsilon}, t) \;\preceq\; c \;=\; f^{\mathrm{top}}_{M,l,g,c}(1).$$

Induction step: Let $k \in \mathbb{N}_+$, $\sigma \in \Sigma^{(k)}$, and $s_1, \ldots, s_k \in T_\Sigma$ such that $s = \sigma(s_1, \ldots, s_k)$ is of height at most n. First let M be not deterministic; i.e., Case (C1).

$$\big(h^{\varepsilon}_{\mu}\big(\sigma(s_1, \ldots, s_k)\big)_q, t\big)$$

$$
\begin{aligned}
&\overset{\text{Def. 3.5(i)}}{=} \Big(\sum_{\substack{w \in Q(X_k)^*,\\ w=(q_1(x_{i_1}),\ldots,q_l(x_{i_l}))}} \mu_k(\sigma)_{q,w} \xleftarrow{\varepsilon} (h^\varepsilon_\mu(s_{i_1})_{q_1}, \ldots, h^\varepsilon_\mu(s_{i_l})_{q_l}), t\Big) \\
&\overset{\star}{=} \sum_{\substack{w=(q_1(x_{i_1}),\ldots,q_l(x_{i_l}))\in Q(X_k)^*,\\ t=t'[t_1,\ldots,t_l], t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall j\in[l])\colon\, t_j\in\mathrm{supp}(h^\varepsilon_\mu(s_{i_j})_{q_j})}} (\mu_k(\sigma)_{q,w}, t') \odot \prod_{j\in[l]} (h^\varepsilon_\mu(s_{i_j})_{q_j}, t_j) \\
&\overset{\text{I.H.}}{\preceq} \sum_{\ldots} c \odot f^{\mathrm{top}}_{M,l,g,c}(n-1)^l \overset{\text{Obs. 4.12}}{\preceq} \sum_{\ldots} c \odot f^{\mathrm{top}}_{M,l,g,c}(n-1)^{v_M} \\
&\qquad \text{where } \sum_{\ldots} \text{ is the sum of } \star \\
&\overset{\dagger}{\preceq} \sum_{\substack{w=(q_1(x_{i_1}),\ldots,q_l(x_{i_l}))\in Q(X_k)^*,\\ t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall j\in[l])\colon\, t_j\in\mathrm{supp}(h^\varepsilon_\mu(s_{i_j})_{q_j})}} a \;\preceq \sum_{j'\in[(d_M)^{1+v_M}\cdot e_M\cdot\prod_{j\in[v_M]} l(n-1)]} a \\
&\qquad \text{where } a = c \odot f^{\mathrm{top}}_{M,l,g,c}(n-1)^{v_M} \\
&\preceq\; g\big((d_M)^{1+v_M}\cdot e_M\cdot l(n-1)^{v_M}, a\big) \;=\; f^{\mathrm{top}}_{M,l,g,c}(n)
\end{aligned}
$$

The step at $\dagger$ is governed by $t \in \mathrm{supp}(h^\varepsilon_\mu(s)_q)$, which implies that there exists at least one non-zero summand of the sum, needed for the application of Observation 2.2(ii). For the next step, note that $e_M \geq \mathrm{card}(\mathrm{supp}(\mu_k(\sigma)_{q,w}))$ and $l(n-1) \geq \mathrm{card}(\mathrm{supp}(h^\varepsilon_\mu(s_{i_j})_{q_j}))$. Finally, let us consider how many $w \in Q(X_k)^*$ there are such that $\mu_k(\sigma)_{q,w} \neq \widetilde{\mathbf{0}}$. Clearly, there are at most $\sum_{j\in[0,v_M]}(d_M)^j$ such w, but if M is also bottom-up then there are at most $(d_M)^k$ because $w = (q_1(x_1), \ldots, q_k(x_k))$ for some $q_1, \ldots, q_k \in Q$. Note that $d_M > 1$ by Definition 4.1 except when M is top-down and bottom-up. If $d_M > 1$, then $\sum_{j\in[0,v_M]}(d_M)^j \leq (d_M)^{1+v_M}$, and if M is bottom-up then $(d_M)^k \leq (d_M)^{1+v_M}$ because $k \leq \mathrm{mx}_\Sigma = v_M$. This concludes the induction step for Case (C1).

Now let M be deterministic.

$$
\begin{aligned}
&\big(h^\varepsilon_\mu(\sigma(s_1,\ldots,s_k))_q, t\big) \\
&\overset{\text{Def. 3.5(i)}}{=} \Big(\sum_{\substack{w \in Q(X_k)^*,\\ w=(q_1(x_{i_1}),\ldots,q_l(x_{i_l}))}} \mu_k(\sigma)_{q,w} \xleftarrow{\varepsilon} (h^\varepsilon_\mu(s_{i_1})_{q_1}, \ldots, h^\varepsilon_\mu(s_{i_l})_{q_l}), t\Big) \\
&= \sum_{\substack{w=(q_1(x_{i_1}),\ldots,q_l(x_{i_l}))\in Q(X_k)^*,\\ t=t'[t_1,\ldots,t_l], t'\in\mathrm{supp}(\mu_k(\sigma)_{q,w}),\\ (\forall j\in[l])\colon\, t_j\in\mathrm{supp}(h^\varepsilon_\mu(s_{i_j})_{q_j})}} (\mu_k(\sigma)_{q,w}, t') \odot \prod_{j\in[l]} (h^\varepsilon_\mu(s_{i_j})_{q_j}, t_j) \\
&\overset{\text{I.H.}}{\preceq} \sum_{j'\in[(d_M)^{1+v_M}\cdot e_M\cdot\prod_{j\in[v_M]} l(n-1)]} c \odot f^{\mathrm{top}}_{M,l,g,c}(n-1)^{v_M} \\
&\preceq\; g\big((d_M)^{1+v_M}\cdot e_M\cdot l(n-1)^{v_M}, a\big) \;=\; f^{\mathrm{top}}_{M,l,g,c}(n)
\end{aligned}
$$

Since M is deterministic, w is fixed by definition, $\mathrm{supp}(\mu_k(\sigma)_{q,w})$ is singleton by definition and the fact that $t \in \mathrm{supp}(h_\mu^\varepsilon(s)_q)$, and $\mathrm{supp}(h_\mu^\varepsilon(s_{i_j})_{q_j})$ is singleton for every $j \in [l]$ by Proposition 3.9 and $t \in \mathrm{supp}(h_\mu^\varepsilon(s)_q)$. Recall that $d_M = 1$ and $e_M = 1$. Moreover, Property (G$\oplus$) holds if $l(n-1) \neq 1$. Together with the induction hypothesis, this justifies the step marked I.H. ∎

Finally, we also derive an ample cardinality majorization for polynomial top-down tree series transducers.

4.14 Definition The *ample cardinality majorization associated with* M is the mapping $l_M^{\mathrm{top}} : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ recursively defined by

$$l_M^{\mathrm{top}}(1) = e_M$$
$$l_M^{\mathrm{top}}(n) = (d_M)^{1+v_M} \cdot e_M \cdot l_M^{\mathrm{top}}(n-1)^{v_M}$$

for every $n \geq 2$. □

4.15 Lemma The ample cardinality majorization associated with M is a cardinality majorization; i.e., $\mathrm{card}(\mathrm{supp}(h_\mu^\varepsilon(s)_q)) \leq l_M^{\mathrm{top}}(n)$ for every $n \in \mathbb{N}_+$, $q \in Q$, and $s \in T_\Sigma$ of height at most n.

Proof. The proof proceeds along the lines of the proof of Lemma 4.10 with just minor changes, most of which were already outlined in the proof of Theorem 4.13. ∎

5. Incomparability results

In the first part of this section we reprove two recent results from [27] concerning growth properties of polynomial bottom-up tree series transducers using our coefficient majorization approach (i.e., using Theorem 4.7). The second part then focuses on some simplified coefficient majorization that allows us to derive incomparability results for classes of m-t-ts transformations computed by polynomial bottom-up as well as top-down tree series transducers.

Let $M = (Q, \Sigma, \Delta, \mathbb{N}_\infty, D, \mu)$ be a polynomial bottom-up tree series transducer. First we reprove a slightly less general version of Lemma 5.14 of [27]. In [27], ∞ may not occur as coefficient in $\tau_M^m(s)$ for every $s \in T_\Sigma$. However, if ∞ occurs in μ but not in $\tau_M^m(s)$, then ∞ can be eliminated from μ.

5.1 Lemma Let $M = (Q, \Sigma, \Delta, \mathbb{N}_\infty, D, \mu)$ be a polynomial bottom-up tree series transducer with $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and $\Delta = \{\alpha^{(0)}\}$ such that ∞ does not occur as coefficient in any tree series of μ. There exists an integer b such that $(\tau_M^{\mathrm{o}}(s), \alpha) \leq b^{\mathrm{height}(s)}$ for every $s \in T_\Sigma$.

Proof. We can instantiate Theorem 4.7, because $\mathbb{N}_\infty$ is totally ordered by $\leq$. Moreover, $\mathbb{N}_\infty$ has property (G$\oplus$). The constants of Definition 4.1 are: $r_M = \mathrm{mx}_\Sigma = 2$, $e_M \leq \mathrm{card}(T_\Delta(X_2)) = 3$, and $u_{M,\mathrm{o}} \leq 1$. Finally, an upper bound $c \in \mathbb{N}$ of the coefficients of μ clearly exists.

We choose the sum majorization $g(n, a) = n \cdot a$ and the cardinality majorization $l(n) = 1$, which is a cardinality majorization due to $\mathrm{card}(T_\Delta) = 1$. Hence

$$\begin{aligned} f^{\mathrm{bot}}_{M,\mathrm{o},l,g,c}(1) &= c \\ f^{\mathrm{bot}}_{M,\mathrm{o},l,g,c}(n) &= (d_M)^2 \cdot e_M \cdot c \cdot f^{\mathrm{bot}}_{M,\mathrm{o},l,g,c}(n-1)^{u_{M,\mathrm{o}}} \end{aligned}$$

for every $n \geq 2$ and thus $f^{\mathrm{bot}}_{M,\mathrm{o},l,g,c}(n) \leq (3 \cdot (d_M)^2 \cdot c)^{n-1} \cdot c$. Because $c \neq \infty$, we have $b = 3 \cdot \mathrm{card}(Q) \cdot (d_M)^2 \cdot c \neq \infty$, and we obtain $(\tau^{\mathrm{o}}_M(s), \alpha) \leq b^{\mathrm{height}(s)}$ by Theorem 4.7. ■

Similarly we can prove a variant of Lemma 5.16 of [27].

5.2 Lemma Let $M = (Q, \Sigma, \Delta, \mathbb{N}_\infty, D, \mu)$ be a polynomial bottom-up tree series transducer with $\mathrm{mx}_\Sigma = 1$ such that ∞ does not occur as coefficient in any tree series of μ. Then there is an integer b such that $(\tau^{\varepsilon}_M(s), t) \leq b^{\mathrm{height}(s)^2}$ for every $s \in T_\Sigma$ and $t \in T_\Delta$.

Proof. Theorem 4.7 is applicable, because $\mathbb{N}_\infty$ fulfils the general restrictions imposed on the semiring. Obviously, the constants of Definition 4.1 are: $r_M = \mathrm{mx}_\Sigma = 1$ and $u_{M,\varepsilon} = 1$. Clearly, there exists an upper bound $c \in \mathbb{N}$ of the coefficients of μ. Using the sum majorization $g(n, a) = n \cdot a$ and ample cardinality majorization $l = l^{\mathrm{bot}}_M$ associated with M of Definition 4.9, which is a cardinality majorization due to Lemma 4.10, we obtain

$$\begin{aligned} f^{\mathrm{bot}}_{M,\varepsilon,l,g,c}(1) &= c \\ f^{\mathrm{bot}}_{M,\varepsilon,l,g,c}(n) &= (d_M)^{n-1} \cdot (e_M)^n \cdot c \cdot f^{\mathrm{bot}}_{M,\varepsilon,l,g,c}(n-1) \end{aligned}$$

for every $n \geq 2$ and thus

$$f^{\mathrm{bot}}_{M,\varepsilon,l,g,c}(n) = (d_M)^{\sum_{i \in [1,n-1]} i} \cdot (e_M)^{\sum_{i \in [2,n]} i} \cdot c^n \leq (d_M \cdot e_M \cdot c)^{\frac{n \cdot (n+1)}{2}},$$

which implies the required bound by setting $b = \mathrm{card}(Q) \cdot d_M \cdot e_M \cdot c_M$ as follows. Since $\mathrm{card}(Q) \cdot (d_M \cdot e_M \cdot c)^{\frac{n \cdot (n+1)}{2}} \leq b^{\mathrm{height}(s)^2}$, we obtain $(\tau^{\varepsilon}_M(s), t) \leq b^{\mathrm{height}(s)^2}$ by Theorem 4.7. ■

Corollary 5.18 in [27] is proved using essentially Lemmata 5.1 and 5.2 together with some examples required to show incomparability.

5.3 Corollary $\mathrm{p\text{–}BOT}^{\varepsilon}(\mathbb{N}_\infty) \bowtie \mathrm{p\text{–}BOT}^{\mathrm{o}}(\mathbb{N}_\infty)$. □

Using the same approach we can also reprove Lemmata 5.19 and 5.21 of [27]. They are used to prove Corollary 5.23 of [27], which essentially states the above for $\mathbb{T}$.

The ample cardinality majorization associated with a deterministic tree series transducer (see Definitions 4.9 and 4.14) is $l(n) = 1$. Now let us consider the first argument of the sum majorization g in the definition of the ample coefficient majorization (see Definitions 4.5 and 4.11) for deterministic tree series transducers. We immediately observe that the first argument is always 1. Next let us consider additively idempotent semirings. Certainly, such semirings fulfil (G⊕) irrespective of the partial order. Moreover, additively idempotent semirings are partially ordered by their natural order. Finally, we can use $g(n, a) = a$ as sum majorization.

Let $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ be a (non-trivial) polynomial tree series transducer that is bottom-up or top-down. The following theorem shows that, provided that M is deterministic or $\mathcal{A}$ is additively idempotent, a very simple mapping, called coefficient approximation, is a coefficient majorization.

5.4 Definition For every $a \in A$ and $z \in \mathbb{N}$ we define the *coefficient approximation* $f_{a,z} : \mathbb{N}_+ \longrightarrow A$ by $f_{a,z}(n) = a^{\sum_{i \in [0,n-1]} z^i}$ for every $n \in \mathbb{N}_+$. □

5.5 Theorem Let $m \in \{\varepsilon, \mathrm{o}\}$, and let $\mathcal{A}$ be a semiring partially ordered by $\preceq$. Moreover, let $M = (Q, \Sigma, \Delta, \mathcal{A}, D, \mu)$ be a polynomial tree series transducer that is top-down or bottom-up, and let c be an upper bound of the coefficients of μ. If (i) $\mathcal{A}$ is additively idempotent or (ii) M is deterministic, then the coefficient approximation $f_{c,z} : \mathbb{N}_+ \longrightarrow A$ with

$$z = \begin{cases} u_{M,m} & \text{if } M \text{ is bottom-up,} \\ v_M & \text{if } M \text{ is top-down,} \end{cases}$$

is a coefficient majorization. Moreover $(\tau_M^m(s), t) \preceq f_{c,z}(n)$ for every $n \in \mathbb{N}_+$, $s \in T_\Sigma$ of height at most n, and $t \in \mathrm{supp}(\tau_M^m(s))$.

Proof. To show that $f_{c,z}$ is a coefficient majorization, we show that $f_{c,z}$ is equal to the ample coefficient majorization $f^{\mathrm{bot}}_{M,m,l,g,c}$ or $f^{\mathrm{top}}_{M,l,g,c}$ (depending on whether M is bottom-up or top-down) for a particular cardinality majorization l and the sum majorization $g(n,a) = \sum_{i \in [n]} a$.

Let us first consider case (ii), in which M is deterministic. We let $l(n) = 1$ for every $n \in \mathbb{N}_+$, which is a cardinality majorization due to Proposition 3.9. Finally, we note that by determinism $d_M = 1$ and $e_M = 1$, thus the first argument of the sum majorization g is always 1. In case (i) we let l be an arbitrary cardinality majorization; e.g., we could set $l = l^{\mathrm{bot}}_M$ if M is bottom-up, and $l = l^{\mathrm{top}}_M$ if M is top-down. Clearly, $g(n,a) = a$ for every $n \in \mathbb{N}_+$ and $a \in A$.

We continue in both cases by showing that $f_{c,z}(n) = h(n)$, where $h = f^{\mathrm{bot}}_{M,m,l,g,c}$ if M is bottom-up, and $h = f^{\mathrm{top}}_{M,l,g,c}$ if M is top-down. Obviously, $h(1) = c = f_{c,z}(1)$ and otherwise

$$\begin{aligned} h(n) &= g\big((d_M)^x \cdot e_M \cdot l(n-1)^y, c \odot h(n-1)^z\big) \\ &\qquad \text{where } x = y = r_M \text{ and } z = u_{M,m} \text{ if } M \text{ is bottom-up,} \\ &\qquad \text{otherwise } x = 1 + v_M \text{ and } y = z = v_M \\ &= c \odot h(n-1)^z \;=\; c^{\sum_{i \in [0,n-1]} z^i} \;=\; f_{c,z}(n). \end{aligned}$$

Thus $h = f_{c,z}$ and by Theorems 4.7 and 4.13 it follows that $f_{c,z}$ is a coefficient majorization. It remains to show the latter statement of the theorem. In case (i) we have

$$(\tau_M^m(s), t) \overset{\text{Thms. 4.7 \& 4.13}}{\preceq} \sum_{q \in D} f_{c,z}(n) \overset{\dagger}{=} f_{c,z}(n),$$

where at † we used that $\mathcal{A}$ is idempotent. In case (ii) we conclude $(\tau_M^m(s), t) \preceq f_{c,z}(n)$ from Theorems 4.7 and 4.13. ■

The following observation shows that $f_{a,z}(n) \preceq f_{a,z'}(n)$ whenever $z \leq z'$. This allows us to use an upper bound of the parameter z in order to obtain an upper bound of the coefficient of an output tree.

5.6 Observation Let $a \in A$ with $\mathbf{1} \preceq a$ and $z, z', n \in \mathbb{N}$ with $z \leq z'$. Then $f_{a,z}(n) \preceq f_{a,z'}(n)$.

Proof. Immediate from Observation 2.2(iii). ∎

Next we establish that the coefficient approximation for deterministic tree series transducers as well as for polynomial tree series transducers over additively idempotent semirings (i.e., in those cases when it is a coefficient majorization according to Theorem 5.5) gives an upper bound that can be reached by a homomorphism tree series transducer. We use this result in our main incomparability result (see Lemma 5.9).

5.7 Lemma Let $\Sigma' = \{\gamma^{(1)}, \alpha^{(0)}\}$, $\Delta' = \{\delta^{(2)}, \alpha^{(0)}\}$, and $\Delta'' = \{\alpha^{(0)}\}$. Moreover, let $z \in \mathbb{N}_+$, $c \in A$ with $\mathbf{1} \preceq c$, and $\Sigma'' = \{\sigma^{(z)}, \alpha^{(0)}\}$. There exists a homomorphism x tree series transducer $N = (\{*\}, \Sigma, \Delta, \mathcal{A}, \{*\}, \nu)$ such that c is an upper bound of the coefficients of ν, $u_{N,m} = z$, and for every $n \in \mathbb{N}_+$ there exist $s \in T_\Sigma$ of height n and $t \in \mathrm{supp}(\tau_N^m(s))$ such that $(\tau_N^m(s), t) = f_{c,z}(n)$, where:

(i) $m = \varepsilon$, $\Sigma = \Sigma''$, $\Delta = \Delta''$, and $x =$ bottom-up;

(ii) $m = \mathrm{o}$, $\Sigma = \Sigma'$, $\Delta = \Delta'$, and $x =$ bottom-up; or

(iii) $m = \mathrm{o}$, $\Sigma = \Sigma'$, $\Delta = \Delta'$, and $x =$ top-down (note that $u_{N,\mathrm{o}} = v_N$).

Proof. We prove the statements individually.

(i) Let $\nu_0(\alpha)_{*,\varepsilon} = c\,\alpha$ and $\nu_z(\sigma)_{*,(*,\ldots,*)} = c\,\alpha$. Note that $u_{N,\varepsilon} = z$. Moreover, let $s \in T_{\Sigma''}$ be the fully balanced tree of height $n \in \mathbb{N}_+$. A straightforward structural induction shows that $(\tau_N^\varepsilon(s), \alpha) = f_{c,z}(n)$ as follows. The induction base is $(\tau_N^\varepsilon(\alpha), \alpha) = c = f_{c,z}(1)$. In the induction step we have for every $s = \sigma(s', \ldots, s')$ with $s' \in T_{\Sigma''}$ being a fully balanced tree of height $n-1$

$$\begin{aligned}
&\big(\tau_N^\varepsilon\big(\sigma(s', \ldots, s')\big), \alpha\big) \\
&\overset{\text{Def. 3.5(ii)}}{=} \big(h_\nu^\varepsilon\big(\sigma(s', \ldots, s')\big)_*, \alpha\big) \\
&= \quad (\nu_z(\sigma)_{*,(*,\ldots,*)}, \alpha) \odot \prod_{i \in [z]} (h_\nu(s')_*, \alpha) \overset{\text{Def. 3.5(ii)}}{=} c \odot (\tau_N^\varepsilon(s'), \alpha)^z \\
&\overset{\text{I.H.}}{=} \quad c \odot f_{c,z}\big(n-1\big)^z \overset{\text{Def. 5.4}}{=} f_{c,z}(n).
\end{aligned}$$

(ii) Let $\nu_0(\alpha)_{*,\varepsilon} = c\alpha$ and $\nu_1(\gamma)_{*,(*)} = c\delta(x_1, \delta(\ldots, \delta(x_1, \alpha)\ldots))$ such that x_1 occurs z times in the latter tree. Clearly, $u_{N,\mathrm{o}} = z$. Moreover, one can easily show by a similar induction as in Item (i) that for every $s \in T_{\Sigma'}$ of height $n \in \mathbb{N}_+$ there exists $t \in T_{\Delta'}$ such that $(\tau_N^{\mathrm{o}}(s), t) = f_{c,z}(n)$.

(iii) Let $\nu_0(\alpha)_{*,\varepsilon} = c\alpha$ and $\nu_1(\gamma)_{*,(*(x_1),\ldots,*(x_1))} = c\delta(x_1, \delta(\ldots, \delta(x_z, \alpha)\ldots))$. Clearly $v_N = u_{N,\mathrm{o}} = z$ and the proof of (iii) is analogous to the previous ones and omitted. ∎

The main theorem states the incomparability of the classes of m-t-ts transformations computed by polynomial bottom-up tree series transducers for $m = \varepsilon$ and $m = \mathrm{o}$ over a semiring $\mathcal{A}$ with an additional property, which we introduce next. Roughly speaking, we require that $\mathcal{A}$ is partially ordered by a partial order $\preceq$ such that for some $a \in A$ we have $a^i \prec a^j$ whenever $i < j$. Moreover, we require that every element that occurs in a decomposition of a^n can be bounded from above by a power of a.

5.8 Definition A partially ordered semiring $\mathcal{A} = (A, \oplus, \odot, \mathbf{0}, \mathbf{1}, \preceq)$ is *weakly growing*, if:

(i) there is $a \in A$ such that $a^i \prec a^j$ for all non-negative integers $i < j$; and

(ii) for every $a_1, a_2, b \in A \setminus \{\mathbf{0}\}$, $d \in A$, and $n \in \mathbb{N}$, if $a^n = a_1 \odot b \odot a_2 \oplus d$, then there is an $m \in \mathbb{N}$ such that $b \preceq a^m$. □

The first condition ensures that $a^0 \prec a^1 \prec a^2 \prec \cdots$. The second condition intuitively requires that the growth is not too slow; i.e., we should at least be able to bound (from above) elements that occur in decompositions. Stronger conditions than (ii) can be obtained, for example, by requiring that whenever $b \neq \mathbf{0}$, then $b \preceq a^m$ for some $m \in \mathbb{N}$, an Archimedian type property for the element a. This would essentially state that the growth of a is unbounded; i.e., $\uparrow \{ a^n \mid n \in \mathbb{N} \} = \emptyset$. Certainly, the non-negative integers (without infinity) $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ fulfil this property for $a = 2$ as well as $\mathbb{A}$ does for $a = 1$. However, already $\mathbb{N}_\infty$ does not satisfy it.

Another strong notion of growth can be obtained by requiring that (i) $\mathcal{A}$ is naturally ordered, (ii) $a^i \sqsubset a^j$ whenever $i < j$, and (iii) $a \sqsubseteq a \odot b$ and $a \sqsubseteq b \odot a$ for every $a, b \in A$ with $b \neq \mathbf{0}$. The semirings $\mathbb{N}$, $\mathbb{N}_\infty$, and $\mathbb{A}$ fulfil this property, but $\mathbb{T}$ does not.

For our incomparability results we only need the weakly growing property. In the deterministic case, a yet weaker property obtained from Definition 5.8 by fixing $d = \mathbf{0}$ in Item (ii) is sufficient for our incomparability results for deterministic tree series transducers. However, we consider this a minor issue and proceed with the definition as given (even for deterministic tree series transducers).

The following semirings are weakly growing:

- $\mathbb{N}_\infty$ with the partial order $\leq$, $a = 2$, and $m = n$;
- $\mathbb{T}$ with the partial order $\leq$, $a = 1$, and $m = \max(n, d)$;
- $\mathbb{A}$ with the partial order $\leq$, $a = 1$, and $m = n$;
- $\mathbb{L}_S$ (S an alphabet) with the partial order $\subseteq$, $a = \{\varepsilon, s\}$ for some $s \in S$, and $m = n$.

The above statements are easily checked. On the other hand, $\mathbb{B}$ and $\mathbb{R}_{\min,\max}$ are not weakly growing because they are multiplicatively periodic.

Next we are going to show that given an additively idempotent and weakly growing semiring, the classes of m-t-ts transformations computed by polynomial bottom-up tree series transducers using ε-substitution and o-substitution are incomparable. Moreover, we also obtain the incomparability of the former class and the class of ε-t-ts transformations computed by polynomial top-down tree series transducers. The same statements for deterministic tree series transducers can be proved independently of additive idempotency.

Before stating the incomparability theorem, we provide a sketch of the proof. Informally speaking, we show both directions by constructing a specific homomorphism tree series transducer N using the particular coefficient $a \in A$ that fulfils the conditions of Definition 5.8. The approximation mapping can be applied to every polynomial tree series transducer M that is supposed to compute the same m-t-ts transformation. By a careful choice of the input and output ranked alphabets we limit the constants $u_{M,m}$ and v_M. We then proceed by showing that N has a higher growth rate than M. This growth argument yields the desired contradiction.

5.9 Lemma Let $\mathcal{A}$ be a weakly growing semiring. Moreover, let $x = \mathrm{p}$ if $\mathcal{A}$ is additively idempotent and $x = \mathrm{d}$ otherwise. Then:

- h–$\mathrm{BOT}^{\varepsilon}(\mathcal{A}) \not\subseteq x$–$\mathrm{BOT}^{\mathrm{o}}(\mathcal{A})$ and h–$\mathrm{BOT}^{\mathrm{o}}(\mathcal{A}) \not\subseteq x$–$\mathrm{BOT}^{\varepsilon}(\mathcal{A})$; and
- h–$\mathrm{BOT}^{\varepsilon}(\mathcal{A}) \not\subseteq x$–$\mathrm{TOP}^{\varepsilon}(\mathcal{A})$ and h–$\mathrm{TOP}^{\varepsilon}(\mathcal{A}) \not\subseteq x$–$\mathrm{BOT}^{\varepsilon}(\mathcal{A})$.

Proof. Let $\mathcal{A}$ be weakly growing with respect to the partial order $\preceq$ and the element $a \in A$ (see Definition 5.8) First we prove h–$\mathrm{BOT}^{\varepsilon}(\mathcal{A}) \not\subseteq x$–$\mathrm{BOT}^{\mathrm{o}}(\mathcal{A})$ and h–$\mathrm{BOT}^{\varepsilon}(\mathcal{A}) \not\subseteq x$–$\mathrm{TOP}^{\varepsilon}(\mathcal{A})$. We consider the ranked alphabets $\Sigma'' = \{\sigma^{(2)}, \alpha^{(0)}\}$ and $\Delta'' = \{\alpha^{(0)}\}$ as input and output ranked alphabet, respectively. Then by Lemma 5.7(i) (with $z = 2$) there is a homomorphism bottom-up tree series transducer $N = (\{\star\}, \Sigma'', \Delta'', \mathcal{A}, \{\star\}, \nu)$ such that a is an upper bound of the coefficients of ν, $u_{N,\varepsilon} = r_N = 2$, and for every $n \in \mathbb{N}_+$ there exist $s \in T_{\Sigma''}$ of height n and $t \in \mathrm{supp}\big(\tau_N^{\varepsilon}(s)\big)$ such that $(\tau_N^{\varepsilon}(s), t) = f_{a,u_{N,\varepsilon}}(n) = a^{2^n - 1}$.

Assume that there exists a (bottom-up or top-down) polynomial tree series transducer $M = (Q, \Sigma'', \Delta'', \mathcal{A}, D, \mu)$ which, in case $\mathcal{A}$ is not additively idempotent, is deterministic, with $\tau_M^{\mathrm{o}} = \tau_N^{\varepsilon}$. Since M is polynomial, there are only finitely many non-zero coefficients $c_1, \dots, c_k \in A$ for some $k \in \mathbb{N}$ occurring in the tree series of μ. Obviously, we can assume that for every c_j with $j \in [k]$ there exist $a_j, \bar{a}_j \in A \setminus \{\mathbf{0}\}$, $b_j \in A$, and $m_j \in \mathbb{N}$ such that $a^{m_j} = a_j \odot c_j \odot \bar{a}_j \oplus b_j$. If there is a c_j not obeying this property, then it cannot influence τ_M^{o} (see Definition 3.5), because $\tau_M^{\mathrm{o}} = \tau_N^{\varepsilon}$ and every coefficient appearing in a tree series in the range of τ_N^{ε} is a power of a. Thus, such coefficients c_j can be changed in μ to $\mathbf{1}$ without effect on τ_M^{o}.

Since $\mathcal{A}$ is weakly growing with respect to a, there is an $e_j \in \mathbb{N}$ such that $c_j \preceq a^{e_j}$. Consequently, $\max_{i \in [k]} a^{e_i} = a^{\max_{i \in [k]} e_i}$ is an upper bound of the coefficients of μ. Let $e = \max_{i \in [k]} e_i$ and $c' = a^e$. By Theorem 5.5 and Observation 5.6 for every $s \in T_{\Sigma''}$ and every $t \in \mathrm{supp}(\tau_M^{\mathrm{o}}(s))$

$$(\tau_M^{\mathrm{o}}(s), t) \preceq f_{c',1}\big(\mathrm{height}(s)\big) = (c')^{\mathrm{height}(s)} = (a^e)^{\mathrm{height}(s)},$$

because $u_{M,\mathrm{o}} \leq 1$ and $v_M \leq 1$ due to the specific form of Δ''. However, there exists an $n' \in \mathbb{N}_+$ such that $e \cdot n' < 2^{n'} - 1$. With this height n' there also exist $s' \in T_{\Sigma''}$ and $t' \in \mathrm{supp}(\tau_N^{\varepsilon}(s'))$ such that $(\tau_N^{\varepsilon}(s'), t') = f_{a,2}(n') = a^{2^{n'}-1}$, whereas $(\tau_M^{\mathrm{o}}(s'), t') \preceq a^{e \cdot n'}$ and $a^{e \cdot n'} \prec a^{2^{n'}-1}$ (because $\mathcal{A}$ is weakly growing with respect to a), which yields a contradiction to the assumption $\tau_M^{\mathrm{o}} = \tau_N^{\varepsilon}$. Consequently, τ_N^{ε} is neither in x–$\mathrm{BOT}^{\mathrm{o}}(\mathcal{A})$ nor in x–$\mathrm{TOP}^{\varepsilon}(\mathcal{A})$.

The statements h–$\mathrm{BOT}^{\mathrm{o}}(\mathcal{A}) \not\subseteq x$–$\mathrm{BOT}^{\varepsilon}(\mathcal{A})$ and h–$\mathrm{TOP}^{\varepsilon}(\mathcal{A}) \not\subseteq x$–$\mathrm{BOT}^{\varepsilon}(\mathcal{A})$ are established using the input ranked alphabet $\Sigma' = \{\gamma^{(1)}, \alpha^{(0)}\}$ and output ranked alphabet $\Delta' = \{\delta^{(2)}, \alpha^{(0)}\}$. By Lemma 5.7(ii) there is a homomorphism bottom-up tree series transducer N such that a is an upper bound of the coefficients of the tree representation of N and $u_{N,\mathrm{o}} = 2$ and by Lemma 5.7(iii) there is a homomorphism top-down tree series transducer N' with a being an upper bound of the coefficients of the tree representation of N' and $v_{N'} = 2$. Moreover, for every $n \in \mathbb{N}_+$ there exist $s, s' \in T_{\Sigma'}$ of height n and $t \in \mathrm{supp}(\tau_N^{\mathrm{o}}(s))$ and $t' \in \mathrm{supp}(\tau_{N'}^{\varepsilon}(s'))$ such that

$$(\tau_N^{\mathrm{o}}(s), t) = f_{a,u_{N,\mathrm{o}}}(n) = a^{2^n - 1} \quad \text{and} \quad (\tau_{N'}^{\varepsilon}(s'), t') = f_{a,v_{N'}}(n) = a^{2^n - 1}.$$

Let $M = (Q, \Sigma', \Delta', \mathcal{A}, D, \mu)$ be a polynomial bottom-up tree series transducer that is deterministic, whenever $\mathcal{A}$ is not additively idempotent, and $\tau_M^{\varepsilon} = \tau_N^{\mathrm{o}}$. An argumentation analogous to the one in the first part of the proof (using $u_{N,\varepsilon} = r_N = 1$) shows that $(\tau_M^{\varepsilon}(s), t) \preceq (c')^{\mathrm{height}(s)}$ for every $s \in T_{\Sigma'}$ and $t \in \mathrm{supp}(\tau_M^{\varepsilon}(s))$, where $c' = a^e$ for some $e \in \mathbb{N}$. This again yields the desired contradiction. ■

5.10 Theorem Let $\mathcal{A}$ be a weakly growing semiring. Moreover, let $\Pi = \{\mathrm{p}, \mathrm{d}, \mathrm{h}\}$ if $\mathcal{A}$ is additively idempotent, and $\Pi = \{\mathrm{d}, \mathrm{h}\}$ otherwise. For every $x, y \in \Pi$

$$x\text{–BOT}^{\mathrm{o}}(\mathcal{A}) \bowtie y\text{–BOT}^{\varepsilon}(\mathcal{A}) \quad \text{and} \quad y\text{–BOT}^{\varepsilon}(\mathcal{A}) \bowtie x\text{–TOP}^{\varepsilon}(\mathcal{A}).$$

Proof. The theorem is an immediate consequence of Lemma 5.9. ■

Consider the additively idempotent semirings $\mathbb{T}$, $\mathbb{A}$, and $\mathbb{L}_S$. For those and $\mathbb{N}_\infty$ we derive the following statements.

5.11 Corollary Let S be an alphabet.

(i) For every $\mathcal{A} \in \{\mathbb{T}, \mathbb{A}, \mathbb{L}_S\}$ and every $x, y \in \{\mathrm{p}, \mathrm{d}, \mathrm{h}\}$

$$x\text{–BOT}^{\varepsilon}(\mathcal{A}) \bowtie y\text{–BOT}^{\mathrm{o}}(\mathcal{A}) \quad \text{and} \quad x\text{–BOT}^{\varepsilon}(\mathcal{A}) \bowtie y\text{–TOP}^{\varepsilon}(\mathcal{A}).$$

(ii) For every $x, y \in \{\mathrm{d}, \mathrm{h}\}$

$$x\text{–BOT}^{\varepsilon}(\mathbb{N}_\infty) \bowtie y\text{–BOT}^{\mathrm{o}}(\mathbb{N}_\infty) \quad \text{and} \quad x\text{–BOT}^{\varepsilon}(\mathbb{N}_\infty) \bowtie y\text{–TOP}^{\varepsilon}(\mathbb{N}_\infty).$$

Proof. Both results are immediate consequences of Theorem 5.10. ■

In fact, the first part of Corollary 5.11(ii) is slightly weaker than Corollary 5.3 (Corollary 5.18 of [27]), because in the latter result classes of polynomial t-ts transformations are compared (and not only deterministic ones). The second part strengthens Proposition 3.14 of [19]. Also note that for $\mathcal{A} = \mathbb{T}$ the first part of Corollary 5.11(i) restates Corollary 5.23 of [27].

Open Problems

It remains to compare bottom-up tree series transducers using o-substitution and top-down tree series transducers. Homomorphism tree series transducers of the above types have been investigated in [27]. Specifically, it was shown in Theorem 5.12 of [27] that for homomorphisms over zero-divisor free and commutative semirings the classes of t-ts transformations computed by the above transducers coincide.

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